C3.4 Algebraic Geometry

Lecture 16. Blowups and desingularization

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The blowup of affine space at the origin

The blow-up of $\mathbb{A}^n$ at the origin is the set

$$B_0\mathbb{A}^n = \{(x, \ell) : \mathbb{A}^n \times \mathbb{P}^{n-1} : x \in \ell\}.$$

Use coordinates $(x_1, \ldots, x_n)$ on $\mathbb{A}^n$ and $[y_1 : \cdots : y_n]$ on $\mathbb{P}^{n-1}$.
$x \in \ell$ if and only if $(x_1, \ldots, x_n)$ and $(y_1, \cdots, y_n)$ are proportional.
Equivalently the matrix

$$\begin{pmatrix} x_1 & \ldots & x_n \\ y_1 & \ldots & y_n \end{pmatrix}$$

has rank 1. This happens if and only if its 2×2 minors vanish.
So we can write the equations

$$B_0\mathbb{A}^n = \mathbb{V}(x_i y_j - x_j y_i) \subset \mathbb{A}^n \times \mathbb{P}^{n-1}.$$

This shows in particular that $B_0\mathbb{A}^n$ is a quasi-projective variety.

The blowup of affine space at the origin

The morphism

$$\pi : B_0\mathbb{A}^n \to \mathbb{A}^n,\ \pi(x, [y]) = x$$

is birational with inverse

$$\mathbb{A}^n \dashrightarrow B_0\mathbb{A}^n, \quad x \mapsto (x, [x])$$

defined on $x \neq 0$: the point is that for $x \neq 0$, x lies in a unique line $[x]$.

The fibre $\pi^{-1}(x)$ is a point for $x \neq 0$. Also

$$E_0 = \pi^{-1}(0) = \{0\} \times \mathbb{P}^{n-1},$$

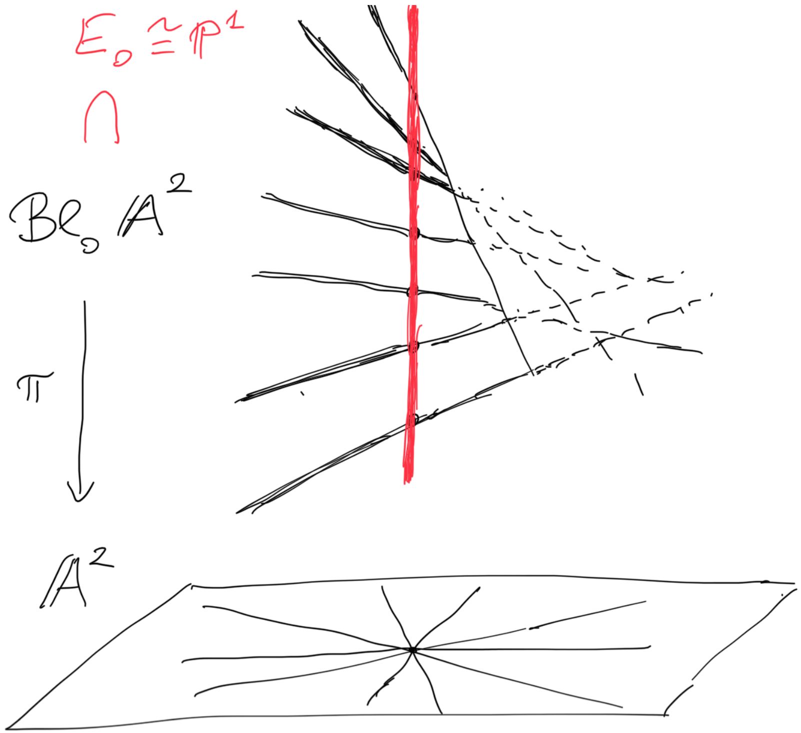
called the **exceptional set** of π.

Thus

$$\pi : B_0\mathbb{A}^n \setminus E_0 \to \mathbb{A}^n \setminus 0$$

is an isomorphsm, and π collapses E_0 to the point 0.

The blowup of affine space at the origin



The blowup of an affine variety in one of its points

Let $X \subset \mathbb{A}^n$ be an afffine variety with $0 \in X$.

Definition The **blow-up** of X at 0 is the Zariski closure

$$B_0X = \overline{\pi^{-1}(X \setminus \{0\})} \subset B_0\mathbb{A}^n.$$

The map $\pi|_{B_0X} : B_0X \to X$ is birational, and we have the **exceptional set**

$$E = \pi^{-1}(0) \cap B_0X.$$

Interpretation The blowup B_0X keeps track of those directions $E \subset E_0$ along which X approaches 0.

Remark The blowup B_0X, the Zariski closure of $\pi^{-1}(X \setminus \{0\})$, is different from

$$\pi^{-1}(X) = B_0X \cup_E E_0.$$

An example

Example Consider

$$X = \mathbb{V}(xy) \subset \mathbb{A}^2,$$

the union of the coordinate axes. Then $0 \in X$, and we can consider its blowup. First,

$$\pi^{-1}(X \setminus 0) = \{((x,y),[a:b]) \in \mathbb{A}^2 \times \mathbb{P}^1 : xb - ya = 0, xy = 0, (x,y) \neq (0,0)\}.$$

We get the following solutions: either

$$((x,0),[1:0]) \subset \mathbb{A}^2 \times \mathbb{P}^1$$

for $x \neq 0$, or

$$((0,y),[0:1]) \subset \mathbb{A}^2 \times \mathbb{P}^1$$

for $y \neq 0$.

Then we get the closure

$$B_0 X = (\mathbb{A}^1 \times 0, [1:0]) \sqcup (0 \times \mathbb{A}^1, [0:1]) \subset B_0\mathbb{A}^2 \subset \mathbb{A}^2 \times \mathbb{P}^1,$$

a disjoint union of two lines.

An example

We have

$$B_0X = (\mathbb{A}^1 \times 0, [1:0]) \sqcup (0 \times \mathbb{A}^1, [0:1]) \subset B_0\mathbb{A}^2 \subset \mathbb{A}^2 \times \mathbb{P}^1,$$

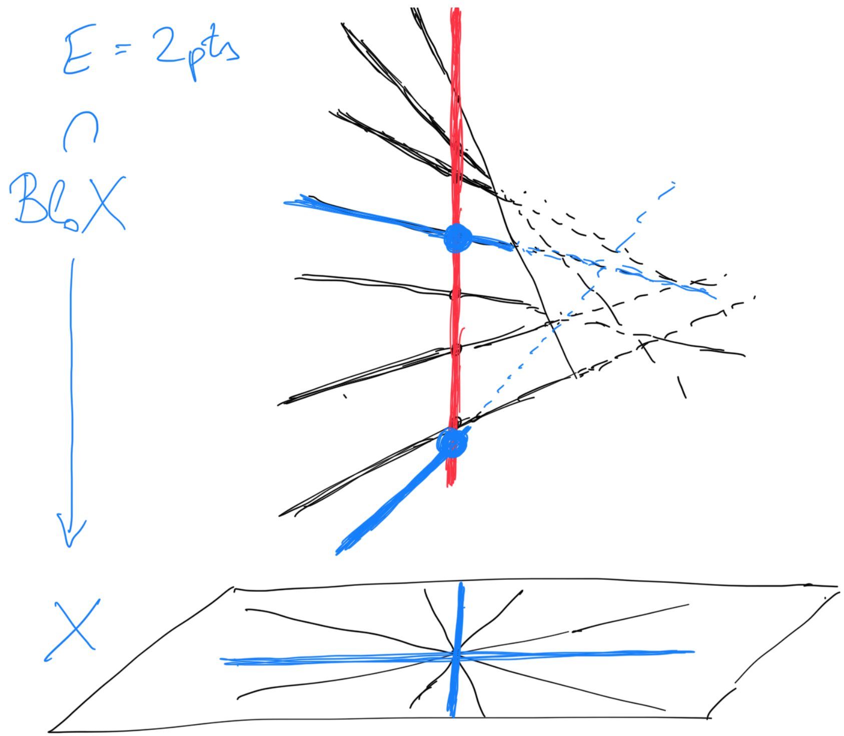
a disjoint union of two lines.

The exceptional set E consists of two points: $((0,0),[1:0])$, $((0,0),[0:1])$, the 2 directions of the lines in X.

Note also that X is **singular** at the origin, while B_0X is **nonsingular**.

Indeed, it is often (though not always) the case that if $0 \in X$ is a singular point, B_0X is 'less singular'.

An example



Another example: the cuspidal curve

Example Consider the cuspidal curve

$$Y = \mathbb{V}(y^2 - x^3) \subset \mathbb{A}^2.$$

It is singular at $0 \in Y$. Let us describe the variety

$$B_0 Y \subset \mathbb{A}^2 \times \mathbb{P}^1.$$

Use coordinates $((x, y), [a : b])$; we have the equation $xb - ya = 0$ as one of the equations of the blowup. We want to compute the Zariski closure of

$$\pi^{-1}(Y) \subset \mathbb{A}^2 \times \mathbb{P}^1.$$

Instead of this mixture of affine and projective coordinates, it is a lot easier to work in affine coordinates (remember quasi-projective varieties are locally affine!).

Another example: the cuspidal curve

Use the usual open cover of $\mathbb{P}^1$. We have the open set

$$(B_0Y)_a = (B_0Y) \cap (a \neq 0) \subset \mathbb{A}^2 \times \mathbb{A}^1,$$

with coordinates $(x, y, u = b/a)$ and equation $y = ux$. Substitute into the equation of the curve Y, to get the equation

$$0 = y^2 - x^3 = x^2u^2 - x^3.$$

The proper transform (Zariski closure) is obtained by dropping the x^2 factor:

$$u^2 - x = 0.$$

We obtain a description of our open set

$$(B_0Y)_a = \{u^2 - x = 0, y = ux\} \subset \mathbb{A}^3_{x,y,u}.$$

This is a nonsingular curve, isomorphic (using projection to the last coordinate u) to $\mathbb{A}^1$.

Another example: the cuspidal curve

The second open set is

$$(B_0Y)_b = (B_0Y) \cap (b \neq 0) \subset \mathbb{A}^2 \times \mathbb{A}^1,$$

with coordinates $(x, y, v = a/b)$ and equation $x = vy$. Substitute into the equation of the curve Y, to get the equation

$$0 = y^2 - x^3 = y^2 - y^3v^3.$$

The proper transform (Zariski closure) is now obtained by dropping the y^2 factor:

$$1 - yv^3 = 0.$$

We get

$$(B_0Y)_b = \{1 - yv^3 = 0, x = vy\} \subset \mathbb{A}^3_{x,y,v}.$$

Again, this is a nonsingular curve.

Another example: the cuspidal curve

The sets $(a \neq 0)$ and $(b \neq 0)$ cover $\mathbb{P}^1$. So the two open sets we looked at cover the blowup:

$$B_0Y = (B_0Y)_a \cup (B_0Y)_b.$$

These open sets are described as explicit affine varieties

$$(B_0Y)_a = \{u^2 - x = 0, y = ux\} \subset \mathbb{A}^3_{x,y,u}$$

and

$$(B_0Y)_b = \{1 - yv^3 = 0, x = vy\} \subset \mathbb{A}^3_{x,y,v}.$$

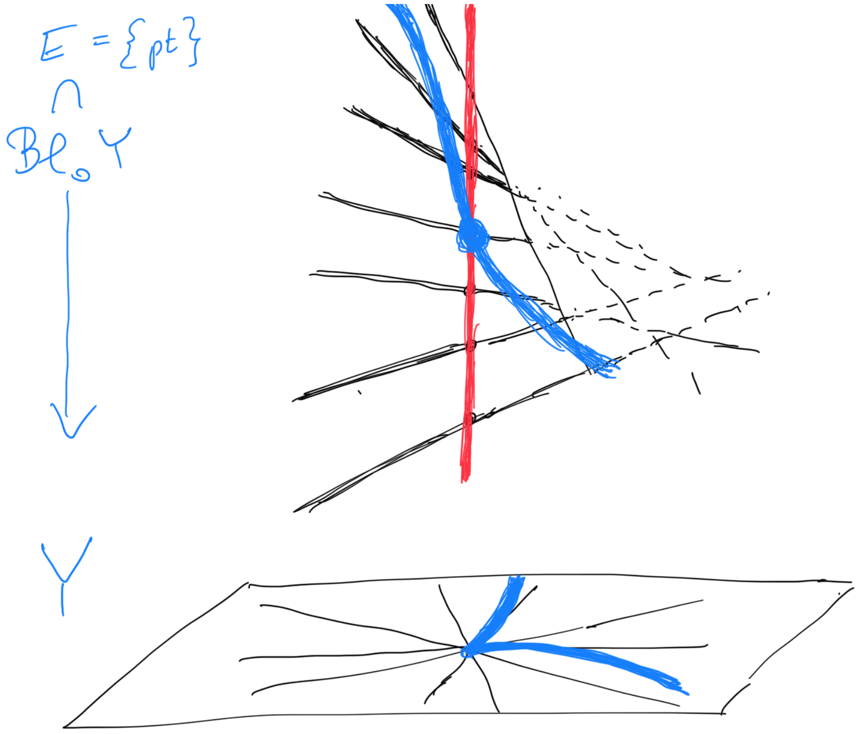
Both are nonsingular, so B_0Y is a nonsingular curve.
Finally the blowup map

$$\pi\colon B_0\mathbb{A}^2 \to \mathbb{A}^2$$

induces a surjective, birational morphism

$$\pi|_{B_0Y}\colon B_0Y \to Y.$$

Another example: the cuspidal curve



Strategy for computing a general blowup

Consider in general
$$0 \in X \subset \mathbb{A}^n.$$
The blowup
$$B_0X \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$$
can be computed into the following steps.

1. Cover $\mathbb{P}^{n-1}$ by the n standard affine open pieces. Introduce sensible names for the local coordinates.

2. Use the simplified quadric equations of $B_0\mathbb{A}^n$ to make substitutions into the equations of X.

3. Remove extra factors of local equations of the exceptional set to find the Zariski closure $(B_0X)_i$ in this local chart.

4. Check for singularities, etc of the resulting open pieces $(B_0X)_i$.

Strategy for computing a general blowup

The whole blowup B_0X will be covered by the n affine pieces

$$B_0X = \bigcup_{i=0}^{n-1} (B_0X)_i.$$

We get a surjective, birational morphism

$$\pi|_{B_0X} \colon B_0X \to X.$$

In some cases, the singularities of B_0X will be better than the singularities of X.

In particularly favourable cases, an iterated blowup of points of X may lead to a chain of surjective, birational morphisms

$$X_m \to X_{m-1} \to \ldots \to X_1 = B_pX \to X$$

with $p \in X$ a singular point, and X_m a **nonsingular** quasi-projective variety.

Resolution of singularities

Here is the general result, a cornerstone of 20th century algebraic geometry.

Hironaka's Resolution of singularities theorem Assume char $k = 0$. Given a quasi-projective variety $X \supset Z = \mathrm{Sing}(X)$, there exists a surjective, birational morphism

$$f\colon Y \to X,$$

from a quasiprojective variety Y, such that with $E = f^{-1}(Z)$, f induces an isomorphism

$$Y \setminus E \cong X \setminus Z.$$

In other words, X is unchanged outside its singular locus.
If X is projective, Y can also be chosen to be projective.
The map f is a composite of certain kinds of blowups (not just that of closed points).