C3.4 Algebraic Geometry

Lecture 14. The function field and rational maps

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The function field of an irreducible variety

All varieties in this lecture will be irreducible.

Let $X \subset \mathbb{A}^n$ be an irreducible affine variety.

Recall that this means that the coordinate ring

$$k[X] = k[x_1, \ldots, x_n]/\mathbb{I}(X)$$

is an integral domain.

Definition The **function field** of X is

$$k(X) = \mathrm{Frac}(k[X]) = \{\tfrac{g}{h} : g, h \in k[X], h \neq 0\},$$

the field of fractions of the integral domain $k[X]$.

Examples

- For $X = \mathbb{A}^1$, we have
$$k[\mathbb{A}^1] \cong k[x],$$
and so
$$k(\mathbb{A}^1) \cong k(x),$$
the field of rational functions in the variable x.
- For $X = \mathbb{A}^1 \setminus 0$, we have
$$k[X] \cong k[x]_x = k[x, x^{-1}],$$
and so
$$k(X) \cong k(x) \cong k[\mathbb{A}^1].$$
- For $X = \mathbb{A}^n$, we have $k[\mathbb{A}^n] \cong k[x_1, \ldots, x_n]$. So
$$k(\mathbb{A}^n) \cong k(x_1, \ldots, x_n).$$

A basic lemma

Let X be an irreducible affine variety.

Lemma For $h \in k[X]$, let

$$D_h = X \setminus \mathbb{V}(h) \subset X$$

be the corresponding basic open subset. Then

$$k(D_h) \cong k(X).$$

Proof By a Lemma in Lecture 12, we have

$$k[D_h] \cong k[X]_h = k[X][h^{-1}].$$

Thus

$$k(D_h) = \mathrm{Frac}(k[X]_h) \cong \mathrm{Frac}(k[X]) \cong k(X).$$

$\square$

Important corollaries

Corollary 1 If $U \subset X$ is any affine open subset in an irreducible affine variety, then

$$k(U) \cong k(X).$$

Proof Basic open sets form a basis of the topology of X.
Let $D_h \subset U$ be a basic open of X for $h \in k[X]$. Consider

$$g = h|_U \in \mathcal{O}_X(U) \cong O_U(U) \cong k[U],$$

the latter isomorphism following from U being affine.

So $D_h = D_g$ is also a basic open of U.

Now we obtain

$$k(U) \cong k(D_g) = k(D_h) \cong k(X).$$

$\square$

Important corollaries

Corollary 2 If $U_1, U_2 \subset X$ are affine open subsets in an irreducible quasi-projective variety $X \subset \mathbb{P}^n$, then

$$k(U_1) \cong k(U_2).$$

Proof As affine opens form a basis of the topology of X, the intersection $U_1 \cap U_2$ contains an affine subvariety V. Then by Corollary 1,

$$k(U_1) \cong k(V) \cong k(U_2).$$

$\square$

The function field of an irreducible quasi-projective variety

Let $X \subset \mathbb{P}^n$ be an irreducible quasi-projective variety.

Definition The **function field** of X is

$$k(X) = k(U) = \mathrm{Frac}(k[U]),$$

where $U \subset X$ is an affine open subset.

Note that by Corollary 2, the answer is independent of the affine open chosen. We also deduce

Corollary 3 If $U \subset X$ is any open subset of an irreducible quasi-projective variety $X \subset \mathbb{P}^n$, then

$$k(U) \cong k(X).$$

Example Let

$$U = \mathbb{A}^2 \setminus \mathbb{V}(x) \subset X = \mathbb{A}^2 \setminus \{(0,0)\} \subset \mathbb{A}^2.$$

Then $U \subset \mathbb{A}^2$ are both affine, X is not. Their function fields agree:

$$k(U) \cong k(X) \cong k(\mathbb{A}^2) \cong k(x,y).$$

The structure of function fields

Theorem

(1) Let X be an irreducible quasi-projective variety. Then $k(X)$ is a finitely generated field extension of k.

(2) Conversely, given any finitely generated field extension K/k, there exists an irreducible quasi-projective variety X with $k(X) \cong K$.

Proof of (1) We may assume that $X \subset \mathbb{A}^n$ is irreducible and affine. Then

$$k[X] = k[x_1, \ldots, x_n]/\mathbb{I}(X).$$

So

$$k(X) = \mathrm{Frac}(k[X])$$

is generated as a field over k by the images of $x_1, \ldots, x_n$. In particular, $k(X)$ is finitely generated over k.

The structure of function fields

Proof of (2) Conversely, let K be a field generated over k by some elements $\alpha_1, \ldots, \alpha_n$. Define a ring homomorphism

$$\varphi\colon R = k[x_1, \ldots, x_n] \to K$$

by mapping x_i to α_i.
Let $J = \ker \varphi$. Then by the Isomorphism Theorem $R/J \hookrightarrow K$, so J is a prime ideal as K is an integral domain.
Let

$$X = \mathbb{V}(J) \subset \mathbb{A}^n$$

be the corresponding irreducible affine variety. Then

$$k(X) = \operatorname{Frac} R/J \hookrightarrow K$$

contains the generators α_i in the image. So $k(X) \cong K$. $\square$

The function field of a hypersurface

Example Let $V_f \subset \mathbb{A}^n$ be an irreducible hypersurface defined by an irreducible polynomial $f \in k[x_1, \ldots, x_n]$.
Let us assume that f involves the variable x_n and is of degree d in that variable:

$$f(x_1, \ldots, x_n) = c_0 x_n^d + c_1(x_0, \ldots, x_{n-1}) x_n^{d-1} + \ldots$$

We have

$$k[V_f] = k[x_1, \ldots, x_n]/\langle f \rangle.$$

From Lecture 9, we also know

$$\dim V_f = n - 1 = \operatorname{trdeg}_k k(V_f).$$

There is a map of rings

$$k[x_1, \ldots, x_{n-1}] \to k[V_f]$$

which is injective, as there is no algebraic relation between $x_1, \ldots, x_n$. So we get an injection

$$k(x_1, \ldots, x_{n-1}) \hookrightarrow k(V_f).$$

The function field of a hypersurface

We have an injection

$$k(x_1, \ldots, x_{n-1}) \hookrightarrow k(V_f).$$

We get a realisation of the function field of V_f as

$$k(V_f) = k(x_1, \ldots, x_{n-1})[x_n]/\langle c_0 x_n^d + c_1(x_0, \ldots, x_{n-1}) x_n^{d-1} + \ldots \rangle,$$

a finite extension of degree d of the purely transcendental field

$$k(\mathbb{A}^{n-1}) = k(x_1, \ldots, x_{n-1}).$$

The same argument also computes the function field of an irreducible projective hypersurface; recall the function field only depends on a dense open subset, so it is enough to consider any affine chart.

For example, the function field of a plane curve of degree d is a degree d extension of $k(x)$.

Rational maps

Let X be an irreducible quasi-projective variety.

Definition A **rational map** $f : X \dashrightarrow Y$ to another quasi-projective variety Y is an equivalence class of pairs (f, U), where $f : U \to Y$ is a regular map defined on a non-empty open subset of X, and we identify pairs of maps which agree on a non-empty open subset.

Example We have a rational map

$$\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$$

given by

$$(x_0 : \cdots : x_n) \mapsto (x_0 : \cdots : x_{n-1}).$$

This is defined on $U = \mathbb{P}^n \setminus \{[0 : \cdots : 0 : 1]\}$.

Rational functions

Definition A **rational function** on an irreducible quasi-projective variety X is a rational map $f : X \dashrightarrow \mathbb{A}^1$.

Lemma There is a natural bijection

$$k(X) \leftrightarrow \{\text{rational functions } f : X \dashrightarrow \mathbb{A}^1\}.$$

Remark Compare this with what we had before: for an irreducible quasi-projective variety X, there is a bijection

$$\mathcal{O}_X(X) \leftrightarrow \{\text{morphisms } f : X \to \mathbb{A}^1\}.$$

In particular, if X is affine, we have a bijection

$$k[X] \leftrightarrow \{\text{morphisms } f : X \to \mathbb{A}^1\}.$$

Rational functions

Lemma There is a natural bijection

$$k(X) \leftrightarrow \{\text{rational functions } f : X \dashrightarrow \mathbb{A}^1\}.$$

Proof Both sides are invariant under passing to an open subset, so we may assume that X is affine. In this case,

$$k(X) = \mathrm{Frac}(k[X]).$$

A natural map is given for $f, g \in k[X]$ by

$$\tfrac{f}{g} \mapsto \left[\left(D_g, \tfrac{f}{g}\right)\right].$$

We need to check that we get every rational function $X \dashrightarrow \mathbb{A}^1$ this way. By the above Remark, any such corresponds to an element $h \in \mathcal{O}_X(U)$ for some open $U \subset X$. Taking a basic open $D_g \subset U$, we have an element $h|_{D_g} \in \mathcal{O}_X(D_g) = k[X]_g$. So we can write $h = f/g^N$ for $f \in k[X]$. $\square$

Rational maps to affine space

Corollary On an irreducible quasi-projective variety X, the data of a rational map $f : X \dashrightarrow \mathbb{A}^n$ is equivalent to n elements $f_i \in k(X)$:

$$f(x) = (f_1(x), \dots, f_n(x)),$$

valid in some open set $x \in U \subset X$.

Proof A rational map $f : X \dashrightarrow \mathbb{A}^n$ determines, and is determined by, n rational functions $f_i \colon X \dashrightarrow \mathbb{A}^1$, the coordinates of f. By the Lemma, each of these can be thought of as an element $f_i \in k(X)$.
Each $f_i \in k(X)$ is a regular function on some open set $U_i \subset X$.
The formula

$$f(x) = (f_1(x), \dots, f_n(x))$$

is valid on the intersection

$$U = \bigcap_i U_i \subset X$$

of these open sets U_i.

Composition of rational maps

Rational maps are not functions in the classical sense (they are only "partially defined").

In general, composition of rational maps may not be possible, even in the simplest cases.

Example Let $f\colon \mathbb{A}^1 \to \mathbb{A}^1$ be defined by $a \mapsto 0$. This is a regular, hence rational map.

Let $g\colon \mathbb{A}^1 \to \mathbb{A}^1$ be $a \mapsto a^{-1}$. This is a rational map.
The image of f is $\{0\} \subset \mathbb{A}^1$. The domain of g is $\mathbb{A}^1 \setminus \{0\}$.

So the composite $g \circ f$ is not defined.

Dominant rational maps

Definition A rational map

$$f = [(U, F)] : X \dashrightarrow Y$$

is **dominant**, if the image $F(U) \subset Y$ is dense.

The point of the definition is that dominant rational maps can always be composed.

Consider $f = [(F, U)] : X \dashrightarrow Y$ and $g = [(G, V)] : Y \dashrightarrow Z$ dominant rational.

Let

$$W = U \cap F^{-1}(V) \subset X,$$

an open subset of X.

Then F is defined on W, and maps to V.

Now we can compose with G to get a representative $[W, G \circ F]$ for $g \circ f$.

Birational equivalence

Definition A **birational equivalence** $f : X \dashrightarrow Y$ is a dominant rational map between irreducible quasiprojective varieties, which has a dominant rational inverse: a rational map $g : Y \dashrightarrow X$ with $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$ (equalities of rational maps).

We say $X \simeq Y$ are **birational**.

Example 1 $\mathbb{A}^n \simeq \mathbb{P}^n$ are birational, via the rational map $\mathbb{A}^n \dashrightarrow \mathbb{P}^n$ defined by the inclusion $\mathbb{A}^n \cong U_0 \subset \mathbb{P}^n$. (This is in fact a morphism.)
It has the rational inverse $\mathbb{P}^n \dashrightarrow \mathbb{A}^n$ defined by

$$[x_0 : \cdots : x_n] \to \left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right)$$

defined on U_0.

Example 2 Similarly, for an irreducible affine variety $X \subset \mathbb{A}^n$, let $\bar{X} \subset \mathbb{P}^n$ be the projective closure. Then the inclusion $X \to \bar{X}$ gives a birational equivalence $X \simeq \bar{X}$.

A famous birational self-equivalence

Consider the **Cremona transformation**

$$f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$

given by $[x : y : z] \mapsto [yz : xz : xy]$.
f is defined on the open set $U \subset \mathbb{P}^2$, where at least two coordinates are non-zero.
This rational map is equivalent to $[x : y : z] \mapsto [\frac{1}{x} : \frac{1}{y} : \frac{1}{z}]$, defined on the open $V \subset \mathbb{P}^2$ where all coordinates are non-zero.
This form shows that

$$f \circ f = \mathrm{id}_{\mathbb{P}^2},$$

so the map f is its own inverse, and in particular is a birational self-equivalence.

Note that f does **not** have the form $[x] \mapsto [Ax]$ for $A \in \mathrm{GL}(3, k)$, which are the regular self-morphisms (automorphisms) of $\mathbb{P}^2$.

Dominant rational maps and field homomorphisms

Theorem Consider quasi-projective varieties X, Y. There is a one-to-one correspondence between dominant rational maps $f\colon X \dashrightarrow Y$ and field homomorphisms $\varphi\colon k(Y) \to k(X)$.

Proof, forward construction We may restrict to open affines, so we may as well assume X, Y are affine. First assume that we have a dominant rational map $f\colon X \dashrightarrow Y$.

Consider the k-algebra homomorphism

$$f^* : k[Y] \to k(X)$$

defined by

$$(y : Y \to \mathbb{A}^1) \mapsto (f^* y = y \circ f : X \dashrightarrow \mathbb{A}^1).$$

We claim that as f is dominant, f^* is injective.

If so, we can define $f^* : k(Y) \to k(X)$ by

$$\frac{g}{h} \mapsto \frac{f^* g}{f^* h}.$$

Dominant rational maps and field homomorphisms

Proof of claim The k-algebra homomorphism

$$f^* : k[Y] \to k(X)$$

is defined by

$$(y : Y \to \mathbb{A}^1) \mapsto (f^* y = y \circ f : X \dashrightarrow \mathbb{A}^1).$$

Suppose that $y \in k[Y]$ is in the kernel of Y. Write $f = [(U, F)]$.
Then $F^* y = 0$ implies that for all $u \in U$,

$$y(F(u)) = 0.$$

Thus for all $u \in U$,

$$F(u) \subset \mathbb{V}(y)$$

and so

$$F(U) \subset \mathbb{V}(y) \subset Y.$$

As f was assumed dominant, $y = 0$.

Dominant rational maps and field homomorphisms

Proof, reverse construction Suppose we have a field homomorphism $\varphi\colon k(Y) \to k(X)$.

As fields have no proper ideals, φ is injective.

Let $y_1, \ldots, y_n$ be generators of $k[Y]$. Then

$$\varphi(y_j) = \frac{g_j}{h_j} \in k(X).$$

Let $U = \cap D_{h_j} = D_{h_1 \cdot \ldots \cdot h_n}$, an affine subvariety of the affine variety X.

We have $\varphi(y_j) \in \mathcal{O}_X(U) = k[U]$.

Since $k[Y]$ is generated by the y_j, we get an inclusion

$$\varphi\colon k[Y] \hookrightarrow k[U].$$

This corresponds to a dominant morphism $\varphi^* : U \to Y$ of affine varieties.

The pair $[(U, \varphi^*)]$ represents a dominant rational map $\varphi^* : X \dashrightarrow Y$. $\square$

Birational varieties

Corollary Two quasi-projective varieties X, Y are birational if and only if their function fields are isomorphic.

Example As we saw before, we have a birational equivalence $\mathbb{A}^2 \simeq \mathbb{P}^2$. We also have $\mathbb{A}^1 \simeq \mathbb{P}^1$ and it is not hard to argue from this that $\mathbb{A}^1 \times \mathbb{A}^1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. We deduce a birational equivalence

$$\mathbb{P}^2 \simeq \mathbb{A}^2 \cong \mathbb{A}^1 \times \mathbb{A}^1 \simeq \mathbb{P}^1 \times \mathbb{P}^1.$$

This is the first example of two **projective** varieties $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ that are birational to each other. In fact these two varieties are not isomorphic!

Sketch proof Suppose there is an isomorphism $\varphi\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$. Consider $C_0 = \{0\} \times \mathbb{P}^1$ and $C_1 = \{1\} \times \mathbb{P}^1$ inside $\mathbb{P}^1 \times \mathbb{P}^1$. These are two curves (one-dimensional varieties) in $\mathbb{P}^1 \times \mathbb{P}^1$, with $C_0 \cap C_1 = \emptyset$.
Let $D_i = \varphi(C_i)$. Since φ is an isomorphism, D_1, D_2 are two curves in $\mathbb{P}^2$ with empty intersection. This contradicts Bezout's theorem.

The explicit birational maps

To conclude, we spell out explicit birational maps between $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$.

In one direction, we can use

$$((x_0 : x_1), (y_0 : y_1)) \mapsto (x_0y_0 : x_1y_0 : x_0y_1).$$

In the other direction, we can use

$$(z_0 : z_1 : z_2) \mapsto ((z_0 : z_1), (z_0 : z_2)).$$

Note that these are both defined on open subsets of the projective varieties.

To understand where these formulae came from, divide in each case by the first coordinate and see what happens!