# C3.4 Algebraic Geometry

Lecture 11. Some rings of functions and localization

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## Regular functions at points and on open sets

Let  $X \subset \mathbb{A}^n$  be an affine variety. Then its coordinate ring

$$k[X] = k[x_1, \dots, x_n]/\mathbb{I}(X)$$

can be thought of as the ring of regular or polynomial functions on X.

Every element  $f \in k[X]$  can be interpreted as a function

$$f: X \to k$$

by evaluating a representing polynomial in  $k[x_1, \ldots, x_n]$ .

We would like to extend this definition to study functions which are regular near a point, or on some open set in X.

Let  $p \in X$ . A function  $f: U \to k$  defined on a neighbourhood of p is called **regular at** p, if on some open set  $p \in W \subset U$ ,

$$f|_{W} = \frac{g}{h}$$
 for some  $g, h \in k[X]$  with  $h(w) \neq 0$  for all  $w \in W$ .

For an open subset  $U \subset X$ , write  $\mathcal{O}_X(U)$  for the k-algebra of regular functions on U, functions  $f: U \to k$  that are regular at all points  $p \in U$ .

## Examples

1. If  $f \in k[X]$  then f is regular at all points of X: use  $f = \frac{f}{1}$ . Thus  $f \in \mathcal{O}_X(X)$ .

We will prove later that in fact this inclusion

$$k[X] \hookrightarrow \mathcal{O}_X(X)$$

is an isomorphism; note that this is not at all obvious!

- 2. If  $X = \mathbb{A}^1$ , then  $1/x^n \in \mathcal{O}_X(U)$  for any open set U not containing the origin.
- 3. More generally, for any X and  $f \in k[X]$ ,

$$1/f^n \in \mathcal{O}_X(D_f),$$

where

$$D_f = X \setminus \mathbb{V}(f)$$

is the basic open set of X defined by the non-vanishing of f.

## Germs of regular functions

Let  $p \in X$ . We want a notion of "ring of regular functions near p on X". We want to regard two functions as the same, if they agree "sufficiently close to p".

Here is how to formalize this idea.

**Definition** A function germ at p is an equivalence class of pairs (U, f), with  $p \in U \subset X$  open, and  $f: U \to k$  a regular function, where we identify

$$(U, f) \sim (V, g)$$

if  $f|_W = g|_W$  on an open  $p \in W \subset U \cap V$ .

Denote by  $\mathcal{O}_{X,p}$  the set of germs of regular functions at p. This is a k-algebra in an obvious way.

## Function germs: an example

Function germs can behave in unexpected ways, especially on reducible varieties.

#### Example Let

$$X = \mathbb{V}(xy) \subset \mathbb{A}^2$$
.

Then X is a reducible variety: it is the union of the x-axis and the y-axis. Let

$$U = X \setminus \mathbb{V}(y)$$

the open subset of X consisting of the x-axis without the origin.

Consider  $f: U \to k$  given by  $f = \frac{y}{x}$ . This is a regular function on U:

$$f \in \mathcal{O}_X(U)$$
.

But along the x-axis away from the origin,

$$(U,f) \sim (U,0)$$

as  $y = 0 : U \to k$ , so [(U, f)] = 0.

## Multiplicative sets in rings

To relate the ring of germs  $\mathcal{O}_{X,p}$  to other rings of functions algebraically, we need to recall a notion from Commutative Algebra. Let A be a ring (commutative with 1).

**Definition** A subset  $S \subset A$  is a multiplicative set if  $1 \in S$  and  $S \cdot S \subset S$ .

### Examples

- 1.  $S = A \setminus \{0\}$  for any integral domain A.
- 2.  $S = A \setminus \wp$  for any prime ideal  $\wp \triangleleft A$ .
- 3.  $S = A \setminus \mathfrak{m}$  for any maximal ideal  $\mathfrak{m} \triangleleft A$ .
- 4.  $S = \{1, f, f^2, \ldots\}$  for any  $f \in A$ .

## Localising rings

Let  $S \subset A$  be a multiplicative subset.

**Definition** The **localisation** of A at (or with respect to) S is the ring

$$S^{-1}A = (A \times S)/\sim$$

where we abbreviate the pairs (r,s) by  $\frac{r}{s}$ , and the equivalence relation is:

$$\frac{r}{s} \sim \frac{r'}{s'} \iff t(rs' - r's) = 0 \text{ for some } t \in S.$$

**Remark** If A is an integral domain, then t in the definition of the equivalence relation is unnecessary.

Indeed, for integral domains the theory of localisation substantially simplifies.

**Notation** For  $f \in A$  and  $S = \{1, f, f^2, \ldots\}$ , denote

$$A_f = S^{-1}(A).$$

For a prime ideal  $\wp \triangleleft A$  and  $S = A \setminus \wp$ , denote

$$A_{\wp} = S^{-1}(A).$$

# Examples of localisation

#### Examples

- 1. If  $A = \mathbb{Z}$  and  $S = \mathbb{Z} \setminus 0$ , then  $S^{-1}A \cong \mathbb{Q}$ .
- 2. More generally, if A is an integral domain and  $S = A \setminus \{0\}$ , then

$$\operatorname{Frac}(A) = S^{-1}A$$

is the (definition of the) field of fractions of A.

3. If A is an integral domain,  $f \in A \setminus \{0\}$  and  $S = \{1, f, f^2, \ldots\}$ , then we have

$$A_f = A[f^{-1}] = \{a/f^r : a \in A\}.$$

If f is a zero-divisor,  $A_f$  can be more complicated to describe.

## A further example

Let us see an example where A is not a domain.

**Example** Let A = k[x, y]/(xy), and consider the multiplicative subset  $S = \{1, x, x^2, \ldots\}$  in A.

In the the localisation  $S^{-1}A$ , the element  $y = \frac{y}{1}$  is zero, since y is annihilated by  $x \in S$ .

Thus

$$k[x]_x = S^{-1}A \cong k[x, x^{-1}] \subset \text{Frac}(k[x]) = k(x).$$

# Some basic algebraic properties

## Theorem (Algebraic Properties of Localisation)

(1) If A is an integral domain, then for any multiplicative set S in A, there are injections

$$A \hookrightarrow S^{-1}(A) \hookrightarrow \operatorname{Frac}(A).$$

(2) For any prime ideal  $\wp \triangleleft A$  and  $S = A \setminus \wp$ , the localisation  $A_{\wp}$  has a unique maximal ideal

$$\wp A_{\wp} = \left\{ \frac{r}{s} : r \in \wp, s \notin \wp \right\} / \sim .$$

(3) For A an integral domain,

$$A = \bigcap_{\max \mathfrak{m} \lhd A} A_{\mathfrak{m}} = \bigcap_{\text{prime } \wp \lhd A} A_{\wp} \subset \operatorname{Frac}(A).$$

(4) For any ideal  $I \triangleleft A$ , and for any prime ideal  $\wp \triangleleft A$  containing I, the localisation  $A_{\wp}$  has an induced prime ideal  $IA_{\wp}$ , and we have an isomorpism

$$(A/I)_{\wp} \cong A_{\wp}/IA_{\wp}.$$

## The ring of germs of functions is a localisation

**Proposition** Let  $X \subset \mathbb{A}^n$  be an affine variety. At a point  $p \in X$ , the ring of germs of functions at p is the localisation

$$\mathcal{O}_{X,p} \cong k[X]_{\mathfrak{m}_p}$$

of the coordinate ring k[X] at its maximal ideal

$$\mathfrak{m}_p = \mathbb{I}(p) = \{ f \in k[X] : f(p) = 0 \}$$

corresponding to p.

**Proof** Consider the map

$$\mathcal{O}_{X,p} \to k[X]_{\mathfrak{m}_p}$$

defined by

$$(U,f)\mapsto \frac{g}{h},$$

where  $f|_U = \frac{g}{h}$  for  $g, h \in k[X], h(p) \neq 0$ .

## The ring of germs of functions is a localisation

The map is well-defined:  $h(p) \neq 0 \Rightarrow h \notin \mathfrak{m}_p \Rightarrow \frac{g}{h} \in k[X]_{\mathfrak{m}_p}$ .

Moreover, if  $(U, f) \sim (U', f')$ , so  $\frac{g}{h} = \frac{g'}{h'}$  on a basic open  $p \in D_s \subset U \cap U'$ , where  $s \in k[X]$ , then gh' - g'h = 0 on  $D_s$ .

Since  $s(p) \neq 0$ , we have  $s \notin \mathfrak{m}_p$ . Thus  $s \cdot (gh' - g'h) = 0$  everywhere on X, so  $s \cdot (gh' - g'h) = 0$  in k[X].

Thus  $\frac{g}{h} = \frac{g'}{h'}$  in  $k[X]_{\mathfrak{m}_p}$ .

We build the inverse map: for  $h \notin \mathfrak{m}_p$ , let  $U = D_h$ , then send  $\frac{g}{h} \mapsto (U, \frac{g}{h})$ .

If  $\frac{g}{h} = \frac{g'}{h'}$  in  $k[X]_{\mathfrak{m}_p}$ , then  $s \cdot (gh' - g'h) = 0$  for some  $s \in k[X] \setminus \mathfrak{m}_p$ .

Then  $s(p) \neq 0$  so  $p \in D_s$ , and gh' - g'h = 0 on  $D_s$ .

Thus  $\frac{g}{h} = \frac{g'}{h'}$  as functions  $D_s \to k$ , as required.

By construction, the two maps are inverse to each other, so they both define isomorphisms.  $\Box$ 

## An example

Let us return to A = k[x, y]/(xy), corresponding to  $X \subset \mathbb{A}^2$  being the union of coordinate axes.

Consider the point p = (1, 0). Its maximal ideal is

$$\mathfrak{m}_p = \langle x - 1, y \rangle \lhd k[x, y]/(xy) = k[X].$$

Naively, we would like to invert all functions that do not vanish at p, so allow denominators from the set

$$S = \{ f \in A : f \notin \langle x - 1, y \rangle \} = A \setminus \mathfrak{m}_p.$$

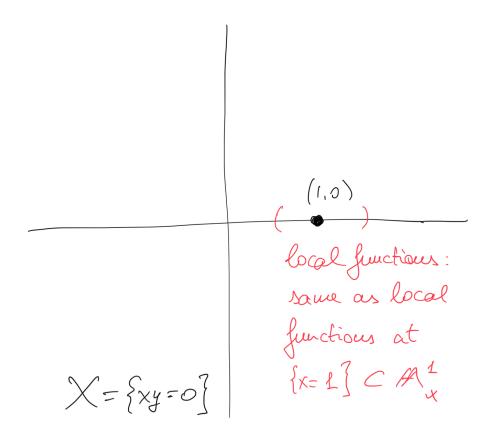
However, while the global functions 0 and y are different in k[X], as we saw before, they become equal near p, and also in the localisation  $S^{-1}A = k[X]_{\mathfrak{m}_p}$ . Indeed, in this example we get

$$k[x]_{\mathfrak{m}_p} = k[x][\frac{1}{h} : h(p) \neq 0] \subset \operatorname{Frac}(k[x]) = k(x).$$

This is the same as what we would have obtained for

$$p = (1,0) \subset Y = \{y = 0\} \subset \mathbb{A}^2.$$

# An example



## An example, continued

Continue with to A = k[x, y]/(xy). Consider the point  $q = (0, 0) \in X$ , the intersection point of the two components. Its maximal ideal is

$$\mathfrak{m}_q = \langle x, y \rangle \lhd k[x, y]/(xy) = k[X].$$

Now we have  $y \neq 0$  near this point, so in the local ring  $k[X]_{\mathfrak{m}_q}$ . Indeed there are surjective restriction maps, obtained by setting y = 0, respectively x = 0:

$$k[X]_{\mathfrak{m}_q} \to k[x]_{\langle x \rangle}$$

and

$$k[X]_{\mathfrak{m}_q} \to k[y]_{\langle y \rangle},$$

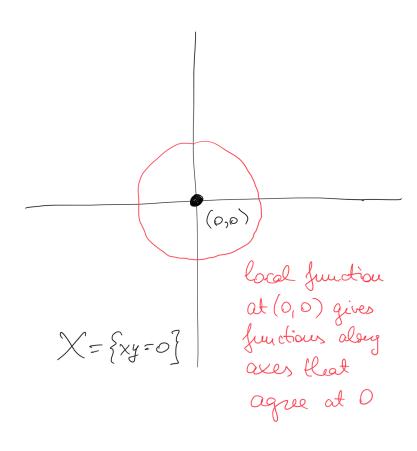
the local rings at the origin in the two copies of the affine line.

Using these two surjective maps, it is an easy exercise to show that

$$k[X]_{\mathfrak{m}_q} \subset k[x]_{\langle x \rangle} \times k[y]_{\langle y \rangle}$$

is the subring of elements of the product ring consisting of pairs of functions with the same value at the origin.

# An example, continued



## Localisation: final general properties on irreducible varieties

For  $X \subset \mathbb{A}^n$ , an **irreducible** variety corresponding to a coordinate ring k[X] which is a domain, we obtain from Algebraic Property (1) above,

$$k[X] \subset \mathcal{O}_{X,p} \cong k[X]_{\mathfrak{m}_p} = k[X][\frac{1}{h}: h(p) \neq 0] \subset \operatorname{Frac}(k[X]) = k(X).$$

We also have, from Algebraic Property (3) above,

$$k[X] = \bigcap_{p \in X} \mathcal{O}_{X,p} \subset k(X).$$

In other words, on an irreducible affine variety, a rational function is regular if and only if it defines a regular germ at each point.

#### Localisation on reducible varieties

For  $X \subset \mathbb{A}^n$  reducible, write

$$X = \bigcup_{i=1}^{N} X_i$$

an irreducible decomposition.

Consider a point

$$p \in X_i \setminus \bigcup_{j \neq i} X_j$$

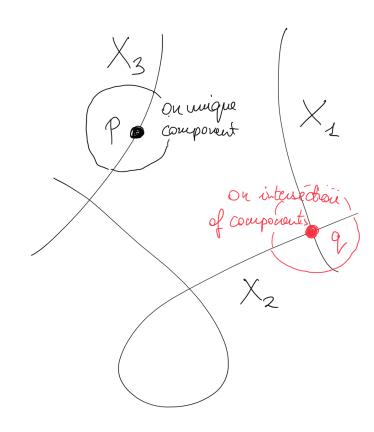
lying on a unique irreducible component.

Then essentially the same argument that was used in the example before proves that the ring of germs at p is

$$\mathcal{O}_{X,p} \cong k[X]_{\mathfrak{m}_p} \cong k[X_i]_{\mathfrak{m}_p}.$$

In other words, for such points the ring of germs "sees" only the (unique) irreducible component p lies on.

# Localisation on reducible varieties



#### Intersection numbers via localization

This is "off-topic" but too nice not to mention.

Let  $f_1, f_2 \in k[x, y]$  be two polynomials without common factors. Let

$$C_i = \{f_i = 0\} \subset \mathbb{A}^2$$

be the two affine plane curves defined by the  $f_i$ .

Let  $p \in C_1 \cap C_2$  be an intersection point. Let  $\mathfrak{m}_p \triangleleft k[x,y]$  be the maximal ideal of this point on the affine plane.

**Definition** The intersection multiplicity of  $C_1$  and  $C_2$  at p is the (finite) quantity

$$\dim_k k[x,y]_{\mathfrak{m}_p}/\langle f_1,f_2\rangle.$$

This is the (local) quantity that goes into the full version of Bezout's theorem.