C3.4 Algebraic Geometry

Lecture 15. Tangent spaces. Non-singular and singular varieties

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Tangent spaces of an affine variety

Consider $f \in k[x_1, \ldots, x_n]$, and $p = (p_1, \ldots, p_n) \in \mathbb{A}^n$.

The linear polynomial $d_p f \in k[x_1, \ldots, x_n]$ is defined by

$$d_p f = df|_{x=p} \cdot (x - p) = \sum \tfrac{\partial f}{\partial x_j}(p) \cdot (x_j - p_j).$$

Consider an affine variety $X \subset \mathbb{A}^n$ with $\mathbb{I}(X) = \langle f_1, \ldots, f_N \rangle$.

Definition The **tangent space** to X at the point p is the space

$$T_p X = \mathbb{V}(d_p f_1, \ldots, d_p f_N) = \cap \ker df_i \subset \mathbb{A}^n.$$

This is an intersection of hyperplanes, so a linear subspace of some dimension.

Trivial example For $X = \mathbb{A}^n$,

$$T_p X = \mathbb{A}^n$$

at every point $p \in \mathbb{A}^n$.

Tangent spaces of a hypersurface

Example Consider a hypersurface $X = \mathbb{V}(f)$ defined by a polynomial f. We have

$$T_pX = \mathbb{V}(d_pf) = \ker(\nabla f_p) \subset \mathbb{A}^n.$$

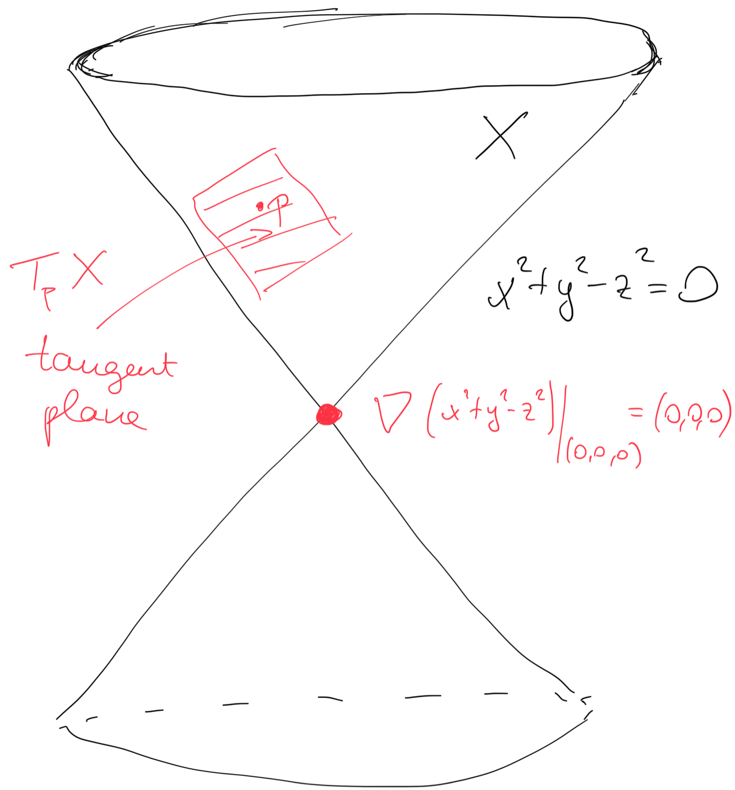
- If the gradient $\nabla f_p \neq 0$, then this is a codimension-one linear subspace of $\mathbb{A}^n$, naturally thought of as the tangent space at p to $X = \mathbb{V}(f)$.
- If the gradient $\nabla f_p = 0$, then

  $$T_pX = \mathbb{A}^n.$$

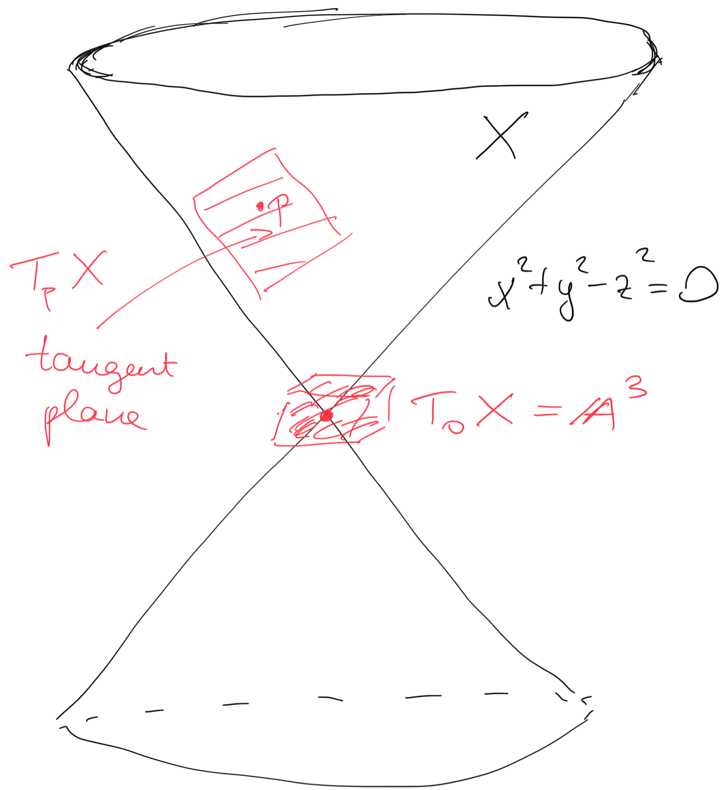
  Algebraic geometry does not say **there are no tangent vectors at such a point** $p \in X$.

  It says **all vectors are tangent vectors at such a point** $p \in X$.

Tangent spaces of a hypersurface



Tangent spaces of a hypersurface



Smooth and singular points of an affine variety

Key fact For any point $p \in X$,

$$\dim_k T_p X \geq \dim_p X.$$

Definiton A point $p \in X$ is a **smooth point** if

$$\dim_k T_p X = \dim_p X.$$

A point $p \in X$ is a **singular point** if

$$\dim_k T_p X > \dim_p X.$$

Let

$$\mathrm{Sing}(X) = \{p \in X : p \text{ is a singular point of } X\} \subset X.$$

The variety X is **nonsingular** if

$$\mathrm{Sing}(X) = \emptyset$$

Smooth and singular points of an affine hypersurface

Example For an irreducible hypersurface $X = \mathbb{V}(f) \subset \mathbb{A}^n$, we have

$$\dim_p X = n - 1$$

at every point $p \in X$.

- If the gradient $\nabla f_p \neq 0$, then

  $$\dim_k T_p X = n - 1 = \dim_p X,$$

  so such a point is smooth.

- If the gradient $\nabla f_p = 0$, then

  $$\dim_k T_p X = n > \dim_p X$$

  so such a point is singular.

Remark For $k = \mathbb{R}$, near a smooth point p, $X = \mathbb{V}(f)$ is a **codimension-one submanifold** of $\mathbb{R}^n$.

The singular set is Zariski closed

Let X be an irreducible affine variety of dimension d with

$$\mathbb{I}(X) = \langle f_1, \ldots, f_N \rangle.$$

Theorem The set $\mathrm{Sing}(X) \subset X \subset \mathbb{A}^n$ is a closed subvariety, given by the vanishing in X of all $(n-d) \times (n-d)$ minors of the Jacobian matrix

$$\mathrm{Jac} = \left(\frac{\partial f_i}{\partial x_j} \right).$$

Proof By definition, T_pX is the zero set of

$$\varphi_p : \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} \left.\frac{\partial F_1}{\partial x_1}\right|_p & \cdots & \left.\frac{\partial F_1}{\partial x_n}\right|_p \\ \vdots & & \\ \left.\frac{\partial F_N}{\partial x_1}\right|_p & \cdots & \left.\frac{\partial F_N}{\partial x_n}\right|_p \end{pmatrix} \cdot \begin{pmatrix} x_1 - p_1 \\ \vdots \\ x_n - p_n \end{pmatrix}.$$

The singular set is Zariski closed

We have that T_pX is the zero set of

$$\varphi_p : \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} \frac{\partial F_1}{\partial x_1}\Big|_p & \cdots & \frac{\partial F_1}{\partial x_n}\Big|_p \\ \vdots & & \\ \frac{\partial F_N}{\partial x_1}\Big|_p & \cdots & \frac{\partial F_N}{\partial x_n}\Big|_p \end{pmatrix} \cdot \begin{pmatrix} x_1 - p_1 \\ \vdots \\ x_n - p_n \end{pmatrix}.$$

Hence

$$p \in \operatorname{Sing} X \Leftrightarrow \dim \varphi_p^{-1}(0) > d \Leftrightarrow \dim \ker \operatorname{Jac}_p > d.$$

Claim The last condition is equivalent to the vanishing of all $(n-d) \times (n-d)$ minors of Jac_p.

For otherwise, we could find $n - d$ linearly independent columns in Jac_p, the columns involved in that minor.

Hence the rank of Jac_p would have dimension at least $n - d$.

So the kernel of Jac_p would have dimension at most d.

$\square$

The tangent space is an intrinsic notion

Let $X \subset \mathbb{A}^n$ be an affine variety, and let $p \in X$.

Recall the ring of germs of functions $\mathcal{O}_{X,p}$ and its maximal ideal

$$\mathfrak{m}_p = \left\{ \frac{f}{g} \in \mathcal{O}_{X,p} : f(p) = 0 \right\} \subset \mathcal{O}_{X,p}.$$

Theorem There is a canonical isomorphism

$$T_pX \cong \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^*.$$

The vector space $\mathfrak{m}_p/\mathfrak{m}_p^2$ is called the **cotangent space**.

Remark What this result shows is that the tangent space can equivalently be defined in a purely local way, only using information about the ring of local germs of functions.

The tangent space is an intrinsic notion

Theorem There is a canonical isomorphism

$$T_pX \cong \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^*.$$

Proof By a translation that does not change anything, we can assume

$$p = 0 \in X \subset \mathbb{A}^n.$$

First, we will look at the case $X = \mathbb{A}^n$ itself. For $F \in k[x_1, \ldots, x_n]$, we can consider the linear functional

$$d_0F = \left(\frac{\partial F}{\partial x_i}\right)\Big|_{x_i=0} \colon \mathbb{A}^n \equiv T_0\mathbb{A}^n \to k;$$

this is basically taking inner product of a vector with $\nabla_0 F$. So we get an element

$$d_0F \in (T_0\mathbb{A}^n)^*,$$

giving us a linear map

$$d_0 \colon k[x_1, \ldots, x_n] \to (T_0\mathbb{A}^n)^*.$$

The tangent space is an intrinsic notion

Have a linear map

$$d_0\colon k[x_1, \ldots, x_n] \to (T_0\mathbb{A}^n)^*.$$

Restricting to the maximal ideal $\mathfrak{m}$ at the origin on the left, we get a linear map

$$d_0|_{\mathfrak{m}} : \mathfrak{m} \to (T_0\mathbb{A}^n)^*.$$

Claim 1 The linear map $d_0|_{\mathfrak{m}}$ is surjective, and its kernel is $\mathfrak{m}^2$.

Claim 1 is easy to check - see Lecture Notes for details.

By the Isomorphism Theorem, we then obtain

$$(T_0\mathbb{A}^n)^* \cong \mathfrak{m}/\mathfrak{m}^2,$$

which is the dual of the isomorphism claimed by the Theorem in this case.

The tangent space is an intrinsic notion

Let us consider the general case $X \subset \mathbb{A}^n$. Let us continue to denote by $\mathfrak{m}$ the maximal ideal of the origin in $\mathbb{A}^n$.
The inclusion $j : T_0X \hookrightarrow T_0\mathbb{A}^n$ is injective, so the dual map is surjective, hence we get a surjection

$$j^* : \mathfrak{m}/\mathfrak{m}^2 \cong (T_0\mathbb{A}^n)^* \to (T_0X)^*.$$

and thus a surjection

$$j^* \circ d_0 \colon \mathfrak{m} \to (T_0X)^*.$$

Claim 2 We have

$$\ker(j^* \circ d_0) = \mathfrak{m}^2 + \mathbb{I}(X).$$

For standard details of the proof of Claim 2, see Lecture Notes for details.

We deduce, once again by the isomorphism theorem,

$$\mathfrak{m}/(\mathfrak{m}^2 + \mathbb{I}(X)) \cong (T_0X)^*.$$

The tangent space is an intrinsic notion

Write $\overline{\mathfrak{m}} \lhd k[X]$ for the image of $\mathfrak{m}$ in the quotient $k[X] = R/\mathbb{I}(X)$; this is the maximal ideal of $p = 0$ in the coordinate ring $k[X]$.
The last isomorphism then implies

$$\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 \cong (T_0 X)^*.$$

Finally, we need to relate this quotient to the corresponding quotient in the localised ring.

Claim 3 The standard inclusion $k[X] \hookrightarrow \mathcal{O}_{X,P}$ gives an isomorphism

$$\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 \cong \mathfrak{m}_p/\mathfrak{m}_p^2,$$

where recall $\mathfrak{m}_p \lhd \mathcal{O}_{X,P}$ is the maximal ideal of non-vanishing germs of regular functions at p.

For standard details of the proof of Claim 3, see Lecture Notes for details.

The proof of the Theorem is complete. $\square$

The tangent space is an intrinsic notion

Remark The argument presented above also proves

$$T_pX \cong (\mathcal{I}_p/\mathcal{I}_p^2)^*,$$

where

$$\mathcal{I}_p = \langle x_1 - p_1, \ldots, x_n - p_n \rangle \subset k[X]$$

is the maximal ideal of an arbitary point $p \in X$ inside the coordinate ring.

Corollary The tangent space T_pX only depends on an open neighbourhood of $p \in X$.

Proof By the Theorem, tangent space only depends on the local ring $\mathcal{O}_{X,p}$ and its unique maximal ideal $\mathfrak{m}_p$. $\square$

Tangent spaces and nonsingularity for quasi-projective varieties

Definition For X a quasi-projective variety, we define the tangent space at $p \in X$ by

$$T_pX = \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^*.$$

A point $p \in X$ is **smooth** if

$$\dim_k T_pX = \dim_p X.$$

Proposition The set of non-smooth (singular) points $p \in X$ forms a Zariski closed subvariety of X.

Proof This is true in the (affine) neighbourhood of any point. $\square$

The quasi-projective variety X is **nonsingular**, if

$$\mathrm{Sing}(X) = \emptyset$$

Example For an irreducible projective hypersurface $X = \mathbb{V}(f) \subset \mathbb{P}^n$, singular points $p \in X$ are located where

$$\nabla f_p = 0.$$

A further example

Recall from Lecture 7 the chain of subvarities

$$\Sigma_{2,2} \subset \Delta \subset \mathbb{P}^8,$$

inside the projective space $\mathbb{P}\mathrm{M}_k(3) \cong \mathbb{P}^8$ of 3×3 matrices over k.

Here $\Delta = \{[A] \colon \det A = 0\} \subset \mathbb{P}^8$ is the projective cubic hypersurface defined by the determinant polynomial.

This contains $\Sigma_{2,2} \cong \mathbb{P}^2 \times \mathbb{P}^2$, the Segre variety in $\mathbb{P}^8$.

Fact The hypersurface Δ is singular, with singular locus

$$\mathrm{Sing}(\Delta) = \Sigma_{2,2}.$$