C3.3 Differentiable Manifolds

Problem Sheet 0

Michaelmas Term 2020–2021

1. For a smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$ (or between open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$) we let $\mathrm{d}f_p : \mathbb{R}^n \to \mathbb{R}^m$ denote the differential of f at $p \in \mathbb{R}^n$. Since $\mathrm{d}f_p$ is a linear map, we can identify it with a matrix: if we write $f = (f_1, \ldots, f_m)$ and let $(x_1, \ldots, x_n)$ denote coordinates on $\mathbb{R}^n$, then the matrix is $(\frac{\partial f_i}{\partial x_j})$.

   (a) Let $f : \mathbb{R} \to \mathbb{R}^2$ be given by $f(t) = (t^2, t^3)$.
   Calculate $\mathrm{d}f_t$ for any $t \in \mathbb{R}$ and show that $\mathrm{d}f_t$ is injective except at $t = 0$. Sketch the image of f in $\mathbb{R}^2$.

   (b) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3$.
   Calculate $\mathrm{d}f_x$ for any $x \in \mathbb{R}^3$ and show that $\mathrm{d}f_x$ is surjective for all $x \in \mathbb{R}^3$.

   (c) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $f(x_1, x_2, x_3) = (x_2x_3, x_3x_1, x_1x_2)$.
   Calculate $\mathrm{d}f_x$ for any $x \in \mathbb{R}^3$ and determine for which $x \in \mathbb{R}^3$ we have that $\mathrm{d}f_x$ is an isomorphism.

   (d) Let $M_n(\mathbb{R})$ be the $n \times n$ real matrices and let $\mathrm{GL}(n, \mathbb{R})$ be the set of invertible $n \times n$ real matrices. Let $f : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$ be given by $f(A) = \det A$.
   Calculate $\mathrm{d}f_A$ for any $A \in \mathrm{GL}(n, \mathbb{R})$ as a map from $M_n(\mathbb{R})$ to $\mathbb{R}$ and show that it is surjective for all $A \in \mathrm{GL}(n, \mathbb{R})$.

2. Show that $\mathbb{R}^n$ and $\mathcal{S}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \ldots + x_{n+1}^2 = 1\}$ are second countable and Hausdorff with respect to their natural topologies.

3. Let $N = (0, 0, 1) \in \mathcal{S}^2$ and $S = (0, 0, -1) \in \mathcal{S}^2$ and define $U_N = \mathcal{S}^2 \setminus \{N\}$ and $U_S = \mathcal{S}^2 \setminus \{S\}$.
   Let $\varphi_N : U_N \to \mathbb{R}^2$ and $\varphi_S : U_S \to \mathbb{R}^2$ be given by
   $$\varphi_N(x_1, x_2, x_3) = \frac{(x_1, x_2)}{1 - x_3} \quad \text{and} \quad \varphi_S(x_1, x_2, x_3) = \frac{(x_1, x_2)}{1 + x_3}.$$

   (a) By constructing explicit inverses, or otherwise, show that φ_N and φ_S are homeomorphisms (i.e. continuous bijections with continuous inverses).

   Let $f = \varphi_S \circ \varphi_N^{-1}$ defined on $\varphi_N(U_N \cap U_S)$.

   (b) Calculate f and show that it defines a diffeomorphism of $\mathbb{R}^2 \setminus \{0\}$ (i.e. it is a smooth map with smooth inverse).

   (c) Calculate the differential $\mathrm{d}f_y$ at any point $y \in \mathbb{R}^2 \setminus \{0\}$. Calculate $\det \mathrm{d}f_y$ when $\mathrm{d}f_y$ is viewed as a matrix, and show that it is never zero.

4. (a) Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x_1, x_2) = (e^{x_1} \cos x_2, e^{x_1} \sin x_2)$.
   Show that f is a local diffeomorphism (i.e. given any point $x \in \mathbb{R}^2$ there is an open set $U \ni x$ and $V \ni f(x)$ so that $f : U \to V$ is a diffemorphism). Is f a diffeomorphism?

   (b) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x_1, x_2) = x_1^3 + x_2^3 + e^{x_1 + x_2}$.
   Show that there is a smooth function $g(x_1)$ so that $f(x_1, x_2) = 0$ if and only if $x_2 = g(x_1)$. Deduce that $f^{-1}(0)$ is a manifold.