C3.3 Differentiable Manifolds

Problem Sheet 3

Michaelmas Term 2020–2021

1. Let (x_0, x_1, x_2, x_3) be coordinates on $\mathbb{R}^4$ and let
$$\omega = \mathrm{d}x_0 \wedge \mathrm{d}x_1 + \mathrm{d}x_2 \wedge \mathrm{d}x_3.$$
Let $u : \mathbb{R}^4 \to \mathbb{R}$ be smooth and consider the vector field
$$X = -\partial_1(u)\partial_0 + \partial_0(u)\partial_1 - \partial_3(u)\partial_2 + \partial_2(u)\partial_3.$$
Show that if ϕ_t^X defines the flow of X and ϕ_1^X is defined, then $(\phi_1^X)^*\omega = \omega$.
[Hint: Consider $\frac{\mathrm{d}}{\mathrm{d}t}(\phi_t^X)^*\omega$.]

2. A Riemann surface is a 2-dimensional manifold with an atlas $\{(U_i, \varphi_i) : i \in I\}$ whose transition maps $\varphi_j \circ \varphi_i^{-1}$ for $i, j \in I$ are maps from an open set $\varphi_i(U_i \cap U_j)$ of $\mathbb{C} = \mathbb{R}^2$ to another open set $\varphi_j(U_i \cap U_j)$ which are holomorphic and invertible. Show that a Riemann surface is orientable.

3. Let M_1 and M_2 be manifolds. Show that $M_1 \times M_2$ is orientable if and only if M_1 and M_2 are both orientable.

4. Let M be a manifold and let G act freely and properly discontinuously by diffeomorphisms f_g for $g \in G$ on M. Let $\pi : M \to M/G$ be the projection map.
   (a) Suppose that M/G is orientable, so that there is a volume form Ω on M/G. Show that $\Upsilon = \pi^*\Omega$ is a volume form on M such that $f_g^*\Upsilon = \Upsilon$ for all $g \in G$.
   (b) Suppose that Υ is a volume form on M such that $f_g^*\Upsilon = \Upsilon$ for all $g \in G$. Show that there is a volume form Ω on M/G such that $\pi^*\Omega = \Upsilon$, and hence that M/G is orientable.
   (c) Suppose that $M = \mathcal{S}^1 \times \mathcal{S}^1 \subseteq \mathbb{R}^4$ is the standard 2-torus and $G = \mathbb{Z}_2 = \{-1, +1\}$ acting on $\mathbb{R}^4$ by diffeomorphisms $f_1 = \mathrm{id}$ and $f_{-1} = -\mathrm{id}$. Is M/G diffeomorphic to the Klein bottle?

5. Define $f : (0,1) \times (0, 2\pi) \to B^2$, where B^2 is the unit ball centred at 0 in $\mathbb{R}^2$, by
$$f(r, \theta) = (r\cos\theta, r\sin\theta)$$
and let (y_1, y_2) be coordinates on B^2. Let $B_s \subseteq B^2$ denote the open ball centred at 0 of radius s, for $s \in (0,1)$, with its standard orientation. Let $k \in \{1, -1\}$.
   (a) Compute
$$f^*\left(4(1 - y_1^2 - y_2^2)^{2k}\mathrm{d}y_1 \wedge \mathrm{d}y_2\right).$$
   (b) Hence, or otherwise, calculate
$$\int_{B_s} 4(1 - y_1^2 - y_2^2)^{2k}\mathrm{d}y_1 \wedge \mathrm{d}y_2$$
in each of the cases $k = 1$ and $k = -1$. What happens as $s \to 1$ in each case?

6. Use Stokes Theorem for manifolds with boundary to prove the following results.

(a) Let $\gamma : \mathcal{S}^1 \to \mathbb{R}^2$ be an embedding and let D be the region in $\mathbb{R}^2$ bounded by $C = \gamma(\mathcal{S}^1)$. Let $u_1, u_2 : \mathbb{R}^2 \to \mathbb{R}$ be smooth functions. Then

$$\int_C u_1 \mathrm{d}x_1 + u_2 \mathrm{d}x_2 = \int_D \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \mathrm{d}x_1 \mathrm{d}x_2.$$

(b) Let V be an open subset of $\mathbb{R}^3$ with compact closure and smooth boundary $S = \partial V$. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be smooth. Then

$$\int_V \mathrm{div} F \, \mathrm{d}V = \int_S F \cdot \mathrm{d}S.$$

(c) Let Σ be a compact oriented surface in $\mathbb{R}^3$ with smooth boundary $\Gamma = \partial \Sigma$. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be smooth. Then

$$\int_\Sigma \mathrm{curl} F \cdot \mathrm{d}\Sigma = \int_\Gamma F \cdot \mathrm{d}\Gamma.$$