C3.4 Algebraic Geometry

Lecture 13. Regular functions on quasi-projective varieties

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Regular functions in the affine case

Start with an affine variety $X \subset \mathbb{A}^n$. For $U \subset X$ open, recall

$$\begin{aligned} \mathcal{O}_X(U) &= \{\text{regular functions } f : U \to k\} \\ &= \{f : U \to k : f \text{ is regular at each } p \in U\}. \end{aligned}$$

Here f regular at p means: on some open $p \in W \subset U$, the following functions $W \to k$ are equal:

$$f = \frac{g}{h} \qquad \text{some } g, h \in k[X] \text{ and } h(w) \neq 0 \text{ for all } w \in W.$$

Examples

1. Let $U = D_x = \mathbb{A}^2 \setminus \mathbb{V}(x) \subset \mathbb{A}^2$, then $f : D_x \to k$ defined by $f(x, y) = \frac{y}{x}$ is a regular function on $U = D_x$:

$$f \in \mathcal{O}_X(D_x).$$

2. More generally, for any $g, h \in k[X]$, with $h \neq 0$, we have $\frac{g}{h} \in \mathcal{O}_X(D_h)$.

Regular functions on quasi-projective varieties

This definition generalises readily to quasi-projective varieties, remembering the slogan "quasi-projective varieties are locally affine".

Let $X \subset \mathbb{P}^n$ be a quasi-projective variety.

Definition For any open subset $U \subset X$, let

$$\mathcal{O}_X(U) = \{F : U \to k : F \text{ is regular at each } p \in U\}$$

be the **ring of regular functions on U**.

Here F **regular at** p means: on some **affine** open $p \in W \subset U$, $F|_W$ is regular at p as previously defined.

Rings of germs of functions on quasi-projective varieties

Let $X \subset \mathbb{P}^n$ be a quasi-projective variety, and $p \in X$. The following definition is now immediate; note that all terms we use here are now defined for general quasi-projective varieties.

Definition A **function germ at** p is an equivalence class of pairs (U, f), with $p \in U \subset X$ open, and $f : U \to k$ a regular function, where we identify

$$(U, f) \sim (V, g)$$

if $f|_W = g|_W$ on an open $p \in W \subset U \cap V$.

Denote by $\mathcal{O}_{X,p}$ the set of germs of regular functions at p. This is a k-algebra in an obvious way.

Rings of germs of functions on quasi-projective varieties

In practice, it is easiest to approach $\mathcal{O}_{X,p}$ on an **affine** open neighbourhood $p \in V \subset X$.
Lecture 12, Theorem shows that affines cover a quasi-projective variety.

Proposition Let $p \in V \subset X$ be an **affine** neighbourhood of p in the quasi-projective variety X. Then

$$\mathcal{O}_{X,p} \cong k[V]_{\mathfrak{m}_p},$$

where $\mathfrak{m}_p \lhd k[V]$ is the maximal ideal of $p \in V$ in the coordinate ring of the affine variety V.

Proof Use Lecture 11, Proposition. $\square$

One way to view this is to say that that the localization on the right hand side is **independent** of the affine neighbourhood of p chosen.

Gluing regular functions

Consider open sets U_1, U_2 in a quasi-projective variety X, and regular functions $f_1 \in \mathcal{O}_X(U_1)$ and $f_2 \in \mathcal{O}_X(U_2)$.

Claim A necessary and sufficient condition to be able to find a regular function $f \in \mathcal{O}_X(U_1 \cup U_2)$ restricting to f_i on U_i is that

$$f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}.$$

Proof Necessity is obvious. To see sufficiency, define $f = f_i$ on U_i. Then $f : U_1 \cup U_2 \to k$ is well-defined. Regularity follows because regularity is a local condition and we already know it is satisfied by f_1, f_2 on U_1, U_2. $\square$

This result is almost a "tautology". But it would play an important role in a different development of the subject; it says that the collection of rings $\mathcal{O}_X(U)$ attached to open sets $U \subset X$ forms a **sheaf**. We will not use this language, but see the Lecture Notes for some more details (non-examinable).

Regular functions are not necessarily global quotients

For $f \in \mathcal{O}_X(U)$, it may not be possible to find a fraction $f = \frac{g}{h}$ that works on all of U.

Example Consider the affine variety

$$X = \mathbb{V}(xw - yz) \subset \mathbb{A}^4.$$

Let $U = D_y \cup D_w \subset X$, a quasi-projective variety.
Consider

$$f = \frac{x}{y} = \frac{z}{w} \in k(X) = \text{Frac } k[X].$$

This is a **regular** function

$$f \in \mathcal{O}_X(U),$$

glued from regular functions on D_y, D_w that agree on the intersection.
It can be proved that there is no a global expression $f = \frac{g}{h}$ defined on all of U that defines f.

For algebra aficionados: this is a reflection of the fact that $k[X]$ is not a UFD.

Regular functions on affine varieties

We noted before that for $X \subset \mathbb{A}^n$ an affine variety, elements of its coordinate ring define regular functions: there is an inclusion

$$k[X] \hookrightarrow \mathcal{O}_X(X)$$

defined by mapping $f \mapsto \frac{f}{1}$.

We now have the means to prove the converse: the locally everywhere regular functions on an affine variety are exactly elements of its coordinate ring.

Theorem Let $X \subset \mathbb{A}^n$ be an affine variety. Then

$$O_X(X) \cong k[X].$$

This is important, as it gives a "local" way to characterise the coordinate ring of an affine (quasi-projective) variety as the ring of everywhere regular functions.

Regular functions on affine varieties: an application

Before we give the proof, let us see an application.

Proposition The quasi-projective variety $X = \mathbb{A}^2 \setminus \{0\}$ is **not** affine.

The proof consists of two parts.

Claim 1 We have

$$\mathcal{O}_X(X) \cong k[x, y].$$

Claim 2 This isomorphism implies that X is not affine.

Colloquial summary

- If X were affine, its coordinate ring would have to be its ring of regular functions.
- But then X would have to be $\mathbb{A}^2$, which it isn't.

Proof of Claim 1

Claim 1 We have

$$\mathcal{O}_X(X) \cong k[x, y].$$

Proof We have $\mathbb{A}^2 \setminus \{0\} = D_x \cup D_y$, so $f \in \mathcal{O}_X(X)$ defines regular functions $f_1 = f|_{D_x} \in k[D_x]$, $f_2 = f|_{D_y} \in k[D_y]$ which agree on the intersection: $f_1|_{D_x \cap D_y} = f|_{D_x \cap D_y} = f_2|_{D_x \cap D_y} \in k[D_x \cap D_y]$.

We know

$$\mathcal{O}_X(D_x) = k[x, y]_x \subset k(x, y)$$

and similarly $\mathcal{O}_X(D_y) = k[x, y]_y$ (Lemma from Lecture 12).
We get the following intersection in $k(x, y)$:

$$\begin{aligned}
\mathcal{O}_X(X) &= k[D_x] \cap k[D_y] \\
&= k[x, y]_x \cap k[x, y]_y \\
&= k[x, x^{-1}, y] \cap k[x, y, y^{-1}] \\
&= k[x, y].
\end{aligned}$$

Proof of Claim 2

Claim 2 The isomorphism $\mathcal{O}_X(X) \cong k[x, y]$ implies that X is not affine.

Proof of Claim 2 Suppose that X is isomorphic to an affine variety

$$X \cong Y \subset \mathbb{A}^n.$$

Then this isomorphism would give us isomorphisms

$$k[Y] \cong \mathcal{O}_Y(Y) \cong \mathcal{O}_X(X) \cong k[x, y].$$

So $Y \cong \mathbb{A}^2$ as affine varieties can be recognised from their coordinate rings. Assume that we have an isomorphism $\varphi : k[\mathbb{A}^2] = \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2) \to \mathcal{O}_X(X)$. The preimage of the prime ideal $I = \langle x, y\rangle \subset \mathcal{O}_X(X)$ yields a prime ideal $J = \varphi^{-1}(I) \lhd k[\mathbb{A}^2]$.

But $\mathbb{V}(I) = \emptyset \subset X$, so $\mathbb{V}(J) = \varphi^*(\mathbb{V}(I)) = \emptyset \subset \mathbb{A}^2$.

So $J = k[x, y]$ by the affine Nullstellensatz.

But φ is an isomorphism, so $I = \varphi(J) = k[x, y]$, contradiction. $\square$

Regular functions on affine varieties: the proof

Theorem Let $X \subset \mathbb{A}^n$ be an affine variety. Then

$$O_X(X) \cong k[X].$$

Proof We will prove this result under the additional assumption that X is irreducible. The general case is treated in the Lecture Notes.
We need to show that any element of $O_X(X)$ comes from the coordinate ring. So let $f \in O_X(X)$.

For all $p \in X$, there exists an open neighbourhood $p \in U_p \subset X$ such that

$$f = \tfrac{g_p}{h_p} \text{ as maps } U_p \to k,$$

where $g_p, h_p \in k[X]$, and $h_p \neq 0$ at all points of U_p.

Note that at this point, we have an infinite collection of (g_p, h_p) representing f, one for each point of X.

Regular functions on affine varieties: the proof

Consider the ideal of denominators

$$J = \langle h_p : p \in X \rangle \subset k[X].$$

We have

$$\mathbb{V}(J) = \emptyset,$$

since $h_p(p) \neq 0$.

By the Nullstellensatz then, $J = \langle 1 \rangle$ the whole coordinate ring. So there exists a finite decomposition

$$1 = \sum_{i=1}^{N} \alpha_i h_{p_i} \in k[X] \tag{1}$$

for some **finite** collection of $p_i \in X$, and $\alpha_i \in k[X]$.

Abbreviate $h_i = h_{p_i}$, $g_i = g_{p_i}$, $D_i = D_{h_{p_i}}$. Note that (1) implies that the D_i are an open cover of X: at each point of X, at least one of the h_i must be nonzero.

Regular functions on affine varieties: the proof

We have now built a finite open cover $\{D_i\}$ of X, such that on each of these open sets, f is represented by g_i/h_i, a ratio of elements of $k[X]$.

On the overlap $D_i \cap D_j$, we have $g_i/h_i = g_j/h_j$, so on this intersection,

$$g_i h_j - h_i g_j = 0 \in \mathcal{O}_X(D_i \cap D_j).$$

However, since X is irreducible, this vanishing must be true over the whole of X, so for all i, j

$$g_i h_j - h_i g_j = 0 \in k[X].$$

We then deduce, on D_j,

$$f = \frac{g_j}{h_j} = 1 \cdot \frac{g_j}{h_j} = \sum_{i=1}^{N} \alpha_i h_i \cdot \frac{g_j}{h_j} = \sum_{i=1}^{N} \alpha_i \frac{h_i g_j}{h_j} = \sum_{i=1}^{N} \alpha_i \frac{h_j g_i}{h_j} = \sum_{i=1}^{N} \alpha_i g_i.$$

Once again, $D_j \subset X$ is dense, so we deduce over the whole of X that

$$f = \sum_{i=1}^{N} \alpha_i g_i \in k[X]. \quad \square$$

Regular functions on projective varieties

Affine varieties admit many global regular functions: elements of their coordinate rings.

Let us conclude this lecture by discussing the opposite case: projective varieties.

Theorem Let $X \subset \mathbb{P}^n$ be an irreducible **projective** variety. Then the only global regular functions on X are the constants:

$$\mathcal{O}_X(X) \cong k.$$

The proof, while not difficult, is somewhat lengthy, so we omit it. We discuss one example, and then sketch the proof of one more general case.

Corollary The variety $X = \mathbb{A}^2 \setminus \{0\}$ is not projective.

Regular functions on the projective line

Example Consider $X = \mathbb{P}^1 = U_0 \cup U_1$, with projective coordinates $(x_0 : x_1)$. The open sets $U_i = \{x_i \neq 0\} \cong \mathbb{A}^1$ intersect in $V = U_i \cap U_j \cong \mathbb{A}^1 \setminus \{0\}$.

Then U_0, U_1 are affine varieties, with

$$\mathcal{O}_X(U_0) \cong k[U_0] = k[x], \quad \mathcal{O}_X(U_1) \cong k[U_1] = k[y].$$

The affine coordinates $x = x_1/x_0$, $y = x_0/x_1$ are related by $y = 1/x$, so we can write

$$\mathcal{O}_X(U_0) = k[x], \quad \mathcal{O}_X(U_1) = k[x^{-1}].$$

We also have

$$\mathcal{O}_X(V) \cong k[V] = k[x]_x = k[x, x^{-1}].$$

We thus deduce

$$\mathcal{O}_X(X) = k[x] \cap k[x^{-1}] \subset k[x, x^{-1}].$$

This intersection is clearly

$$\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = k.$$

Regular functions on projective space

Theorem The only global regular functions on $\mathbb{P}^n$ are the constants:

$$\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) \cong k.$$

Sketch Proof Write $\mathbb{P}^n = \cup_{i=0}^n U_i$ for the standard open cover with $U_i \cong \mathbb{A}^n$. We have

$$\mathcal{O}_{\mathbb{P}^n}(U_i) \cong k[U_i] \cong k\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right].$$

A non-constant f would have to restrict as a non-constant polynomial to each of the U_i. That means that its expression as a rational function in the homogeneous coordinates x_i will have a denominator. But this denominator has to vanish somewhere on $\mathbb{P}^n$, and f cannot be regular there.

$\square$