C3.3 Differentiable Manifolds

Problem Sheet 1

Michaelmas Term 2020–2021

1. Using the regular value theorem, or otherwise, show that the following are manifolds and give their dimensions.

   (a) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3^2 = c\}$ where $c \neq 0$ is constant. What happens if $c = 0$?

   (b) $\{(z_1, z_2) \in \mathbb{C}^2 : z_1^2 = z_2(z_2 - \alpha)(z_2 - \beta)\}$ where $\alpha, \beta \in \mathbb{C}$ such that $\alpha\beta \neq 0$ and $\alpha \neq \beta$.

   (c) $\mathrm{SL}(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) : \det A = 1\}$.

   (d) $\mathrm{U}(n) = \{A \in M_n(\mathbb{C}) : \overline{A^{\mathrm{T}}}A = I\}$.

   (e) $\mathrm{Sp}(2n, \mathbb{R}) = \{A \in M_{2n}(\mathbb{R}) : A^{\mathrm{T}}JA = J\}$ where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and I is the $n \times n$ identity matrix.

2. For $i = 1, \ldots, n+1$ let

$$U_i = \{[x] = [(x_1, \ldots, x_{n+1})] \in \mathbb{RP}^n : x_i \neq 0\}$$

   and $\varphi_i : U_i \to \mathbb{R}^n$ be

$$\varphi_i([x]) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right).$$

   Show that $\{(U_i, \varphi_i) : i = 1, \ldots, n+1\}$ defines an atlas for $\mathbb{RP}^n$.

3. (a) Let M be an m-dimensional manifold and let N be an n-dimensional manifold. Show that $M \times N$ is an $(m+n)$-dimensional manifold.

   (b) Use part (a) to show that $T^n = \{(\cos\theta_1, \sin\theta_1, \ldots, \cos\theta_n, \sin\theta_n) \in \mathbb{R}^{2n} : \theta_1, \ldots, \theta_n \in \mathbb{R}\}$, the standard n-torus in $\mathbb{R}^{2n}$, is an n-dimensional manifold.

4. (a) Let $a \geq 0$ and for $(n_1, n_2) \in \mathbb{Z}^2$ define $f_{(n_1,n_2)} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f_{(n_1,n_2)}(x_1, x_2) = (x_1 + n_1 + n_2 a, x_2 + n_2)$$

   Show that this leads to a free and properly discontinuous action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ by diffeomorphisms, so that the quotient $\mathbb{R}^2/\mathbb{Z}^2$ is a 2-dimensional manifold.

   (b) Show the manifold constructed in (a) is diffeomorphic to $T^2 \subseteq \mathbb{R}^4$.

5. Let $\{(U_N, \varphi_N), (U_S, \varphi_S)\}$ be the atlas for $\mathcal{S}^2$ given in lectures, and let $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ be the atlas for $\mathbb{CP}^1$ given in lectures.

   (a) Find maps $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ so that

$$\varphi_N^{-1} \circ f_1 \circ \varphi_1 : U_1 \to U_N \quad \text{and} \quad \varphi_S^{-1} \circ f_2 \circ \varphi_2 : U_2 \to U_S$$

   are smooth functions with smooth inverses which agree on $U_1 \cap U_2$.

   (b) Deduce that $\mathbb{CP}^1$ is diffeomorphic to $\mathcal{S}^2$.

6. Find the following tangent spaces using the regular value theorem (or otherwise).

(a) $T_x T^n$ for $T^n \subseteq \mathbb{R}^{2n}$ and $x = (1, 0, \ldots, 1, 0) \in T^n$.

(b) $T_x M$ for $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3^2 = -1,\ x_3 > 0\}$ and $x = (0, 0, 1) \in M$.

(c) $T_I\, \mathrm{O}(n)$ where $\mathrm{O}(n) = \{A \in M_n(\mathbb{R}) : A^{\mathrm{T}} A = I\}$.

(d) $T_I\, \mathrm{U}(n)$.

(e) $T_I \mathrm{Sp}(2n, \mathbb{R})$.