C3.3 Differentiable Manifolds

Problem Sheet 4

Michaelmas Term 2020–2021

1. Let L be a compact, oriented k-dimensional manifold, let N be an n-dimensional manifold with $n \geq k$ and let M be a compact, oriented $(k+1)$-dimensional manifold with boundary $\partial M = L$.

   (a) Let $f : L \to N$ be a smooth map. Show that, by integrating $f^*\alpha$ on L where $\alpha \in \mathcal{Z}^k(N)$, that f defines a linear map $L_f : H^k(N) \to \mathbb{R}$.

   (b) Let $g : M \to N$ be a smooth map such that $g|_L = f$. Show using Stokes Theorem that $L_f = 0$.

2. Let ξ be the restriction to $\mathcal{S}^1$ of the 1-form

$$x_1 \mathrm{d}x_2 - x_2 \mathrm{d}x_1$$

   on $\mathbb{R}^2$. Writing $T^n = \mathcal{S}^1 \times \cdots \times \mathcal{S}^1$, let $\pi_i : T^n \to \mathcal{S}^1$ be the projection onto the i^{th} factor.

   (a) Show that the de Rham cohomology classes $\pi_i^*[\xi]$ for $i = 1, \ldots, n$ are linearly independent in $H^1(T^n)$.

   (b) Let $n > 1$ and let $f : \mathcal{S}^n \to T^n$ be a smooth map. Show that the degree of f is zero.

3. The *quaternions* consist of the four-dimensional associative algebra $\mathbb{H}$ of expressions $q = x_0 + ix_1 + jx_2 + kx_3$ where $x_i \in \mathbb{R}$ and i, j, k satisfy the relations

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

   (a) Show that $f(q) = q^2$ defines a smooth map from $\mathbb{R}^4 \cup \{\infty\} \cong \mathcal{S}^4$ to itself.

   (b) How many solutions are there to the equation $q^2 = 1$?

   (c) What is the degree of f?

   (d) How many solutions are there to the equation $q^2 = -1$?

4. Let

$$X = a_1 \partial_1 + a_2 \partial_2$$

   be a vector field on $\mathbb{R}^2$ where $a_1, a_2 : \mathbb{R}^2 \to \mathbb{R}$ are smooth such that X is a Killing field on $\mathbb{R}^2$ with the Euclidean metric $\mathrm{d}x_1^2 + \mathrm{d}x_2^2$.

   (a) Solve the Killing equation

$$\mathcal{L}_X(\mathrm{d}x_1^2 + \mathrm{d}x_2^2) = 0$$

   for a_1 and a_2.

   (b) Show that the flow of X is

$$\phi_t^X(\mathbf{x}) = A_t \mathbf{x} + \mathbf{c}_t$$

   where A_t is a rotation and $\mathbf{c}_t$ is a constant vector in $\mathbb{R}^2$.

5. Let B^2 be the unit ball in $\mathbb{R}^2$ and let

$$g = 4\frac{\mathrm{d}y_1^2 + \mathrm{d}y_2^2}{(1-(y_1^2+y_2^2))^2}$$

(a) Let $L \in (0,1)$ and let $\alpha : [0,L] \to B^2$ be the curve $\alpha(t) = (t,0)$. Calculate the length $L(\alpha)$ of the curve α and show that $L(\alpha) \to \infty$ as $L \to 1$.

(b) Show that $\alpha(t) = (\tanh \frac{t}{2}, 0)$ is a geodesic through $(0,0)$ which is parametrized by arclength.

(c) Let H^2 be the upper half-plane in $\mathbb{R}^2$ with the Riemannian metric

$$h = \frac{\mathrm{d}x_1^2 + \mathrm{d}x_2^2}{x_2^2}.$$

Let $f : B^2 \to H^2$ be given by

$$f(y_1, y_2) = \frac{(2y_1, 1 - y_1^2 - y_2^2)}{y_1^2 + (y_2+1)^2}$$

as in Problem Sheet 2. Show that $f : (B^2, g) \to (H^2, h)$ is an isometry.

(d) Let B^2 and H^2 have their standard orientations. Find the Riemannian volume forms Ω on (B^2, g) and Υ on (H^2, h) compatible with the standard orientations, and compute the Hodge duals of $\mathrm{d}y_1$ in (B^2, g) and $\mathrm{d}x_1$ in (H^2, h). Hence, using the results of Problem Sheet 2 and (c), or otherwise, show that f is orientation reversing (i.e. $f^*\Upsilon = -\lambda\Omega$ for some $\lambda : B^2 \to \mathbb{R}^+$).

6. Consider $(\mathcal{S}^{2n+1}, g)$ where g is the standard round metric and let E be the vector field on $\mathcal{S}^{2n+1}$ given by

$$E = \sum_{j=1}^{n+1} x_{2j-1}\partial_{2j} - x_{2j}\partial_{2j-1}.$$

Let $\pi : \mathcal{S}^{2n+1} \to \mathbb{CP}^n$ be the projection map.

(a) Show that $\pi_*(E) = 0$ and that E is a Killing field on $(\mathcal{S}^{2n+1}, g)$.

(b) For $z \in \mathcal{S}^{2n+1}$ let

$$H_z = \{X \in T_z\mathcal{S}^{2n+1} \,:\, g(X, E(z)) = 0\}.$$

Show that $\Phi_z = \mathrm{d}\pi_z : H_z \to T_{\pi(z)}\mathbb{CP}^n$ is an isomorphism.

[*You may assume that π is a submersion.*]

(c) Define h on $\mathbb{CP}^n$ by

$$h_{\pi(z)}(X, Y) = g_z(\Phi_z^{-1}(X), \Phi_z^{-1}(Y)).$$

Show that h is a well-defined Riemannian metric on $\mathbb{CP}^n$.