C3.4 Algebraic Geometry

Lecture 10. Degree theory

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Definition of degree

Recall that a **linear subspace** of $\mathbb{P}^n = \mathbb{P}(k^{n+1})$ is a projectivisation $L = \mathbb{P}(U)$ of a vector subspace $U \subset k^{n+1}$.

Consider a projective variety $X \subset \mathbb{P}^n$ of dimension d. Denote $m = n - d$, the **complementary dimension**.

Definition We define the **degree** of $X \subset \mathbb{P}^n$ to be

$$\deg(X) = \#(X \cap L) \colon L \subset \mathbb{P}^n \text{ a general linear subspace with } \dim L = m.$$

This needs an explanation. Consider $\mathrm{Gr}(m+1, n+1)$, the Grassmannian of $(m+1)$-dimensional subspaces of k^{n+1}.

Fact There is an open subset $U \subset \mathrm{Gr}(m+1, n+1)$ such that for $L \in U$, there is a finite number of intersection points $(X \cap L)$, and the number $\#(X \cap L)$ is maximal. We define this maximal value to be the degree $\deg X$.

Note: As we know, $\mathrm{Gr}(m+1, n+1)$ is irreducible, so every open set is automatically dense in it.

Linear subspaces

Example 1 Let $X = \mathbb{P}(V)$ a linear subvariety itself, of dimension d, with $V \subset k^{n+1}$ a vector subspace of dimension $d+1$.

Then the dimension of intersection formula in projective geometry says that for any other projective linear subspace $L = \mathbb{P}U$,

$$\dim(X \cap L) = \dim X + \dim L - \dim\langle X, L\rangle.$$

So if $\dim L = n - \dim X$, the complementary dimension, we get that

$$\dim(X \cap L) = 0$$

as long as the linear span $\langle X, L\rangle = \mathbb{P}^n$.

The latter will be the case as long as among the $m+1$ vectors giving a basis of U and the $d+1$ vectors giving a basis of V, we can find a basis of k^{n+1}. This will happen "most of the time" (in an open set of the corresponding Grassmannian).

The number of intersection points $\#(X \cap L) = 1$ in this case, so $\deg X = 1$.

Example 2 Let $X = \mathbb{V}(F) \subset \mathbb{P}^n$ a hypersurface defined by a homogeneous polynomial of degree d without multiple factors. Then $\dim X = n-1$, the complementary dimension is thus $m = 1$.

Let $L \subset \mathbb{P}^n$ be a line. We know $L \cong \mathbb{P}^1 \subset \mathbb{P}^n$. For example, as before, we can choose a basis of $\mathbb{P}^n$ so that

$$L = \{x_2 = \ldots = x_n = 0\} \subset \mathbb{P}^n.$$

Then

$$X \cap L = \{[x_0 \colon \ \ldots : x_n] \colon F(x_0, x_1) = 0, x_2 = \ldots = x_n = 0\}.$$

Here $F(x_0, x_1)$ is a homogeneous polynomial of degree d in two variables. The Fact referred to in the definition of degree translates here to the fact that if we choose a general L, then this polynomial will have d distinct roots.

Corollary For $X = \mathbb{V}(F) \subset \mathbb{P}^n$ as above, $\deg X = d$.

Extreme cases

At one extreme, let $p_1, \ldots, p_d \in \mathbb{P}^n$ be a finite set of points and

$$X = \{p_1, \ldots, p_d\} \subset \mathbb{P}^n.$$

Note that this has the structure of a projective variety, with $\dim X = 0$, as discussed in the last lecture.

The complementary dimension is $m = n$.

So we have

$$\deg X = \#(X \cap \mathbb{P}^n) = d.$$

At the other extreme, for $X = \mathbb{P}^n \subset \mathbb{P}^n$, the complementary dimension is $m = 0$, so $L = \{p\}$ a point. So

$$\deg \mathbb{P}^n = \#(\mathbb{P}^n \cap L) = 1.$$

This is of course also covered by the case of general linear subspaces.

Weak Bézout theorem

We state the following result without proof.

Theorem (Weak Bézout's Theorem) Let $X, Y \subset \mathbb{P}^n$ be projective varieties of pure dimension with

$$\dim X \cap Y = \dim X + \dim Y - n.$$

Then

$$\deg X \cap Y \leq \deg X \cdot \deg Y.$$

Note: the dimension assumption says that $X \cap Y$ is "of the expected dimension", given by the dimension of intersection formula for linear subspaces spanning $\mathbb{P}^n$.

Weak Bézout for plane curves

Example: Weak Bézout for plane curves
Let $C_1, C_2 \subset \mathbb{P}^2$ be plane curves, given by homogeneous polynomials $F_i \in k[x_0, x_1, x_2]$ of degrees d_i. Then as discussed before, $\dim C_i = 1$ and $\deg C_i = d_i$.

The dimension assumption above translates to the fact that

$$\dim C_1 \cap C_2 = \dim C_1 + \dim C_2 - 2 = 0,$$

so $C_1 \cap C_2$ is a finite set of points. In other words, it means that C_1, C_2 should have no common components.

Corollary (Weak Bézout for curves) With these assumptions, we have

$$\#(C_1 \cap C_2) \leq d_1 \cdot d_2.$$

Proof Use the general Weak Bézout theorem, together with

$$\deg(C_1 \cap C_2) = \#(C_1 \cap C_2)$$

for the finite set $(C_1 \cap C_2)$. □

Grassmannians

Recall once again

$$\mathrm{Gr}(2,4) \cong \mathbb{V}(y_0y_5 - y_1y_4 + y_2y_3) \hookrightarrow \mathbb{P}^5.$$

This being a hypersurface, we get $\deg \mathrm{Gr}(2,4) = 2$.

Theorem In general, let

$$\mathrm{Gr}(d,n) \hookrightarrow \mathbb{P}^{\binom{n}{d}-1}$$

be the Plücker embedding. Then in this embedding, we have

$$\deg \mathrm{Gr}(d,n) = (d(n-d))! \frac{1! \cdot 2! \cdot \ldots \cdot (d-1)!}{(n-d)! \cdot (n-d+1) \cdot \ldots \cdot (n-1)!}.$$

The proof, as you can guess from the formula, is nontrivial. Note that even the fact that the right hand side is an integer needs a moment's reflection.

The Hilbert function of a projective variety

Let $X \subset \mathbb{P}^n$ be a projective variety. Its homogeneous ideal

$$\mathbb{I}^h(X) \lhd R = k[x_0, \ldots, x_n]$$

is graded:

$$\mathbb{I}^h(X) = \bigoplus \mathbb{I}^h(X)_m.$$

We also have the graded homogeneous coordinate ring

$$S(X) = R/\mathbb{I}^h(X) \cong \bigoplus S(X)_m,$$

with

$$S(X)_m = k[x_0, \ldots, x_n]_m / \mathbb{I}^h(X)_m.$$

Definition The **Hilbert function** of the graded ring $S(X)$ is the function

$$h_X : \mathbb{N} \to \mathbb{N}$$

defined by

$$h_X(m) = \dim_k S(X)_m.$$

The Hilbert function of a projective variety: first example

Easy Lemma We have the formula

$$h_X(m) = \binom{m+n}{m} - \dim_k \mathbb{I}^h(X)_m.$$

Example 1 Let $X = \mathbb{P}^n$. Then $\mathbb{I}^h(X) = (0)$, so

$$\begin{aligned} h_{\mathbb{P}^n}(m) &= \binom{m+n}{m} \\ &= \frac{(m+n)\cdots(m+1)}{n!} \\ &= \frac{1}{n!}m^n + \text{lower order terms in } m \end{aligned}$$

The Hilbert function of a plane curve

Example 2 Let $C = \mathbb{V}(F) \subset \mathbb{P}^2$, for F irreducible homogeneous of degree d. Then $\mathbb{I}^h(X) = \langle F \rangle$, so for $m \geq d$,

$$\mathbb{I}^h(X)_m \cong k[x_0, x_1, x_2]_{m-d}.$$

We get, for $m \geq d$,

$$\begin{aligned} h_C(m) &= \binom{m+2}{m} - \binom{m-d+2}{m-d} \\ &= \frac{(m+2)(m+1)}{2} - \frac{(m-d+2)(m-d+1)}{2} \\ &= \frac{1}{2}\big(m^2 + 3m + 2 - (m^2 - 2md + 3m) - (d-2)(d-1)\big) \\ &= dm - \frac{(d-1)(d-2)}{2} + 1. \end{aligned}$$

The Hilbert function of a plane curve

Introduce the quantity

$$g(C) = \frac{(d-1)(d-2)}{2}.$$

Corollary The Hilbert function of a plane curve $C = \mathbb{V}(F) \subset \mathbb{P}^2$ of degree d is given by

$$h_C(m) = d \cdot m - g(C) + 1$$

for $m \geq d$.

Note The quantity $g(C)$ is the **genus** of C, an invariant of fundamental importance in the study of (plane) algebraic curves. There are many other definitions.

Examples

- For $d \leq 2$, we have $g(C) = 0$. Indeed, for $d = 1$ we have $C = L = \mathbb{P}^1$. For $d = 2$, we have a quadric which (if nonsingular) is isomorphic to $\mathbb{P}^1$.
- For $d = 3$, we have $g(C) = 1$. This is the case of cubic elliptic curves.

The Hilbert polynomial

We state the following fundamental results also without proof.

Theorem For any projective variety $X \subset \mathbb{P}^n$, there exists a polynomial $p_X \in \mathbb{Q}[x]$, such that **for sufficiently large m**,

$$h_X(m) = p_X(m).$$

p_X is called the **Hilbert polynomial** of $X \subset \mathbb{P}^n$. The leading term of p_X is

$$\frac{\deg X}{(\dim X)!} \cdot m^{\dim X}.$$

In other words, the Hilbert function is "eventually polynomial", of degree $\dim X$.

The Hilbert polynomial: examples

Example 1, revisited We computed

$$h_{\mathbb{P}^n}(m) = \frac{1}{n!}m^n + \text{lower order terms in } m$$

which agrees with the statement of the Theorem: $\dim \mathbb{P}^n = n$ and $\deg \mathbb{P}^n = 1$.

Example 2, revisited For $C = \mathbb{V}(F) \subset \mathbb{P}^2$ a plane curve of degree d, we computed

$$h_C(m) = d \cdot m - g(C) + 1$$

for $m \geq d$. This also agrees with the Theorem, as $\dim C = 1$ and $\deg C = d$.

An important caveat

Note that both the degree $\deg X$ and the Hilbert polynomial p_X of a projective variety $X \subset \mathbb{P}^n$ were defined **in terms of the embedding**.
They are **not** invariants of X up to isomorphism!
An honest, though clumsy, notation would be $\deg(X \subset \mathbb{P}^n)$ and $p_{X \subset \mathbb{P}^n}$.

Example Recall the Veronese embedding $\nu_2 \colon \mathbb{P}^1 \to \mathbb{P}^2$. Its image is a quadric plane curve

$$C = \{x_0 x_2 - x_1^2 = 0\} \subset \mathbb{P}^2.$$

The Veronese map and its inverse on C give an isomorphism of projective varieties $C \cong \mathbb{P}^1$.
Yet we have $\deg \mathbb{P}^1 = 1$ and $\deg C = 2$, and the Hilbert polynomials are also different.

Contrast this with the quantity $\dim X$ which, as we discussed before, is an invariant of X up to isomorphism. Indeed, $\dim C = \dim \mathbb{P}^1 = 1$.