C3.3 Differentiable Manifolds

Problem Sheet 2

Michaelmas Term 2020–2021

1. Show that the following smooth maps are immersions. Are they embeddings? Justify your answer in each case.

   (a) $f : \mathcal{S}^1 \to \mathbb{R}^2$ given by
   $$f(\cos\theta, \sin\theta) = (\sin 2\theta \cos\theta, \sin 2\theta \sin\theta)$$

   (b) $f : (-\pi, \pi) \to \mathbb{R}^2$ given by
   $$f(x) = (\sin 2x, \sin x)$$

   (c) $f : \mathcal{S}^2 \to \mathbb{R}^4$ given by
   $$f(x_1, x_2, x_3) = \frac{(x_1, x_1x_3, x_2, x_2x_3)}{1 + x_3^2}$$

   (d) $f : \mathbb{RP}^2 \to \mathbb{R}^5$ given by
   $$f([(x_1, x_2, x_3)]) = \left(\frac{x_2x_3}{\sqrt{3}}, \frac{x_3x_1}{\sqrt{3}}, \frac{x_1x_2}{\sqrt{3}}, \frac{x_1^2 - x_2^2}{2\sqrt{3}}, \frac{1}{6}(x_1^2 + x_2^2 - 2x_3^2)\right)$$
   for $(x_1, x_2, x_3) \in \mathcal{S}^2$.

2. Consider $\mathcal{S}^3 \subseteq \mathbb{R}^4$.

   (a) Show that $\mathcal{S}^3$ is parallelizable.

   Define $f : \mathcal{S}^3 \to \mathcal{S}^2$ by
   $$f(x_0, x_1, x_2, x_3) = \left(x_0^2 + x_1^2 - x_2^2 - x_3^2, 2x_0x_3 + 2x_1x_2, 2x_1x_3 - 2x_0x_2\right).$$

   (b) Show that $f^{-1}\{y\} \subseteq \mathcal{S}^3$ is a circle for all $y \in \mathcal{S}^2$.

   (c) Show that f is a submersion.

3. Define a vector field on the upper-half plane H^2 by
   $$X = \frac{1}{x_2}\partial_1 + x_2\partial_2.$$

   (a) Compute the integral curves and hence the flow of X.

   (b) Use the definition of the Lie derivative to compute $\mathcal{L}_X\partial_1$ and $\mathcal{L}_X\partial_2$.

   (c) Compute $[X, \partial_1]$ and $[X, \partial_2]$ and verify that $\mathcal{L}_X\partial_1 = [X, \partial_1]$ and $\mathcal{L}_X\partial_2 = [X, \partial_2]$.

4. Let V be a vector space of dimension n and let $\alpha \in \Lambda^k V^*$. Consider the linear map $A_\alpha : \Lambda^{n-k}V^* \to \Lambda^n V^*$ defined by $A_\alpha(\beta) = \alpha \wedge \beta$.

   (a) Show that if $\alpha \neq 0$, then $A_\alpha \neq 0$.

   (b) Prove that $\alpha \mapsto A_\alpha$ is an isomorphism from $\Lambda^k V^*$ to the vector space $\mathrm{Hom}(\Lambda^{n-k}V^*, \Lambda^n V^*)$ of linear maps from $\Lambda^{n-k}V^*$ to $\Lambda^n V^*$. So, if we choose an isomorphism $\Lambda^n V^* \cong \mathbb{R}$, then $\Lambda^k V^* \cong (\Lambda^{n-k}V^*)^*$.

5. Let B^2 be the unit ball in $\mathbb{R}^2$ and let H^2 be the upper-half plane.

(a) Define $f : B^2 \to H^2$ by

$$f(y_1, y_2) = \frac{(2y_1, 1 - y_1^2 - y_2^2)}{y_1^2 + (y_2 + 1)^2}.$$

Show that f is a diffeomorphism.

[Hint: What is $f(f(y_1, y_2))$?]

(b) Compute $f_*(\partial_1)$ and $f_*(\partial_2)$.

(c) Compute

$$f^* \left(\frac{\mathrm{d}x_1 \wedge \mathrm{d}x_2}{x_2^2} \right).$$

6. Let (x_0, x_1, x_2, x_3) be coordinates on $\mathbb{R}^4$. Let

$$X = -x_1\partial_0 + x_0\partial_1 - x_3\partial_2 + x_2\partial_3$$

and

$$\omega = -x_2\mathrm{d}x_0 + x_3\mathrm{d}x_1 + x_0\mathrm{d}x_2 - x_1\mathrm{d}x_3.$$

(a) Compute the flow of X and hence $\mathcal{L}_X\omega$ using the definition of Lie derivative.

(b) Compute $\mathrm{d}\omega$ and $\mathrm{d}(i_X\omega)$ and hence compute $\mathcal{L}_X\omega$ using Cartan's formula.