C3.3 Differentiable Manifolds

Problem Sheet 0: Solutions

Michaelmas Term 2020–2021

1. For a smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$ (or between open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$) we let $\mathrm{d}f_p : \mathbb{R}^n \to \mathbb{R}^m$ denote the differential of f at $p \in \mathbb{R}^n$. Since $\mathrm{d}f_p$ is a linear map, we can identify it with a matrix: if we write $f = (f_1, \ldots, f_m)$ and let $(x_1, \ldots, x_n)$ denote coordinates on $\mathbb{R}^n$, then the matrix is $(\frac{\partial f_i}{\partial x_j})$.

   (a) Let $f : \mathbb{R} \to \mathbb{R}^2$ be given by $f(t) = (t^2, t^3)$.
   Calculate $\mathrm{d}f_t$ for any $t \in \mathbb{R}$ and show that $\mathrm{d}f_t$ is injective except at $t = 0$. Sketch the image of f in $\mathbb{R}^2$.

   We calculate
$$\mathrm{d}f_t = \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix}.$$
   This matrix always has rank 1 (i.e. is not the zero matrix in this case) if $t \neq 0$, and therefore $\mathrm{d}f_t$ is injective except for $t = 0$.

   The image of f is a classic cusp curve, where the cusp is at 0.

   (b) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3$.
   Calculate $\mathrm{d}f_x$ for any $x \in \mathbb{R}^3$ and show that $\mathrm{d}f_x$ is surjective for all $x \in \mathbb{R}^3$.

   We see that
$$\mathrm{d}f_x = (2x_1 \; 2x_2 \; -1).$$
   This matrix always has full rank (i.e. 1) because the last entry is never zero, and hence $\mathrm{d}f_x$ is surjective for all $x \in \mathbb{R}^3$.

   (c) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $f(x_1, x_2, x_3) = (x_2x_3, x_3x_1, x_1x_2)$.
   Calculate $\mathrm{d}f_x$ for any $x \in \mathbb{R}^3$ and determine for which $x \in \mathbb{R}^3$ we have that $\mathrm{d}f_x$ is an isomorphism.

   We compute
$$\mathrm{d}f_x = \begin{pmatrix} 0 & x_3 & x_2 \\ x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{pmatrix}.$$
   We see that $\det \mathrm{d}f_x = 2x_1x_2x_3$ for all $x \in \mathbb{R}^3$ and so $\mathrm{d}f_x$ is an isomorphism if and only if all of x_1, x_2, x_3 are non-zero.

   (d) Let $M_n(\mathbb{R})$ be the $n \times n$ real matrices and let $\mathrm{GL}(n, \mathbb{R})$ be the set of invertible $n \times n$ real matrices. Let $f : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$ be given by $f(A) = \det A$.
   Calculate $\mathrm{d}f_A$ for any $A \in \mathrm{GL}(n, \mathbb{R})$ as a map from $M_n(\mathbb{R})$ to $\mathbb{R}$ and show that it is surjective for all $A \in \mathrm{GL}(n, \mathbb{R})$.

To compute $\mathrm{d}f_A$ we see that

$$f(A+B) - f(A) = \det(A+B) - \det A = \det(A(I + A^{-1}B)) - \det A = \det A(\det(I + A^{-1}B) - 1).$$

We then notice that

$$\det(I + A^{-1}B) = 1 + \operatorname{tr}(A^{-1}B) + o(\|B\|).$$

From here, we then use the definition of $\mathrm{d}f_A$ as the unique linear map so that

$$\frac{\|f(A+B) - f(A) - \mathrm{d}f_A(B)\|}{|B|} \to 0$$

as $\|B\| \to 0$. In other words, we see that

$$f(A+B) - f(A) = \det A \operatorname{tr}(A^{-1}B) + o(|B|)$$

and so

$$\mathrm{d}f_A(B) = \det A \operatorname{tr}(A^{-1}B).$$

Taking $B = cA$ for any $c \in \mathbb{R}$ we see that

$$\mathrm{d}f_A(cA) = \det A \operatorname{tr}(cI) = nc \det A.$$

Since $\det A \neq 0$ we can choose c as we wish to ensure $\mathrm{d}f_A$ is surjective onto $\mathbb{R}$ for any A.

2. Show that $\mathbb{R}^n$ and $\mathcal{S}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \ldots + x_{n+1}^2 = 1\}$ are second countable and Hausdorff with respect to their natural topologies.

   To show that $M = \mathbb{R}^n$ or $\mathcal{S}^n$ is Hausdorff, suppose $x, y \in M$ are distinct. Then $x_i \neq y_i$ for some i. If $x_i < y_i$, pick $c \in (x_i, y_i)$ and set

$$U = \{z \in \mathcal{S}^n : z_i < c\}, \quad V = \{z \in \mathcal{S}^n : z_i > c\}.$$

   Then U, V are disjoint open sets in M with $x \in U$, $y \in V$. If $x_i > y_i$, swap U, V. Thus M is Hausdorff. [All we are doing here, of course, is a special case of showing that metric spaces are Hausdorff.]

   To see that $\mathbb{R}^n$ is second countable, note that

$$\mathcal{B} = \{(a_1, b_1) \times \cdots \times (a_n, b_n) : a_i, b_i \in \mathbb{Q}, \ \ a_i < b_i\}$$

   is a countable basis for its topology. Another option would be

$$\mathcal{B} = \{B_r(x) \, : \, x \in \mathbb{Q}^n, \, r \in \mathbb{Q}^+\}$$

   where $B_r(x)$ denotes the Euclidean ball of radius r and centre x.

   Hence, if $\mathcal{B}$ is a countable basis for $\mathbb{R}^n$, then $\{U \cap \mathcal{S}^n : U \in \mathcal{B}\}$ is a countable basis for the topology of $\mathcal{S}^n$, so $\mathcal{S}^n$ is also second countable. [This just says that subspaces of second countable spaces are second countable.]

3. Let $N = (0,0,1) \in \mathcal{S}^2$ and $S = (0,0,-1) \in \mathcal{S}^2$ and define $U_N = \mathcal{S}^2 \setminus \{N\}$ and $U_S = \mathcal{S}^2 \setminus \{S\}$.

   Let $\varphi_N : U_N \to \mathbb{R}^2$ and $\varphi_S : U_S \to \mathbb{R}^2$ be given by

$$\varphi_N(x_1, x_2, x_3) = \frac{(x_1, x_2)}{1 - x_3} \quad \text{and} \quad \varphi_S(x_1, x_2, x_3) = \frac{(x_1, x_2)}{1 + x_3}.$$

   (a) By constructing explicit inverses, or otherwise, show that φ_N and φ_S are homeomorphisms (i.e. continuous bijections with continuous inverses).

We have explicit inverses:

$$\varphi_N^{-1}(y_1, y_2) = \frac{(2y_1, 2y_2, y_1^2 + y_2^2 - 1)}{1 + y_1^2 + y_2^2}$$

and

$$\varphi_S^{-1}(y_1, y_2) = \frac{(2y_1, 2y_2, 1 - y_1^2 - y_2^2)}{1 + y_1^2 + y_2^2}.$$

Both φ_N, φ_S and their inverses are clearly continuous, so they are homeomorphisms.

Let $f = \varphi_S \circ \varphi_N^{-1}$ defined on $\varphi_N(U_N \cap U_S)$.

(b) Calculate f and show that it defines a diffeomorphism of $\mathbb{R}^2 \setminus \{0\}$ (i.e. it is a smooth map with smooth inverse).

We see that $U_N \cap U_S = \mathcal{S}^2 \setminus \{N, S\}$ and $\varphi_N(U_N \cap U_S) = \mathbb{R}^2 \setminus \{0\} = \varphi_S(U_N \cap U_S)$. We may compute that $f = \varphi_S \circ \varphi_N^{-1} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ is

$$f(y_1, y_2) = \frac{(y_1, y_2)}{y_1^2 + y_2^2}.$$

This is smooth, because we are excluding the origin from $\mathbb{R}^2$, and $f = f^{-1}$, so it is a diffeomorphism.

(c) Calculate the differential $\mathrm{d}f_y$ at any point $y \in \mathbb{R}^2 \setminus \{0\}$. Calculate $\det \mathrm{d}f_y$, viewed as a matrix with respect to the standard basis of $\mathbb{R}^2$, and show that it is never zero.

We may calculate that

$$\mathrm{d}f_y = \frac{1}{(y_1^2 + y_2^2)^2} \begin{pmatrix} y_2^2 - y_1^2 & -2y_1y_2 \\ -2y_1y_2 & y_1^2 - y_2^2 \end{pmatrix}.$$

We see that

$$\det \mathrm{d}f_y = -\frac{1}{(y_1^2 + y_2^2)^2} < 0.$$

4. (a) Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x_1, x_2) = (e^{x_1} \cos x_2, e^{x_1} \sin x_2)$.
Show that f is a local diffeomorphism (i.e. given any point $x \in \mathbb{R}^2$ there is an open set $U \ni x$ and $V \ni f(x)$ so that $f : U \to V$ is a diffemorphism). Is f a diffeomorphism?

We calculate that $\mathrm{d}f_x$ is given by the matrix

$$\mathrm{d}f_x = \begin{pmatrix} e^{x_1} \cos x_2 & -e^{x_1} \sin x_2 \\ e^{x_1} \sin x_2 & e^{x_1} \cos x_2 \end{pmatrix}.$$

We quickly calculate that $\det \mathrm{d}f_x = e^{2x_1} > 0$ so $\mathrm{d}f_x$ is invertible for all $x \in \mathbb{R}^2$. Therefore, by the Implicit Function Theorem, f is a local diffeomorphism.

We see that f is not a diffeomorphism because $f(x_1, x_2 + 2\pi) = f(x_1, x_2)$ for all x_1, x_2, so f is not injective. It is also not surjective because $f(x_1, x_2)$ is never zero as $|f(x_1, x_2)|^2 = e^{2x_1} > 0$.

(b) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x_1, x_2) = x_1^3 + x_2^3 + e^{x_1 + x_2}$.
Show that there is a smooth function $g(x_1)$ so that $f(x_1, x_2) = 0$ if and only if $x_2 = g(x_1)$. Deduce that $f^{-1}(0)$ is a manifold.

We calculate that

$$\frac{\partial f}{\partial x_2} = 3x_2^2 + e^{x_1 + x_2} > 0$$

for all x_1, x_2. So, by the Implicit Function Theorem, there is a smooth function $g(x_1)$ so that $f(x_1, x_2) = 0$ if and only if $x_2 = g(x_1)$.

Therefore $f^{-1}(0) = \{(x, g(x)) : x \in \mathbb{R}\}$. We may therefore take a single chart $U = f^{-1}(0)$ and $\varphi(x, g(x)) = x$. Then $\varphi : f^{-1}(0) \to \mathbb{R}$ is continuous and has inverse $\varphi^{-1}(x) = (x, g(x))$. The transition function condition is trivially satisfied, since the only transition function is $\varphi \circ \varphi^{-1} = \mathrm{id}$ which is obviously a diffeomorphism. Hence $f^{-1}(0)$ is a 1-dimensional manifold.