C3.4 Algebraic Geometry

Lecture 12. Quasi-projective varieties

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Why quasiprojective varieties?

We would like to consider a class (category, if you wish) of “algebraic varities” that includes affine and projective varieties, but also (closed and) open subsets of those.

We would also like to define (to get a category), what morphisms of these are.

Example 1 The open set $U = \mathbb{A}^1 \setminus \{0\}$ of $\mathbb{A}^1$ can be thought of as an affine variety: the first projection of the affine variety

$$Y = \mathbb{V}(xy - 1) \subset \mathbb{A}^2$$

is a bijection (*isomorphism*) between Y and U.

Example 2 The open set $V = \mathbb{A}^2 \setminus \{0\}$ can **not** be thought of as an affine variety: it is not (*isomorphic to*) an affine variety $Y \subset \mathbb{A}^n$.

We will see that $\mathbb{A}^2 \setminus \{0\}$ is an example of a quasiprojective variety which is not affine, nor projective.

Quasiprojective varieties

Let $R = k[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of projective space $\mathbb{P}^n$.

Definition A **quasi-projective variety** $X \subset \mathbb{P}^n$ is any open subset of a projective variety, so there exists ideals $I, J \lhd R$ with

$$X = U_J \cap \mathbb{V}(I)$$

where

$$U_J = \mathbb{P}^n \setminus \mathbb{V}(J).$$

Notice we can also write X as the difference of two closed sets:

$$X = \mathbb{V}(I) \setminus \mathbb{V}(I + J).$$

A **quasi-projective subvariety** Y of X is a subset of X which is also a quasi-projective variety.

Examples

The following are all examples of quasi-projective varieties.

1. Any projective variety $X \subset \mathbb{P}^n$ is a quasi-projective variety as

$$X = \mathbb{P}^n \cap X.$$

2. Any affine variety $X \subset \mathbb{A}^n$ is a quasi-projective variety as

$$X = U_0 \cap \overline{X},$$

   where $\overline{X} \subset \mathbb{P}^n$ is the projective closure of X, and $U_0 = \mathbb{A}^n$ is one of the covering open affines of $\mathbb{P}^n$.

3. We have

$$\mathbb{A}^2 \setminus \{0\} = (U_0 \cap (U_1 \cup U_2)) \cap \mathbb{P}^2,$$

   where $U_i = \{x_i \neq 0\}$ are the covering affine open subsets of $\mathbb{P}^2$.

4. Any open subset of a quasi-projective variety is also a quasi-projective variety, since

$$U_{J'} \cap (U_J \cap \mathbb{V}(I)) = (U_{J'} \cap U_J) \cap \mathbb{V}(I).$$

Morphisms

Morphisms are defined in the same way as for projective varieties. "Morphisms in projective geometry are defined locally."

Fix quasi-projective varieties $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$. We work with $(m+1)$-tuples of homogeneous polynomials of the projective coordinates $x_0, \ldots, x_n$ on X. Let $R = k[x_0, \ldots, x_n]$.

Definition A **morphism** $F : X \to Y$ of quasi-projective projective varieties is a function F such that for every $p \in X$, there is an open neighbourhood $p \in U \subset X$, and **homogeneous** polynomials $f_0, \ldots, f_m \in R$ **of the same degree**, with

$$F : U \to Y \text{ given by } F([a_0 : \cdots : a_n]) = [f_0(a) : \cdots : f_m(a)].$$

Morphisms of affine varieties

We need to check that for $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ affine, this definition agrees with the definition of a morphism of affine varieties.

Suppose a morphism $f\colon X \to Y$ between affine varieties is given by an m-tuple $f_1, \ldots, f_m$ of polynomials of the variables $x_1, \ldots, x_n$ of X:

$$\begin{array}{rccc} f: & X & \to & Y \\ & (x_1, \ldots, x_n) & \mapsto & (f_1(x_i), \ldots, f_m(x_i)). \end{array}$$

Let $d = \max \deg f_i$, $F_0(x) = X_0^d$, $F_i(X_i) = X_0^d f_i(x_i)$, where the affine variables $(x_1, \ldots, x_n)$ and the projective variables $[X_0 : \ldots : X_n]$ are related as usual by $x_i = X_i/X_0$ for $i \neq 0$.

Then we can use the quasi-projective representation

$$[F_0(X_i) : \cdots : F_m(X_i)] = [X_0^d : X_0^d f_1(x_i) : \cdots : X_0^d f_m(x_i)].$$

On the open set $\mathbb{A}^n$ of $\mathbb{P}^n$ where we can set $X_0 = 1$, we indeed get

$$[1 : x_1 : \cdots : x_n] \mapsto [1 : f_1(x_i) : \cdots : f_m(x_i)].$$

Affine quasi-projective varieties

As we have a notion of morphism, we automatically have a notion of isomorphism for quasi-projective varieties.

The following terminology is unfortunate, but completely standard in the subject.

Definition A quasi-projective variety X is called **affine**, if it is isomorphic (as a quasi-projective variety) to a Zariski closed subset $Y \subset \mathbb{A}^m$.

We could call these "affine quasi-projective varieties" but everybody calls these "affine varieties". We may sometimes write $k[X]$ for $k[Y] = k[\mathbb{A}^m]/\mathbb{I}(Y)$.

As we will see later, this ring has an intrinsic definition in terms of X.

Example The open set $V = \mathbb{A}^2 \setminus \{0\}$ is not an affine (quasi-projective) variety: it is not isomorphic to a closed subset $Y \subset \mathbb{A}^n$.

An example of an affine quasi-projective variety

Example 1 Consider the quasi-projective variety

$$U = \mathbb{A}^1 \setminus \{0\} = \mathbb{P}^1 \setminus \{0, \infty\}.$$

Let

$$Y = \mathbb{V}(xy - 1) \subset \mathbb{A}^2,$$

which can be written as a quasi-projective variety as

$$Y = \mathbb{V}(x_1 x_2 - x_0^2) \cap U_0 \subset \mathbb{P}^2.$$

Then the formula

$$[x_0 \colon x_1 \colon x_2] \to [x_0 \colon x_2]$$

defines a well-defined map of quasi-projective varieties $F \colon Y \to U$.
Its inverse $G \colon U \to Y$ can be written in the coordinates z_0, z_1 of $\mathbb{P}^1$ as

$$G \colon [z_0 \colon z_1] \mapsto [z_0 z_1 \colon z_0^2 \colon z_1^2].$$

Check that these indeed define mutual inverses and behave as expected!

Basic open sets in affine varieties are affine

Lemma Let $X \subset \mathbb{A}^n$ be an affine variety, $f \in k[X]$. Then the basic open set

$$D_f = X \setminus \mathbb{V}(f)$$

is an affine (quasi-projective) variety with

$$k[D_f] \cong k[X]_f.$$

The proof is just a more general version of the previous example.

Proof Let $\mathbb{I}(X) \lhd k[x_1, \ldots, x_n]$ be the ideal of X. Let

$$\widetilde{I} = \langle \mathbb{I}(X), x_{n+1}f - 1 \rangle \subset k[x_1, \ldots, x_n, x_{n+1}].$$

Let

$$Y = \mathbb{V}(\widetilde{I}) \subset \mathbb{A}^{n+1}$$

be the affine variety corresponding to the ideal $\widetilde{I}$. The proof will be concluded by the following two Claims.

Basic open sets in affine varieties are affine

Claim 1 We have $k[Y] \cong k[X]_f$.

Proof of Claim 1 There is a natural map from left to right given by x_{n+1} mapping to $\frac{1}{f} \in k[X]_f$ and $g \in k[X]$ mapping to $\frac{g}{1} \in k[X]_f$.
There is also a natural map from right to left given by $\frac{g}{f^a} \mapsto g x_{n+1}^a$.
It can be checked easily that these maps are mutual inverses.

Claim 2 We have $Y \cong D_f$ as quasi-projective varieties.

Proof of Claim 2 The first projection $(a_1, \dots, a_n, a_{n+1}) \mapsto (a_1, \dots, a_n)$ restricts to a map $Y \to \mathbb{A}^n$ whose image is D_f.
It has an inverse which is the map

$$(a_1, \dots, a_n) \mapsto \left(a_1, \dots, a_n, \frac{1}{f(a)}\right)$$

from D_f to Y.
Introducing homogeneous coordinates, it is easy to write these as genuine maps of quasi-projective varieties. For details, see Lecture Notes.
The proof of the Lemma is complete. $\square$

Quasi-projective varieties are locally affine

Theorem Every quasi-projective variety X has a finite open cover by affine (quasi-projective) subvarieties. In particular, affine open subsets form a basis for the Zariski topology on X.

Proof We have, picking generators for ideals,

$$\mathbb{P}^n \supset X = \mathbb{V}(F_1, \ldots, F_N) \setminus \mathbb{V}(G_1, \ldots, G_M).$$

Because $\mathbb{P}^n$ has an affine open cover by its standard open sets U_i, it suffices to check the claim on the open subset $U_0 \cap X$ of X.
Then $U_0 \cap X$ is

$$\mathbb{V}(f_1, \ldots, f_N) \setminus \mathbb{V}(g_1, \ldots, g_M) = \bigcup_j (\mathbb{V}(f_1, \ldots, f_N) \setminus \mathbb{V}(g_j)) = \cup_j D_{g_j},$$

where

$$D_{g_j} = \{g_j \neq 0\} \subset \mathbb{V}(f_1, \ldots, f_N)$$

is a basic open subset in an affine variety.
Now apply the Lemma from the previous pages. $\square$

Quasi-projective varieties as glued from affine varieties

The Theorem shows that quasi-projective varieties are basically affine varieties "glued together". We could take this as a starting point!

$$X = \bigcup_i U_i, \quad \text{with} \quad U_i \subset \mathbb{A}^{n_i} \text{ affine.}$$

This point of view takes algebraic geometry closer to other subjects like differential geometry: we specify a set of "local models" and build global objects from local ones.

This point of view expands the list of possible objects further, allowing so-called "abstract varieties" and eventually "schemes":

$\{$affine varieties$\} \subset \{$quasi-proj. varieties$\} \subset \{$abstract varieties$\} \subset \{$schemes$\}$

Objects in algebraic geometry

We have as possible objects

$$\{\text{affine varieties}\} \subset \{\text{quasi-proj. varieties}\} \subset \{\text{abstract varieties}\} \subset \{\text{schemes}\}$$

Here

- affine varieties (corresponding to finitely generated reduced k-algebras) are basic building blocks;
- quasi-projective varieties are what we can glue inside a convenient global model (projective space);
- abstract varieties are what we can glue abstractly, without a global model;
- schemes are what we can glue when we do not insist that our algebras should be reduced.

This list could go on!

Morphisms in algebraic geometry

Morphisms for our objects

$\{$affine varieties$\} \subset \{$quasi-proj. varieties$\} \subset \{$abstract varieties$\} \subset \{$schemes$\}$

are then defined as follows:

- morphisms of affine varieties are algebra homomorphisms of the corresponding to finitely generated reduced k-algebras (backwards);
- for all other types of objects, morphisms are collections of local morphisms compatible with glueing.

This point of view is explained further in the Lecture Notes (non-examinable), and in second courses in Algebraic Geometry.

Examples of different types of objects in algebraic geometry

Examples:

{affine varieties} $\subset$ {quasi-proj. varieties} $\subset$ {abstract varieties} $\subset$ {schemes}

$\{x^3+y^3+1=0\} \subset \mathbb{A}^2$ $\mathbb{A}^2 \setminus \{0\}$ $\mathbb{A}^1 \cup_{\mathbb{A}^1\setminus\{0\}} \mathbb{A}^1$ $\mathrm{Spec}\, k[x,y]/(x^2,xy)$