C3.1 Algebraic Topology

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Please be aware there are likely typos in these notes: comments/corrections are welcome!

Course Book

- Hatcher, Algebraic Topology — Chp. 2 & 3

  This is also freely available from the author's website.

Expectations

- You are expected to read chapters 2 & 3 of Hatcher
- You should read the technical remarks about orientation signs in these notes: we will likely not have time for those in lectures.
- This course will not discuss intersection numbers rigorously. The notes often mention these in order to develop your intuition. The books by Bott & Tu and Guillemin & Pollack discuss these ideas rigorously.

Other references

- Ulrike Tillmann's C3.1 notes — see course page
- The discussion of orientations in my B3.2 Geometry of Surfaces notes may be helpful.

Other books

Massey, A basic course in Algebraic Topology

(Not "Algebraic Topology: An Introduction" ← does not treat homology)

MORE BASIC but full of ideas:

Fulton, Algebraic Topology: a first course.

MORE ADVANCED:

May, A concise course in Algebraic Topology

Davis & Kirk, Lecture Notes in Algebraic Topology

Bredon, Topology and Geometry

Classics by **Spanier**, **Dold**, also see references in May's book

Bott & Tu, Differential forms in Algebraic Topology

Guillemin & Pollack, Differential Topology

CONTENTS

0. OVERVIEW OF THE COURSE

Motivation, category theory, functors H_* and H^*: some computations

why functors are useful: Invariance of dimension, Brower fixed pt thm

1. ALGEBRAIC PRELIMINARIES: CHAIN COMPLEXES

Graded abelian groups

chain complexes, chain maps, subcomplex, quotient complex

chain map induces map on homology

exact sequence, snake lemma: SES induces LES on H_*, naturality of LES

5-Lemma, SES splits $\Leftrightarrow$ direct sum

2. Δ-COMPLEXES AND SIMPLICIAL HOMOLOGY

Δ^n, n-simplices, Δ-complex (structure), simplicial cx, triangulation

simplicial chain complex, $H_*^\Delta(S^n)$, $H_*^\Delta(T^2)$, remark about orientations

$H_*^\Delta(\sqcup \text{ conn.comp.}) \cong \bigoplus H_*^\Delta(\text{conn. comp.})$, $H_0^\Delta(X) \cong \mathbb{Z}^{\#\text{conn.comp.}}$

3. SINGULAR HOMOLOGY

Motivation, singular chain cx

naturality / functoriality, $H_*(\text{point})$

4. CHAIN HOMOTOPIES AND HOMOTOPY INVARIANCE

chain homotopy, prism operator

homotopic maps $f \simeq g$ (relative A), homotopy equivalent spaces $X \simeq Y$

contractible space, homotopic maps induce equal maps on homology

homotopy equivalence induces iso on H_*, $H_*(\mathbb{R}^n) = H_*(\mathbb{D}^n) = H_*(pt)$

pairs of spaces, relative homology $H_*(X,A)$, LES in H_* for pair

reduced homology $\widetilde{H}_*(X)$, LES for $\widetilde{H}_*$, $H_*(\mathbb{D}^n, S^{n-1}) \cong \widetilde{H}_{*-1}(S^{n-1})$

naturality of LES for pairs

5. EXCISION THEOREM AND QUOTIENTS

retractions, deformation retractions, excision thm, quotients,

good pairs $\Rightarrow H^*(X,A) \cong \widetilde{H}_*(X/A)$, generator of $H_n(S^n) \cong H_n(\mathbb{D}^n, S^{n-1}) \cong \mathbb{Z}$

another remark about orientations

6. MAYER-VIETORIS SEQUENCE

MV LES, $H_*(S^n)$

wedge sum $X \vee Y$, cone CX, suspension ΣX, connected sum $X \# Y$

7. DEGREE OF MAPS OF SPHERES

degree, application to tangent vector fields on sphere, hairy ball theorem

local degree, proof of fundamental thm of algebra

8. CELLULAR HOMOLOGY

CW complexes, cellular complex, rank $H_n^{CW} \leqslant$ # n-cells

$H_*^{CW}(\mathbb{D}^1 \times \mathbb{D}^1)$, $H_*^{CW}(\mathbb{RP}^n)$, $H_*^{CW}(S^n)$, $H_*^{CW}(\Sigma_g)$

Δ-cx $\Rightarrow$ CW cx, $H_*^{CW}(X) \cong H_*^{\Delta}(X) \cong H_*(X)$, Axioms for homology

9. COHOMOLOGY

cochains, cohomology, $H^*(X)$, $H^*_{CW}(X)$, $H^*_{\Delta}(X)$, $H^*(\mathbb{RP}^3)$

functoriality, homotopy invariance, cochain homotopy, dual of a SES

excision, LES, Mayer-Vietoris for H^*, axioms for cohomology

10. CUP PRODUCT

Cup product, $H^*(X)$ unital graded-commutative ring, pull-back is ring hom,

examples: $H^*(T^2)$, $H^*(\Sigma_2)$, remarks about intersection theory

11. KÜNNETH FORMULA AND PRODUCT SPACES

Tensor products of R-mods, tensor product of chain cxes,

algebraic Künneth thm, product spaces $X \times Y$, Euler characteristic χ

CW-cx for product space, Künneth thm, $H^*(S^n \times S^m)$, $H^*(T^n)$

12. UNIVERSAL COEFFICIENTS THEOREM

Universal coeff. thm, Background on Ext groups and free resolutions

(Co)homology with coefficients in a ring/field/module, $H^*(\mathbb{RP}^2; \mathbb{Z}/2)$

Univ. coeff. thm for PID R, Duality $H^*(X;\mathbb{F}) \simeq H_*(X;\mathbb{F})$ over fields

Structure thm for f.g. mods M over PID R, $\mathrm{Ext}^1_R(M;R)$, torsion shift H_* to H^{*+1}

13. MANIFOLDS: POINCARÉ-LEFSCHETZ DUALITY

Local orientation, orientation, fundamental class, degree, P. duality, L. duality,

Locally finite homology H_*^{lf}, cohomology with compact supports H^*_c, Cap product and P.D.,

Alexander duality, knot complements, Jordan curve thm

0. OVERVIEW OF THE COURSE

Motivation

Space X $\xrightarrow{\text{associate}}$ Algebraic object $A(X)$
like numbers, groups, rings, ...

Isomorphism of spaces $X \cong Y$ $\Longrightarrow$ Isomorphism $A(X) \cong A(Y)$

PAY-OFF Can tell spaces apart:
compute $A(X), A(Y) \rightsquigarrow$ if $A(X) \ncong A(Y)$ then $X \ncong Y$

Examples

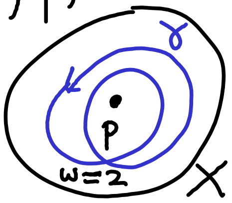
1) Set X $\longrightarrow$ $A(X) = \# X \in \mathbb{N} \cup \{\infty\}$
(bijection $X \to Y$) $\Longrightarrow$ same size

2) Vector space X $\longrightarrow$ $A(X) = \dim X \in \mathbb{N} \cup \{\infty\}$
(linear iso $X \to Y$) $\Longrightarrow$ same dim

3) Topological Space X
- $\longrightarrow$ $\# \pi_0(X) =$ **# path components** $\in \mathbb{N} \cup \{\infty\}$
- $\longrightarrow$ **#Connected components** $\in \mathbb{N} \cup \{\infty\}$
- $\longrightarrow$ $\chi(X) =$ **Euler characteristic** $\in \mathbb{Z}$
- $\longrightarrow$ (for $X \subseteq \mathbb{R}^2$) Function $X \times \overbrace{\mathcal{L}X}^{\text{loops} = C^0(S^1, X)} \longrightarrow \mathbb{Z} \cup \{\infty\}$

$$(p, \gamma) \longmapsto w(\gamma; p)$$

Winding number of γ around p.



(Homeomorphism $X \to Y$) $\longrightarrow$ $A(X) = A(Y)$

CONVENTION for this course:

Space = topological space

Map of spaces = continuous map = cts map

For spaces, "$\cong$" means homeomorphism

"id" = identity map

All diagrams commute unless we say otherwise, e.g.

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\delta"'] & D \end{tikzcd}$$

means $\beta \circ \alpha = \delta \circ \gamma$

Category theory is the best language to phrase all this

Algebraic topology builds functors such as

$$\begin{tikzcd} & \left\{\substack{\text{Groups}\\ \text{and group homs}}\right\} \\ F : \left\{\substack{\text{Topological spaces}\\ \text{and continuous maps}}\right\} \arrow[ur] \arrow[r] \arrow[dr] & \left\{\substack{\text{Abelian groups}\\ \text{and group homs}}\right\} \\ & \left\{\text{Rings and ring homs}\right\} \end{tikzcd}$$

We will not use much category theory, just basic terminology:

Def A **category** C consists of the data:

$Ob(C)$ = a collection of **objects**

$Hom(A,B)$ = a set of **morphisms** between any $A,B \in Ob C$ ("**arrows**")

- with **composition** rule $Hom(B,C) \times Hom(A,B) \xrightarrow{\circ} Hom(A,C)$ which is associative.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

- with **identity** morphs $id_A \in Hom(A,A)$ s.t. $f \circ id_A = id_B \circ f = f$ $\forall (f: A \to B) \in Hom(A,B)$

Example
Sets = { sets with all maps between sets }
Top = { topological spaces with continuous maps }
Gps = { groups with group homs }

Def A (covariant) **functor** $F: C_1 \longrightarrow C_2$ is the data:

- an assignment $(A \in Ob\, C_1) \longmapsto (F(A) \in Ob\, C_2)$
- an assignment $\underset{\substack{\cap \\ Hom_{C_1}(A,B)}}{(A \xrightarrow{f} B)} \longmapsto \underset{\substack{\cap \\ Hom_{C_2}(F(A),F(B))}}{(F(A) \xrightarrow{F(f)} F(B))}$

Compatible with identities and compositions.

$$F(id_A) = id_{F(A)} \qquad F(g\circ f) = F(g) \circ F(f)$$

A **contravariant functor** is defined similarly except it reverses the direction of arrows: $(F(A) \xleftarrow{F(f)} F(B)) \in Hom(F(\underline{B}), F(\underline{\underline{A}}))$
(so $F(g\circ f) = F(f) \circ F(g)$ reverses order of compositions)

Examples

1) $F: \text{Top} \longrightarrow \text{Sets}$, $A \longmapsto A$, $f \longmapsto f$ "forget the topology and continuity"

2) $F: \text{Sets} \longrightarrow \text{Gps}$, $A \longmapsto$ free abelian group generated by A

$$\mathbb{Z}\langle A\rangle = \left\{ \sum_{\text{finite}} n_i \cdot a_i \ : \ \begin{array}{l} a_i \in A \\ n_i \in \mathbb{Z} \end{array} \right\}$$

$$(A \xrightarrow{f} B) \longmapsto \left(\begin{array}{c} F(A) : \mathbb{Z}\langle A\rangle \longrightarrow \mathbb{Z}\langle B\rangle \\ \sum n_i \cdot a_i \longmapsto \sum n_i \cdot f(a_i) \end{array} \right)$$

When we say a construction is **natural** we mean functorial:

$$\begin{tikzcd}
X \arrow[r, "A"] \arrow[d, "f"'] \arrow[dd, bend right=60, "g\circ f"'] & A(X) \arrow[d, "A(f)"] \arrow[dd, bend left=60, "{A(g\circ f) = A(g)\circ A(f)}"] \\
Y \arrow[r, "A"] \arrow[d, "g"'] & A(Y) \arrow[d, "A(g)"] \\
Z \arrow[r, "A"] & A(Z)
\end{tikzcd}$$

A : (a category of spaces) $\longrightarrow$ (a cat. of algebraic objects)

The algebraic objects we assigned are assigned compatibly with maps of spaces, and the compatibility maps $A(f)$ are also compatible w.r.t. composition.

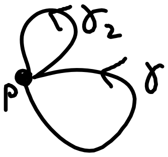
So we made compatible choices in constructing A.

Not to be confused with natural transformations of functors (later) which is about relating two such constructions A_1, A_2 in a compatible way

Example of a functor in algebraic topology (see B3.5 Topology and Groups course)

$$\underset{\substack{\uparrow \\ \text{topological space}}}{\pi_1(X}, \underset{p\in X}{p)} = \text{Fundamental group} = \left\{ \begin{array}{l} \gamma \in \mathcal{L}X \\ \gamma(1) = p \end{array} \right\} \Big/ \begin{array}{l}\text{continuous deformations} \\ \text{of loops based at } p\end{array}$$

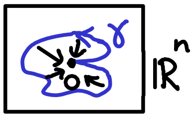
Group multiplication: concatenate loops $\gamma_1 * \gamma_2$



(each travelling twice as fast)

Examples

$\pi_1(\mathbb{R}^n) = 0 \longleftarrow$ (for basepoint $= 0 \in \mathbb{R}^n$: deform: $h: S^1 \times [0,1] \longrightarrow \mathbb{R}^n$, $h(t,s) = (1-s)\cdot\gamma(t)$

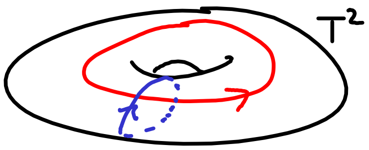


$\pi_1(S^1) \cong \mathbb{Z} \longleftarrow$ total # times wind around circle

$\pi_1(S^n) \cong 0$ $n \geq 2$ (not obvious)

$\nwarrow$ n-sphere

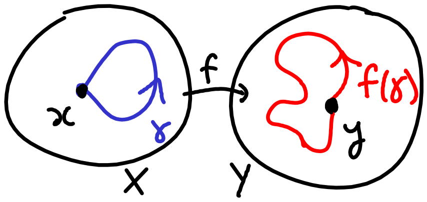
$\pi_1(\text{torus}) \cong \mathbb{Z}^2 \longleftarrow$



those loops generate π_1

FUNCTOR

$$\text{Based Top} = \left\{ \begin{array}{l} \text{Topological spaces with choice of basepoint,} \\ \text{and continuous basepoint-preserving maps} \end{array} \right\} \xrightarrow{\pi_1} \text{Gps}$$



$$(X, p) \longmapsto \pi_1(X, p)$$

$$\left((X, x) \xrightarrow[f]{\text{cts}} (Y, y)\right) \longmapsto \left(\begin{array}{c} \pi_1(X, x) \xrightarrow{\text{gp. hom.}} \pi_1(Y, y) \\ \gamma \longmapsto f \circ \gamma \end{array} \right)$$

Lemma Functors map isomorphisms to isomorphisms (iso. means $\exists$ inverse w.r.t. composition)

Pf

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "id"'] & B \arrow[r, "g"] & A \end{tikzcd} \Rightarrow \begin{tikzcd} FA \arrow[r, "Ff"] \arrow[rr, bend right, "F(id)=id"'] & FB \arrow[r, "Fg"] & FA \end{tikzcd}$$

, similarly for

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[rr, bend right, "id"'] & A \arrow[r, "f"] & B \end{tikzcd} . \ \square$$

Def **Natural transformation** $\alpha : F \longrightarrow G$ between functors $\begin{tikzcd} C_1 \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & C_2 \end{tikzcd}$

is an association $(A \in \mathrm{Ob}\, C_1) \longmapsto (\alpha_A : F(A) \longrightarrow G(A))$

$\in \mathrm{Hom}_{C_2}(F(A), G(A))$

such that $(A \xrightarrow{f} B) \Rightarrow$

$\rotatebox{90}{$\in$}$

$\mathrm{Hom}_{C_1}(A,B)$

$$\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\alpha_B"] & G(B) \end{tikzcd}$$

(commutes)

It is called a **natural isomomorphism** if each α_A is an isomorphism in C_2

Example of a natural transformation in algebraic topology

Let $H_1(X,p)$ = abelianisation of $\pi_1(X,p)$ (want to identify $ab=ba$ so quotient by $\langle aba^{-1}b^{-1} \rangle$) ← **commutators**

$\Rightarrow$ natural trans. $(\text{Based Top} \xrightarrow{\pi_1} \text{Gps}) \xrightarrow{\alpha} (\text{Based Top} \xrightarrow{H_1} \text{Gps})$

which associates $\underset{\in \text{BasedTop}}{(X,p)} \longmapsto (\alpha_{(X,p)} : \pi_1(X,p) \xrightarrow{\text{quotient}} H_1(X,p))$

Cultural Rmk **higher homotopy groups** $\pi_n(X,p) = \left\{ \begin{array}{l} S^n \xrightarrow{\text{cts}} X \\ \text{basept} \mapsto p \end{array} \right\} \Big/ \text{cts deform}^n$

FACT abelian for $n \geqslant 2$, but hard: e.g. $\pi_k(S^n)$ not all known.

We will not study these in this course.

We will study simpler invariants called HOMOLOGY groups $H_n(X)$

FACT (Hurewicz) $\exists$ natural transformation $\pi_n \longrightarrow H_n$

which will make sense at the end of course:

$f : S^n \xrightarrow{\text{cts}} X$ gives rise to a class $f_*[S^n] \in H_n(X)$.

Exercise to practice these notions from category theory:

Summarise your undergraduate linear algebra as follows:

1) $\exists$ functor $F : \underbrace{\left\{ \begin{array}{l} \mathbb{R}^n \text{ for } n \in \mathbb{N} \\ \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) = \left\{ \begin{array}{c} m \times n \\ \text{matrices} \end{array} \right\} \end{array} \right\}}_{\text{Mat}} \to \underbrace{\left\{ \begin{array}{c} \text{finite dimensional real vector spaces} \\ \text{and linear maps} \end{array} \right\}}_{\text{Vect}}$

2) A choice of basis for each vector space V determines a functor $G : \text{Vect} \to \text{Mat}$

3) Construct natural isomorphisms $G \circ F \xrightarrow{\alpha} Id_{\text{Mat}}$, $F \circ G \xrightarrow{\beta} Id_{\text{Vect}}$

When functors satisfying such natural isos exist, the categories are called **equivalent** (not isomorphic). So Mat, Vect are equivalent categories.

Aim of the course: build a functor

HOMOLOGY $H_*: \mathbf{Top} \longrightarrow \textbf{Graded abelian groups}$

$$X \longmapsto H_*(X) \quad \longleftarrow \text{grading } * \in \mathbb{Z}$$

$$(X \to Y) \longmapsto \left(\begin{array}{c} H_*(X) \longrightarrow H_*(Y) \\ \text{grading preserving hom} \end{array} \right)$$

and a contravariant functor

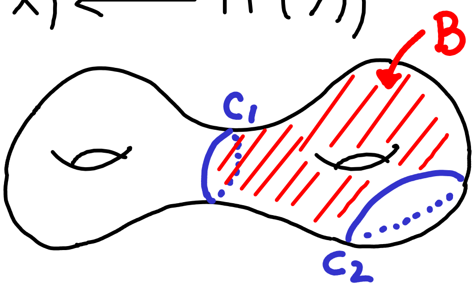
COHOMOLOGY $H^*: \mathbf{Top} \longrightarrow \textbf{Graded rings}$

$$X \longmapsto H^*(X)$$

$$(X \to Y) \longmapsto (H^*(X) \longleftarrow H^*(Y))$$

Rough idea:

$H_* X$ is generated by "nice" subspaces $C \subseteq X$ which have no boundary: $\partial C = \emptyset$, modulo identify C_1, C_2 if $C_1 \cup C_2$ arises as a boundary ∂B. Call such C_1, C_2 **homologous**.



Facts

- $H_0(X) \cong \bigoplus_{\pi_0 X} \mathbb{Z}$ ← generated by a point in each path-comp. ← $\pi_0 X = \{\text{path-connected components}\}$
- $X = \bigsqcup X_i$ path-components $\Longrightarrow H_*(X) \cong \bigoplus H_*(X_i)$
- $\chi(X) = \sum_{d \geq 0} (-1)^d \operatorname{rank} H_d(X)$

↑ Euler characteristic (for $\chi(X)$)

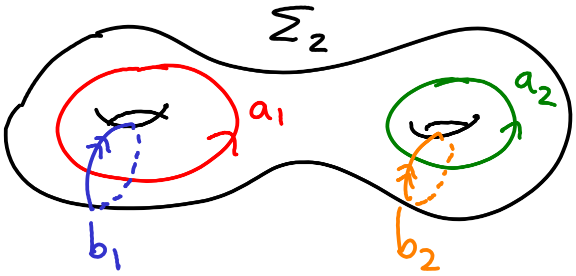
↑ max # $\mathbb{Z}$-linearly independent elements (for rank)

Example: compact surfaces

$$H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}^{2g} & * = 1 \\ \mathbb{Z} & * = 2 \\ 0 & \text{else} \end{cases}$$

↑ orientable surface genus g

$\chi = 2 - 2g$

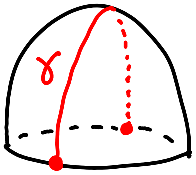


We will show that those 4 loops generate $H_1(\Sigma_2)$

$$H_*(N_h) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}^{h-1} & * = 1 \\ 0 & \text{else} \end{cases}$$

↑ non-orientable surface S^2 with h Möbius bands attached

$\chi = 2 - h$



$N_1 = \mathbb{RP}^2 = S^2/\pm \mathrm{Id}$

Notice γ is a loop. It generates $H_1(N_1)$

Examples of homology calculations

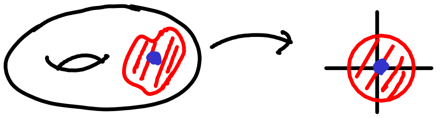
n-dimensional ball $\mathbb{D}^n = \{x : \|x\| \leq 1\} \subseteq \mathbb{R}^n$

$$H_*(\mathbb{R}^n) \cong H_*(\mathbb{D}^n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$$

$$H_*(S^n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z} & * = n \\ 0 & \text{else} \end{cases}$$

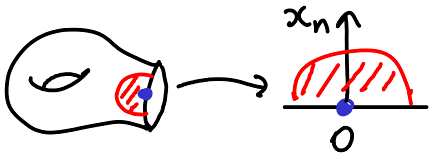
S^n: $\{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ n-dim sphere

Hausdorff top. space s.t. each pt has an open neighbourhood homeo to an open ball in $\mathbb{R}^n$



$$H_*(X) = \begin{cases} 0 & \text{for } * < 0 \\ 0 & \text{for } * > n \text{ for } \textbf{n-dimensional manifolds } X \\ \mathbb{Z} & \text{for } * = n \text{ for connected orientable compact manifold} \\ 0 & \text{for } * = n \text{ for } \begin{cases} \text{non-orientable} & '' \\ \text{non-compact} & '' \\ \text{connected } \textbf{manifolds with boundary} \neq \emptyset \end{cases} \end{cases}$$

boundary point has an open nbhd homeo to open nbhd of $0 \in$ half-space: $\{x \in \mathbb{R}^n : x_n \geq 0\}$

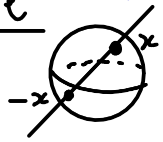


Rmk M compact connected n-mfd

$$\Rightarrow H_{n-1}(M) \cong \begin{cases} \mathbb{Z}^k & \text{some } k \geq 0 \text{ if orientable} \\ \mathbb{Z}^k \oplus \mathbb{Z}/2 & '' \quad '' \text{ non-orientable} \end{cases}$$

$$H_*(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & \text{odd } * = 1, 3, 5, \cdots < n \\ \mathbb{Z} & * = n \text{ if } n \text{ odd} \\ 0 & \text{else} \end{cases}$$

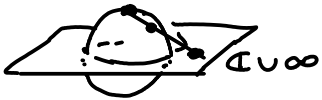
$\mathbb{RP}^n$: $S^n / \pm id$ real projective space



$\mathbb{RP}^n$ orientable $\Leftrightarrow$ n odd (e.g. $\mathbb{RP}^1 \cong S^1$)

$$H_*(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & \text{even } * = 0, 2, 4, \ldots, 2n \\ 0 & \text{else} \end{cases}$$

e.g. $\mathbb{CP}^1 \cong S^2$ stereographic projection



$\mathbb{CP}^n$: space of complex lines through $0 \in \mathbb{C}^{n+1}$

complex projective space

$$\left(\cong (\mathbb{C}^{n+1} \setminus 0) / \mathbb{C}^*\text{-rescaling} \atop = \{[z_0, z_1, \ldots, z_n] : z_j \in \mathbb{C} \text{ not all } 0\} / [z] = [\lambda z] \text{ for } \lambda \in \mathbb{C}^* \right)$$

Examples of cohomology calculations

$$H^0(X) = \prod_{\pi_0 X} \mathbb{Z} \quad \leftarrow \text{ if } \pi_0 X \text{ finite, then } \cong \bigoplus_{\pi_0 X} \mathbb{Z} \cong H_0 X$$

but if infinite then not: here $\bigoplus$ allow only finite sums

$$H^*(X) \cong \prod H^*(X_i) \quad \leftarrow X_i \text{ path-components of } X$$

FACT

If $H_n(X)$ finitely generated abelian gp, so

$$H_n(X) \cong \mathbb{Z}^{r_n} \oplus T_n \quad \leftarrow T_n = \text{torsion elements} = \text{elements of finite order}$$

Then $H^n(X) \cong \mathbb{Z}^{r_n} \oplus T_{\underline{\underline{n-1}}}$ as abelian groups

$\Longrightarrow$ $H^*(\Sigma_g)$, $H^*(\mathbb{R}^n)$, $H^*(\mathbb{D}^n)$, $H^*(S^n)$, $H^*(\mathbb{C}P^n)$ same as for H_*, but :

$$H^*(N_h) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}^{h-1} & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$$

$$H^*(\mathbb{R}P^2) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases} \quad (h=1)$$

$$H_*(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & * \text{ even} = 2, 4, \ldots \leq n \\ \mathbb{Z} & * = n \text{ if } n \text{ odd} \\ 0 & \text{else} \end{cases}$$

and $H^n(\text{non-orientable compact } n\text{-mfd}) \cong \mathbb{Z}/2$.

$\Longrightarrow$ The interesting feature is the <u>ring structure</u> :

$$H^*(\mathbb{C}P^n) \cong \mathbb{Z}[x]/x^{n+1}$$

grading : $|x| = 2$

$\mathbb{Z}[x]$ = polynomials in x with $\mathbb{Z}$-coefficients

$$H^*(S^n) \cong \mathbb{Z}[x]/x^2 \qquad |x| = n$$

$$H^*(T^n) \cong \Lambda[x_1, \ldots, x_n] \qquad |x_i| = 1$$

$T^n = S^1 \times \cdots \times S^1$ **n-torus**

e.g. $n=2$ (torus)
$$\begin{cases} \mathbb{Z} \cdot 1 & * = 0 \\ \mathbb{Z} x_1 \oplus \mathbb{Z} x_2 & * = 1 \\ \mathbb{Z}\, x_1 \wedge x_2 & * = 2 \\ 0 & \text{else} \end{cases}$$

exterior algebra generated$/\mathbb{Z}$ by symbols $x_{i_1} \wedge \ldots \wedge x_{i_k}$ with $i_1 < \ldots < i_k$
product given by $\wedge$ using relations $x_i \wedge x_j = -x_j \wedge x_i$
and $x_i \wedge x_i = 0$. (and $\wedge$ is bilinear)

$$H^*(\mathbb{R}P^{2n}) \cong \mathbb{Z}[x]/(2x, x^{n+1})$$

where $|x| = 2$

$$H^*(\mathbb{R}P^{2n+1}) \cong \mathbb{Z}[x,y]/(2x, x^{n+1}, y^2, xy) \qquad |y| = 2n+1$$

$$H^*(\Sigma_g) \cong \Lambda[a_1, \ldots, a_g, b_1, \ldots, b_g] / \langle a_i b_j \text{ for } i \neq j,\ a_i b_i - a_j b_j,\ a_i a_j,\ b_i b_j \rangle$$

$|a_i| = |b_i| = 1$ $\leftarrow$ exterior alg. instead of poly. alg since $a_i b_i = -b_i a_i$

Why more information?

connected sum: remove a ball in each, glue along ∂ball

$S^2 \times S^2$ and $\mathbb{C}P^2 \# \mathbb{C}P^2$ have same $H_* = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & * = 2 \\ \mathbb{Z} & * = 4 \end{cases}$

but the <u>rings</u> H^* are not iso, hence $S^2 \times S^2 \not\cong \mathbb{C}P^2 \# \mathbb{C}P^2$.

Example of why such functors are useful

Suppose $\exists$ F_* : Top $\longrightarrow$ Gps functors s.t.

① $F_*(S^n) \neq 0 \Leftrightarrow * = n$ and ② $F_*(\mathbb{D}^n) = 0$ all $*$

Rmk We'll build such an F_* : reduced homology $\tilde{H}_*$

s.t. $\tilde{H}_* = H_*$ for $* \neq 0$, and $\tilde{H}_0 \cong \mathbb{Z}^{\#(\text{path-components})-1}$

Theorem Invariance of dimension

(Brower ~1910)

$$\boxed{\begin{array}{ccc} S^n \cong S^m & \Leftrightarrow & n=m \\ \mathbb{R}^n \cong \mathbb{R}^m & \Leftrightarrow & n=m \end{array}}$$

by ①

"homeomorphisms preserve dimension"

Non-trivial result because there are space-filling curves.
e.g. Peano (1890)
$\exists$ cts surjection
$$\underset{\text{interval}}{\underset{\shortparallel}{[0,1]}} \to \underset{\text{square}}{\underset{\shortparallel}{[0,1]^2}}$$
The theorem implies this is not injective.
(cts. bij: compact $\to$ Hausdorff $\Rightarrow$ homeo)

Pf Lemma $\Rightarrow$ $F_n(S^n \xrightarrow{\cong} S^m)$ is iso $\underset{\neq 0}{F_n(S^n)} \xrightarrow{\cong} \underset{\underset{0 \text{ if } n \neq m\ \checkmark}{\shortparallel}}{F_n(S^m)}$ of gps.

If $\mathbb{R}^n \xrightarrow[\varphi]{\cong} \mathbb{R}^m$, then can extend φ to the one-point compactifications : $S^n \cong \mathbb{R}^n \cup \{\infty\} \xrightarrow{\cong} \mathbb{R}^m \cup \{\infty\} \cong S^m$, $\infty \mapsto \infty$. □

("Alexandroff extension")

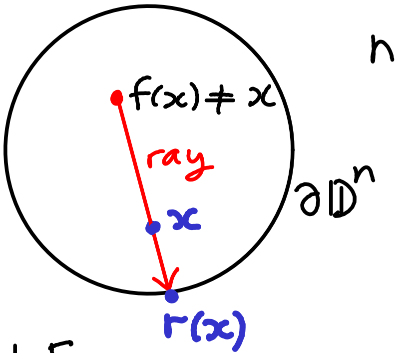
stereographic projection $(x_0, \dots, x_n) \mapsto \dfrac{(x_1, \dots, x_n)}{1 - x_0}$

Rmk new open neighbourhoods at ∞ are $\{\infty\} \cup (\mathbb{R}^n \setminus C)$ where C is (closed &) compact.
The extended map is cts since $\varphi^{-1}(C)$ is (closed &) compact since φ^{-1} is homeo.

Theorem Brower fixed point thm by ① & ②

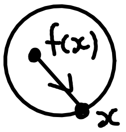
$f : \mathbb{D}^n \longrightarrow \mathbb{D}^n$ continuous $\Rightarrow$ f has a fixed point ($f(p) = p$ some p)

Proof Suppose not. Let $r(x) = (\text{ray from } f(x) \text{ to } x) \cap \partial \mathbb{D}^n$



notice : • $r : \mathbb{D}^n \longrightarrow \partial \mathbb{D}^n = S^{n-1}$ continuous

• $r|_{\partial \mathbb{D}^n} = \mathrm{id}_{S^{n-1}}$

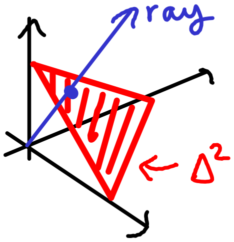


$$\begin{tikzcd} S^{n-1} = \partial \mathbb{D}^n \arrow[r, "\text{inclusion } i"] \arrow[rr, bend right, "r \circ i = id"'] & \mathbb{D}^n \arrow[r, "r"] & S^{n-1} \end{tikzcd}$$

apply F_{n-1} $\Longrightarrow$ $F_{n-1}(r) \circ F_{n-1}(i) = id$ $\Longrightarrow$ $F_{n-1}(i)$ injective $\overset{\neq 0}{F_{n-1}(S^{n-1})} \to \overset{= 0}{F_{n-1}(\mathbb{D}^n)}$ ↯ □

Example (Brower) $A = n \times n$ matrix, $A_{ij} > 0$ real $\Longrightarrow$ $\exists$ real evalue $\lambda > 0$ with real evector $(v_1, \dots, v_n)$ with $v_i \geq 0$

Hint $\mathbb{R}^3 \supseteq$



$X = \{\text{rays in "positive octant"}\}$ ← $x \in \mathbb{R}^n : x_i \geq 0\ \forall i$

notice $AX \subseteq X$

notice $X \cong \Delta^n = \{x \in \text{octant} : \sum x_i = 1\} \cong \mathbb{D}^n$

ray $\mapsto$ ray $\cap\, \Delta^n$

1. ALGEBRAIC PRELIMINARIES : CHAIN COMPLEXES

Graded abelian groups

Def A **$\mathbb{Z}$-graded abelian group** C is an abelian group together with a direct sum decomposition

$$\boxed{C = \bigoplus_{n\in\mathbb{Z}} C_n}$$

C_n ← abelian group

Convention: always grade by $\mathbb{Z}$ unless say otherwise.

Example $C = \mathbb{Z}[x] =$ integer polynomials in x, $C_n = \mathbb{Z}\cdot x^n$ ← so grading by degree

A graded ab.gp. A is a **graded subgp** of C if
- subgp
- $A_n \subseteq C_n$.

A **homomorphism** $h: C \longrightarrow D$ of gr.ab.gps is hom of gps s.t.

$$h(C_n) \subseteq D_n$$

A **hom of degree k** is hom with

$$h(C_n) \subseteq D_{n+k}$$

Shift by k: $\mathbb{Z}$-gr.ab.gp. $C[k]$ with

$$\boxed{C[k]_n = C_{k+n}}$$

Notice: $C[k]_0 = C_k$ is now in degree zero, so shifted down by k

⇒ Can view gr. hom of deg k as a gr. hom

$$h: C \longrightarrow D[k]$$

Abelian groups which are finitely generated

← recall f.g. means $\exists$ surjection $\mathbb{Z}^m \to G$ for some m

FACT Finitely generated abelian groups are classified:

$$G \cong \underbrace{\mathbb{Z}^{r}}_{\text{free part}} \oplus \underbrace{\mathbb{Z}/_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}/_{p_k^{n_k}}}_{\text{torsion part}}$$

r called **rank G**

$n_i \neq 0 \in \mathbb{N}$

p_i primes (possibly not distinct)

compare finite dimensional vectorspaces/field $\mathbb{F}$: $V \cong \mathbb{F}^r$ ← $r = \dim V$

Chain complexes

differential or **boundary homomorph**

Def A **chain complex** (C_*,∂_*) is a gr. ab. gp. C together with a hom ∂ of degree -1 such that $\partial\circ\partial=0$.

Thus:

$$\boxed{\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[rr, bend right, "\partial_n\circ\partial_{n+1}=0"'] & C_n \arrow[r, "\partial_n"] & C_{n-1} \arrow[r] & \cdots \end{tikzcd}}$$

hence $\underset{\substack{\| \\ B_n \\ \text{n-boundaries}}}{\mathrm{Im}\,\partial_{n+1}} \subseteq \underset{\substack{\| \\ Z_n \\ \text{n-cycles}}}{\mathrm{Ker}\,\partial_n}$

n-chains = elements of C_n

Now consider "cycles modulo boundaries":

Def The **homology** of (C_*,∂_*) is the gr. ab. gp.

$$\boxed{H_n(C_*,\partial_*) = \mathrm{Ker}\,\partial_n \big/ \mathrm{Im}\,\partial_{n+1}}$$

often abbreviate by $H_*(C)$

Two cycles are called **homologous** if they differ by a boundary

Def A **chain map** $h:(C_*,\partial_*)\longrightarrow(\tilde{C}_*,\tilde{\partial}_*)$ is a graded hom such that $\boxed{h\circ\partial_* = \tilde{\partial}_*\circ h}$

Example A chain **subcomplex** $C_*\subseteq\tilde{C}_*$ is a graded subgp with $\partial_*=$ restriction of $\tilde{\partial}_*$ to C_*.

So the inclusion $\mathrm{incl}:(C_*,\partial_*)\longrightarrow(\tilde{C}_*,\tilde{\partial}_*)$ is a chain map.

Also get **quotient complex** $\tilde{C}_*/C_*$

with $\tilde{\partial}_*[\tilde{c}]=[\tilde{\partial}_*\tilde{c}]$ (well-defined: $\tilde{\partial}_*C_*=\partial_*C_*\subseteq C_*$)

Lemma A chain map induces a hom on homology

$$H_*(h) = h_* : H_*(C_*, \partial_*) \longrightarrow H_*(\tilde{C}_*, \tilde{\partial}_*)$$
$$[x] \longmapsto [h(x)]$$

Proof $\operatorname{Ker} \partial_n \xrightarrow{\text{hom}} \operatorname{Ker} \tilde{\partial}_n$

$x \longmapsto h(x)$ since $\tilde{\partial}(h(x)) = h(\underbrace{\partial x}_{=0}) = 0$

Need $\operatorname{Im} \partial_n \longrightarrow \operatorname{Im} \tilde{\partial}_n$ to get well-defined hom

$$H_n(C) = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1} \longrightarrow \operatorname{Ker} \tilde{\partial}_n / \operatorname{Im} \tilde{\partial}_{n+1} = H_n(\tilde{C})$$

Proof: $h(b) \underset{\uparrow\, b = \partial c \in \operatorname{Im}(\partial)}{=} h(\partial c) = \tilde{\partial} h(c) \in \operatorname{Im}(\tilde{\partial})$. □

The last step was a very simple example of a proof by **"diagram chasing"**

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "h_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "h_n"] & C_{n-1} \arrow[r] \arrow[d, "h_{n-1}"] & \cdots \\
\cdots \arrow[r] & \tilde{C}_{n+1} \arrow[r, "\tilde{\partial}_{n+1}"'] & \tilde{C}_n \arrow[r, "\tilde{\partial}_n"'] & \tilde{C}_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

commutativity of this diagram is the definition of h being a chain map

$$\begin{tikzcd}
c \arrow[r, "\partial"] \arrow[d, "h"'] & \partial c = b \arrow[d, "h"] \\
hc \arrow[r, "\tilde{\partial}"] & \tilde{\partial}(hc) = h\partial c = h(b)
\end{tikzcd}$$

□

Def (C_*, ∂_*) is **exact** (or **acyclic**) if $H_*(C) = 0$

so $\operatorname{Im} \partial_{n+1} = \operatorname{Ker} \partial_n$

In general **exact sequence** means $\boxed{\text{"Im(previous map) = Ker (next map)"}}$

A **short exact sequence (SES)** is an exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \longrightarrow 0$$

Easy exercise

$$\Big(\underset{\text{exact}}{0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0}\Big) \Leftrightarrow \begin{cases} i & \text{injective} \\ \pi & \text{surjective} \\ B/_{i(A)} \cong C & \text{via } [b] \longmapsto \pi(b) \end{cases}$$

Examples

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2 \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\text{inclusion}} \mathbb{Z} \oplus \mathbb{Z}/2 \xrightarrow{\text{project}} \mathbb{Z}/2 \longrightarrow 0$$

Note A, C do not determine B.

Snake Lemma A SES of chain complexes and chain maps yields a **long exact sequence (LES) on homology**:

$$\boxed{\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{\pi_*} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*[-1]} \cdots}$$

↑ degree -1 map $H_*(C) \to H_*(A)[-1]$ called **connecting map**

(So **exact triangle**:

$$\begin{tikzcd} H_*(A) \arrow[rr] & & H_*(B) \arrow[dl] \\ & H_*(C) \arrow[ul, "{[-1]}"] & \end{tikzcd}$$

)

Pf simplify notation by identifying A with $i(A) \subseteq B$: $\overset{\in A}{a} \equiv \overset{\in B}{i(a)}$, $A \overset{i}{\hookrightarrow} B$

$\partial a \equiv i \partial a = \partial i a$

$\Longrightarrow$ now $A_* \subseteq B_*$ inclusion of subcomplex:

$$0 \longrightarrow (A_*, \partial_*) \xrightarrow{\text{incl}} (B_*, \partial_*) \xrightarrow{\pi} (C_*, \tilde{\partial}_*) \longrightarrow 0$$

$$\begin{tikzcd} 0 \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} & \exists\, b \arrow[r, "\text{surj.}"] \arrow[d] & \overset{\text{cycle}}{c} = \pi(b) \arrow[d] \\ \partial b \arrow[r] & \partial b \arrow[r] & \tilde{\partial} c = 0 \end{tikzcd}$$

↖ lifts to A by exactness

Define $\delta : H_*(C) \longrightarrow H_*(A)[-1]$ (typically b is not in A, so ∂b need not be a bdry in A)

$c \longmapsto \partial b$ ← where $b \in \pi^{-1}(c)$

Well-defined? • $\pi^{-1}(c) = \{b+a : a \in A\}$ and $\partial(b+a) = \partial b + \underbrace{\partial a}_{\text{boundary in } A}$ ✓

- cycle → cycle : $\partial(\partial b) = 0$ ✓
- boundary → boundary :

$$\begin{tikzcd}
\exists \beta \arrow[r, "\text{surj.}"] \arrow[d] & x \in C_{n+1} \arrow[d] \\
\partial \beta \arrow[r] & \text{boundary } c = \tilde{\partial} x \arrow[d] \\
 & 0
\end{tikzcd}$$

$\Rightarrow$ can pick $b = \partial \beta$

$\Rightarrow \partial b = \partial \partial \beta = 0$ ✓

Exactness at $H_n(C)$ (exercise: check exactness at H_*A, H_*B):

Need $\operatorname{Im} \pi_* = \operatorname{Ker} \delta$:

$\subseteq$: $\delta(\pi_* \underset{\text{cycle}}{b}) = \partial b = 0$ ✓

$\supseteq$:

$$\begin{tikzcd}
\exists a \arrow[d] & & \underset{\text{not necessarily cycle!}}{b} \arrow[r] \arrow[d] & \overset{\in \operatorname{Ker}\delta}{c} = \pi_* b \arrow[d] \\
\partial a \underset{\text{assumption } \delta c = 0 \in H_*A}{=} \delta c = \partial b \arrow[rr] & & \partial b \arrow[r] & 0
\end{tikzcd}$$

$\pi_*(b-a) \overset{\pi_* A = 0}{=} c$

$\partial \underbrace{(b-a)}_{\text{thus cycle!}} = \partial b - \partial a = 0$

$\Rightarrow c = \pi_*(b-a) \in \operatorname{Im} \pi_*$ □

Rmk $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \longrightarrow 0$ SES $\Rightarrow$ the connecting map of LES is

$$\boxed{\begin{aligned} \delta : H_*(C) &\longrightarrow H_*(A)[-1] \\ c &\longmapsto i^{-1}(\partial b) \end{aligned}}$$

$\forall\, b \in B$ with $\pi(b) = c$.

Lemma The construction of δ is natural (i.e. functorial)

Pf

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[r, "\pi"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & \tilde{A} \arrow[r, "\tilde{i}"] & \tilde{B} \arrow[r, "\tilde{\pi}"] & \tilde{C} \arrow[r] & 0
\end{tikzcd}$$

all chain maps

$$\begin{tikzcd}
\delta c = a \arrow[r] \arrow[d, "f"] & \partial b \arrow[d, "g"] & b \arrow[r] \arrow[d, "g"] & c \arrow[d, "h"] \\
fa \arrow[r] & g\partial b = \tilde{\partial} g b = \delta h c & gb \arrow[r] & hc
\end{tikzcd}$$

$$\Rightarrow \delta h c = \tilde{i}^{-1} \tilde{\partial} g b = \tilde{i}^{-1} g \partial b = fa = f \delta c \qquad \square$$

Exercise Deduce the LES is natural, so

$$\begin{tikzcd}
\cdots \arrow[r] & H_*A \arrow[r, "i_*"] \arrow[d, "f_*"] & H_*B \arrow[r, "\pi_*"] \arrow[d, "g_*"] & H_*C \arrow[r, "\delta"] \arrow[d, "h_*"] & H_{*-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_*\tilde{A} \arrow[r] & H_*\tilde{B} \arrow[r, "\tilde{\pi}_*"] & H_*\tilde{C} \arrow[r, "\delta"] & H_{*-1}(\tilde{C}) \arrow[r] & \cdots
\end{tikzcd}$$

5-Lemma

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha", "\cong"'] & B \arrow[r] \arrow[d, "\beta", "\cong"'] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta", "\cong"'] & E \arrow[d, "\varepsilon", "\cong"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

exact rows $\Longrightarrow$ γ also iso.

Pf exercise (diagram chase) □

Splitting Lemma

Cor $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ SES of abelian gps

If
$$\begin{tikzcd}
B \arrow[r, "\beta"] & C \arrow[l, bend left, "\exists\gamma"]
\end{tikzcd}$$
s.t. $\beta \circ \gamma = id_C$ then the SES **splits**: $B \cong A \oplus C$

(converse is obvious)

Pf
$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & A\oplus C \arrow[r] \arrow[d, "\alpha+\gamma"] & C \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$
□

Exercise If
$$\begin{tikzcd}
A \arrow[r, "\alpha"] & B \arrow[l, bend left, "\exists\mu"]
\end{tikzcd}$$
s.t. $\mu \circ \alpha = id_A$ then it splits: $B \xrightarrow[\mu\oplus\beta]{\cong} A \oplus C$

Exercise If C is a free abelian group ($C \cong \bigoplus_{i\in I} \mathbb{Z}$) then the SES splits.

Rmk A free $\not\Rightarrow$ splits, e.g. $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \longrightarrow 0$

Cultural Rmk Splitting Lemma generalises the **rank-nullity theorem** from linear algebra: $V \xrightarrow{\beta} W$ linear map of vector spaces $\Rightarrow \mathrm{Im}\,\beta \oplus \mathrm{Ker}\,\beta \cong V$

Pf $0 \to \mathrm{Ker}\,\beta \xrightarrow{incl} V \xrightarrow{\beta} \mathrm{Im}\,\beta \to 0$ is SES, and splits since $\mathrm{Im}\,\beta$ free.

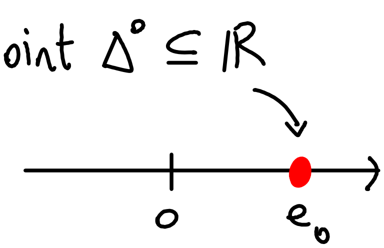
2. Δ-COMPLEXES AND SIMPLICIAL HOMOLOGY

standard n-simplex
$$\Delta^n = \left\{ \underset{\textstyle\sum t_i e_i}{\underset{\parallel}{(t_0, \dots, t_n)}} \in \mathbb{R}^{n+1} : \begin{array}{l} t_i \geq 0 \\ \sum t_i = 1 \end{array} \right\}$$

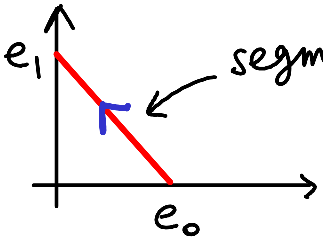
↖ standard basis of $\mathbb{R}^{n+1}$ $e_0, \dots, e_n$ $(e_0 = (1,0,..,0), \dots$

Examples

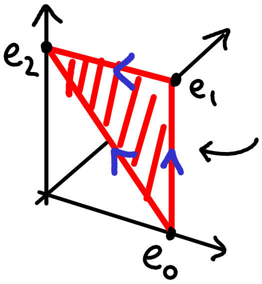
point $\Delta^0 \subseteq \mathbb{R}$



segment $\Delta^1 \subseteq \mathbb{R}^2$



triangle $\Delta^2 \subseteq \mathbb{R}^3$



Def For $\{v_0, \dots, v_n\} \subseteq \mathbb{R}^{n+k}$ ← any $k \geq 0$ s.t.

$v_1 - v_0, \dots, v_n - v_0$ $\mathbb{R}$-linearly independent

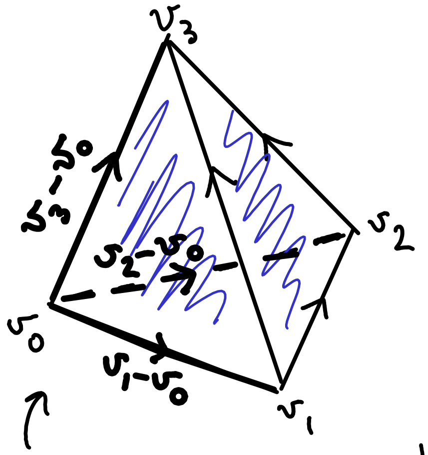
$$[v_0,\dots,v_n] = \underline{\text{n-simplex}} \text{ spanned by } v_0,\dots,v_n$$

$$= \text{convex hull of } v_0,\dots,v_n$$

$$= \left\{ \sum t_i v_i \;:\; t_i \geq 0 \text{ and } \sum t_i = 1 \right\}$$

$$= \text{Image of linear homeo } \sigma : \Delta^n \longrightarrow \mathbb{R}^{n+k}$$

$$\sigma(e_i) = v_i$$



(Solid prism: includes inside)

canonical homeomorphism

Will often blur the distinction between map σ and its image,

$$\sigma \equiv [\sigma e_0, \dots, \sigma e_n]$$

but the <u>ordering of the v_j</u> will be important (so the map σ is more precise)

we encode this extra data by <u>orienting the edges</u> $\overset{v_i}{\bullet} \longrightarrow \overset{v_j}{\bullet}$ if $i<j$

Def **d-dimensional faces** $[v_{i_0}, \dots, v_{i_d}]$ for $i_0 < \dots < i_d$

<u>Example</u> 0-dim faces are the vertices $v_0, \dots, v_n$

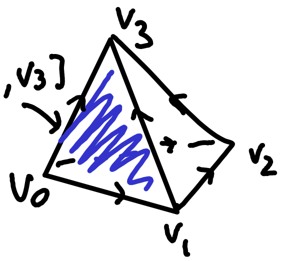
facets $= (n-1)$-dimensional faces

$$= [v_0, \dots, \hat{v}_\ell, \dots, v_n] \text{ where we } \underline{\text{omit}}\ v_\ell$$

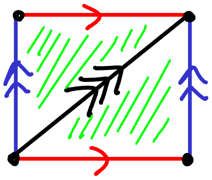
$$= \left\{ \sum t_i v_i \in [v_0,\dots,v_n] \;:\; t_\ell = 0 \right\}$$

$$= \text{Image } \sigma\big|_{\Delta^{n-1}_\ell} : \underset{\{t \in \Delta^n : t_\ell = 0\}}{\underset{\parallel}{\Delta^{n-1}_\ell}} \longrightarrow \mathbb{R}^{n+k}$$

$[v_0, v_1, \hat{v}_2, v_3]$
$\parallel$
$[v_0, v_1, v_3]$



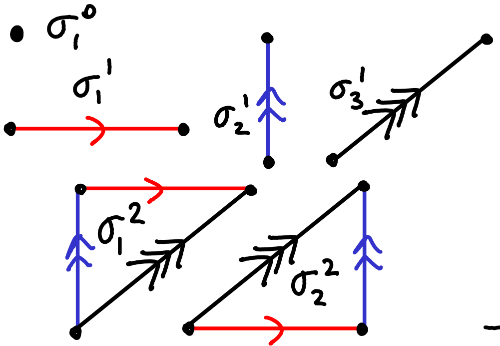
Example Can build a torus out of simplices:

$T^2 =$ 

1 0-simplex • σ_1^0

3 1-simplices

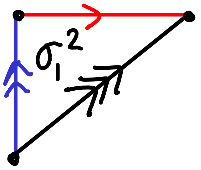
2 2-simplices

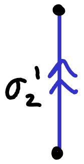


each facet is associated to another simplex, and we identify them linearly

don't confuse the abstract simplices with their images in T^2=quotient space

T^2= quotient space $\bigsqcup \sigma_i^n$ / canonical homeos associated to the facets

for example identify facet  of 

with 

via linear homeo (orientation-preserving)

Def **Δ-complex** is determined by data

- indexing set I_n, for each $n \in \mathbb{N}$
- choice of n-simplex σ_α^n (not necessarily standard) for each $\alpha \in I_n$
- gluing data: for each $\alpha \in I_n$, $0 \leq i \leq n$, associate some $\beta(\alpha, i) \in I_{n-1}$
- consitency condition (see later)

The Δ-complex is the quotient space

$$\boxed{X = \bigsqcup_{\substack{\alpha\in I_n \\ n\in\mathbb{N}}} \sigma_\alpha^n \Big/ \begin{array}{l} i\text{-th facet of } \sigma_\alpha^n \text{ is identified with } \sigma^{n-1}_{\beta(\alpha,i)} \\ \text{via the order-preserving canonical linear homeo} \end{array}}$$

(quotient topology: $U \subseteq X$ is open $\Leftrightarrow$ U intersects σ_α^n in an open set, $\forall \alpha, n$)

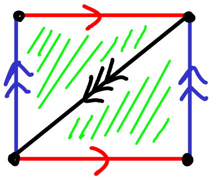
A **Δ-complex structure** on a top. space Y is a homeo from a Δ-cx $X \cong Y$.

Explicit description of the facet identification

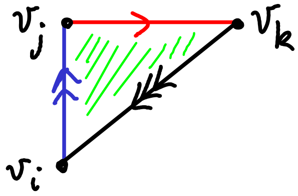
$$\begin{tikzcd}
\{\sum s_j w_j\} = [w_0,\dots,w_{n-1}] \arrow[r] & {[v_0,\dots,v_n]} & \{\sum t_j v_j\} \arrow[l, phantom, "="] \\
& & \{s_0v_0+\dots+s_{i-1}v_{i-1}+s_iv_{i+1}+\dots+s_{n-1}v_n\} \arrow[u, phantom, sloped, "\subseteq"] \\
& & = [v_0,\dots,\hat{v}_i,\dots,v_n] \\
\Delta^{n-1} \arrow[r] \arrow[uuu, "\sigma^{n-1}_{\beta(\alpha,i)}"] & \Delta^{n-1}_i \subseteq \Delta^n \arrow[uuu, "\sigma_\alpha^n"'] \arrow[ruu, "\sigma_\alpha^n|_{\Delta_i^{n-1}}"'] & \\
(s_0,\dots,s_{n-1}) \arrow[r, mapsto] & (s_0,\dots,s_{i-1},0,s_i,\dots,s_{n-1}) &
\end{tikzcd}$$

Non-example

This decomposition for T^2 is not a Δ-complex.



because:

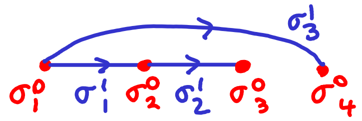


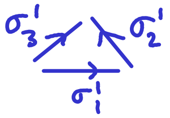
vertices are not totally ordered: $i<j<k<i$ ↯

Consistency condition

We want to additionally ensure that each point of X lies in the interior of exactly one σ_α^n, because we want to avoid unexpected identifications.

Example: $X =$ 



then glue $\sigma_1^2 =$  via 

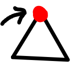
notice how σ_3^0, σ_4^0 get identified in the quotient, but we only notice this after gluing σ_1^2 (If you try to run the definition of simplicial homology – defined later – you notice that the differential cannot satisfy $\partial_1 \circ \partial_2 = 0$)

Equivalently: the facet gluing maps are compatible under double restriction: $\forall i<j$

$$\begin{tikzcd}
{[v_0,\dots,v_n]} \arrow[r, squiggly, "\text{facet}"] \arrow[dr, squiggly, "\text{facet}"'] & {[v_0,\dots,\hat{v}_i,\dots,v_n]} \arrow[r, phantom, "\overset{\text{identify}}{\cong}"] & {[w_0,\dots,w_{n-1}]} \arrow[r, squiggly, "\text{facet}"] & {[w_0,\dots,\hat{w}_{j-1},\dots,w_{n-1}]} \arrow[dr, phantom, "\overset{\text{identify}}{\cong}"] & \\
 & {[v_0,\dots,\hat{v}_j,\dots,v_n]} \arrow[r, phantom, "\overset{\text{identify}}{\cong}"] & {[z_0,\dots,z_{n-1}]} \arrow[r, squiggly, "\text{facet}"] & {[z_0,\dots,\hat{z}_i,\dots,z_{n-1}]} \arrow[r, phantom, "\underset{\text{identify}}{\cong}"] & {[x_0,\dots,x_{n-2}]}
\end{tikzcd}$$

this ensures that $[v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n]$ is identified with the same $[x_0,\dots,x_{n-2}]$ whether we first restrict to $t_i=0$ (omit v_i) or first restrict to $t_j=0$ (omit v_j).

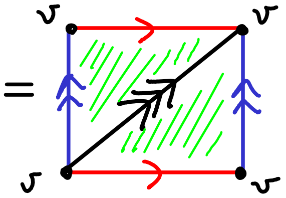
Another equivalent condition: can define the **k-th skeleton** of Δ-cx X, X^k = quotient space you get by gluing all simplices of dimensions $\leq k$. Consistency is the condition that the boundary of each σ_α^n should map continuously into X^{n-1}

(in the above Example consider the vertex →  $= \partial\sigma_1^2$)

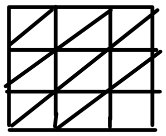
(more precisely, the "topological realisation" of a simpl. complex)

Rmk (see Part A Topology) A **simplicial complex** is a Δ-complex in which each d-dim face is uniquely determined by d distinct vertices. A homeo from such a complex to X is a **triangulation** of X.

Non-example

$T^2 =$ 

both 2-simplices have vertices v, v, v

whereas $T^2 =$  is a triangulation.

Simplicial chain complex

Def For a Δ-complex X, let X_n = set of n-simplices of X

$$C_n^\Delta(X) = \text{free abelian group generated by the set } X_n$$
$$= \left\{ \sum_{\alpha \in I_n} c_\alpha \cdot \sigma_\alpha^n \; : \; c_\alpha \in \mathbb{Z} \text{ and only finitely many } c_\alpha \neq 0 \right\}$$

differential: $\boxed{\partial_n \sigma_\alpha = \sum (-1)^i \cdot \sigma_{\beta(\alpha,i)}}$

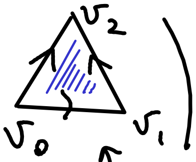
so: $\boxed{\partial_n [v_0,\dots,v_n] = \sum (-1)^i \cdot [v_0,\dots,\hat{v}_i,\dots,v_n]}$

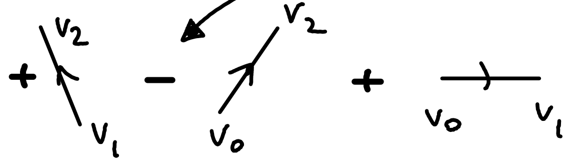
} and extend linearly

will show $\partial \circ \partial = 0$, so get **simplicial homology**: $\boxed{H_*^\Delta(X) = H_*(C_*^\Delta, \partial_*)}$

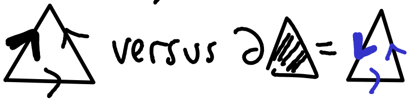
Examples

$\partial_1\left(\overrightarrow{v_0 \qquad v_1}\right) = \overset{\bullet}{-v_0} \quad \overset{\bullet}{+v_1}$

∂_2 () =



Later: The $(-1)^i$ signs keep track of whether the orientation agrees/disagrees with geometric boundary orientation, so



$\partial_2 \circ \partial_1$ (this) $= +(v_2 - v_1) - (v_2 - v_0) + (v_1 - v_0) = 0$

$\partial \circ \partial = 0$ fails for  (not Δ-complex), try!

Lemma $\boxed{\partial \circ \partial = 0}$

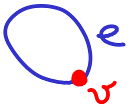
Pf $\partial_{n-1}(\partial_n [v_0,\dots,v_n]) = \sum (-1)^i \partial_{n-1}[v_0,\dots,\hat{v}_i,\dots,v_n]$

$$= \sum_{j<i} (-1)^i (-1)^j [v_0,\dots,\hat{v}_j,\dots,\hat{v}_i,\dots,v_n]$$

↺ antisymmetric if swap i,j

$$+ \sum_{j>i} (-1)^i \underline{\underline{(-1)^{j-1}}} [v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n]$$

$$= 0 \qquad \square$$

Example $S^1 =$ 

Δ-cx: X_0: 1 0-simplex • $e^0_1 = e_{\beta(1,0)} = e_{\beta(1,1)}$

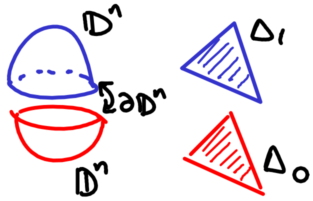
X_1: 1 1-simplex → e^1_1

$$\begin{tikzcd} 0 \arrow[r] & C_1^\Delta \arrow[r] \arrow[d, phantom, "\shortparallel"] & C_0^\Delta \arrow[r] \arrow[d, phantom, "\shortparallel"] & 0 \\ & \mathbb{Z}e & \mathbb{Z}v & \end{tikzcd}$$

$e \longmapsto v - v = 0$

$$\Rightarrow H_*^\Delta(S^1) = \begin{cases} \mathbb{Z} & * = 0,1 \\ 0 & \text{else} \end{cases}$$

Example Δ-cx structure on S^n:



$S^n = \Delta^n \sqcup \Delta^n /$ glue along $\partial \Delta^n$

call this Δ_1 ↑ this Δ_0

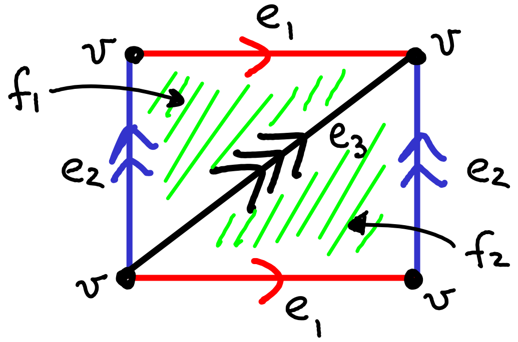
} One can deduce: ← but messy!

pick any vertex

$$H_* S^n = \begin{cases} \mathbb{Z}\langle \text{point} \rangle \cong \mathbb{Z} & * = 0 \\ \mathbb{Z}\langle \Delta_1 - \Delta_0 \rangle \cong \mathbb{Z} & * = n \\ 0 \quad \text{else} & \end{cases}$$

Example

$X = T^2 =$



$$\begin{tikzcd} 0 \arrow[r] & C_2^\Delta \arrow[r] \arrow[d, phantom, "\|"] & C_1^\Delta \arrow[r] \arrow[d, phantom, "\|"] & C_0^\Delta \arrow[r] \arrow[d, phantom, "\|"] & 0 \\ & \mathbb{Z}f_1+\mathbb{Z}f_2 & \mathbb{Z}e_1+\mathbb{Z}e_2+\mathbb{Z}e_3 & \mathbb{Z}v & \end{tikzcd}$$

$$f_1 \mapsto e_1 - e_3 + e_2 \qquad e_1, e_2, e_3 \longmapsto v - v = 0$$
$$f_2 \mapsto e_2 - e_3 + e_1$$

$$H_*^\Delta(T^2) = \begin{cases} \mathbb{Z}v & *=0 \\ (\mathbb{Z}e_1+\mathbb{Z}e_2+\mathbb{Z}e_3)/\mathbb{Z}(e_1-e_3+e_2) & *=1 \\ \mathbb{Z}\cdot(f_1-f_2) & *=2 \\ 0 & \text{else} \end{cases} \leftarrow \text{freely generated by } e_1, e_2$$

$$\cong \begin{cases} \mathbb{Z} & *=0,2 \\ \mathbb{Z}^2 & *=1 \\ 0 & \text{else} \end{cases}$$

(Alternative useful method using a trick from algebra:
Smith normal form of ∂_2:

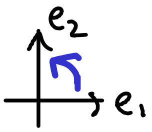
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix} \xrightarrow{\text{row op.}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \xrightarrow{\text{col. op.}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

so after $\mathbb{Z}$-isos of C_2, C_1 we get $\mathbb{Z}^2 \to \mathbb{Z}^3$, $(a,b) \mapsto (a,0,0)$) hence: $H_2 \cong \ker\partial_2 \cong \mathbb{Z}$, $H_1 \cong \operatorname{coker}\partial_2 \cong \mathbb{Z}^2$

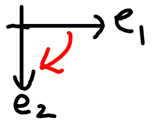
Remark about orientations (see also my B3.2 Geometry of Surfaces notes)

For a vector space an **orientation** is a choice of basis modulo linear endomorphisms of $\det > 0$

Example $\mathbb{R}^2$



right-hand orientation (positive) $\xrightarrow[\det<0]{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$



left-hand orientation (negative)

Fact $GL(n,\mathbb{R})$ has 2 path-components $\Big\langle \begin{matrix} A : \det A > 0 \\ A : \det A < 0 \end{matrix}$ so can always continuously deform a basis to another within same orientation

Canonical orientation on $\mathbb{R}^n$: $e_1, \ldots, e_n$ standard basis $\leftarrow$ "positive orientation"

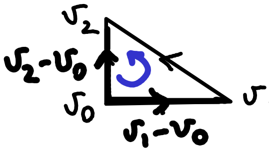
Example $[v_0, \ldots, v_n]$ simplex $\Rightarrow$ $v_1 - v_0, \ldots, v_n - v_0$ is a basis of vector subspace

$$V = \{\Sigma a_i v_i : \Sigma a_i = 0\} \subseteq \mathbb{R}^{n+k}$$

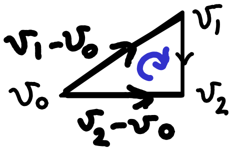
hence a choice of orientation of V,
and each transposition of vertices $v_0, \ldots, v_n$ switches the orientation class.

If $v_0,..,v_n \in \underline{\underline{\mathbb{R}^n}}$ then $V=\mathbb{R}^n$ so simplex's orientation can be compared with $\mathbb{R}^n$-orient.

<u>Example</u> In $\mathbb{R}^2$:



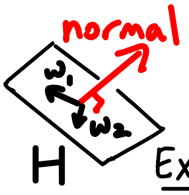
positively oriented



negatively oriented

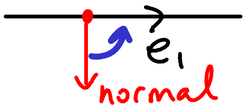
- No canonical choice of orientation for an abstract vector space. Need choose basis $v_1 \to v_n$ then <u>declare</u> another basis <u>positively oriented</u> if the change of basis matrix has $\det > 0$.

- For hyperplane $H \subseteq \mathbb{R}^n$ <u>with choice of normal</u> can <u>declare</u> orientation of basis $w_1, \to w_{n-1}$ of H <u>positive</u> if normal, $w_1,\dots,w_{n-1}$ is positive $\mathbb{R}^n$-basis



<u>convention</u> "outward normal first"

<u>Example</u>



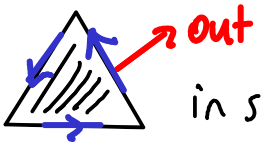
$H \subseteq \mathbb{R}^2$ $\Rightarrow$ e_1 positive basis for H

$$(\text{normal}, e_1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \det = +1 > 0$$

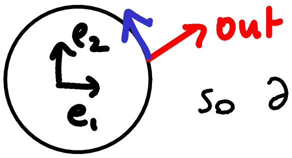
<u>Example</u> $\Delta^n \subseteq \mathbb{R}^{n+1}$ with normal $(1,1,\dots,1)$ is positively oriented.

<u>UPSHOT</u> For an n-simplex $[v_0,\dots,v_{\underline{\underline{n}}}]$ in $\underline{\underline{\mathbb{R}^n}}$, each facet lies in a hyperplane and have canonical choice of normal : <u>outward normal.</u> Hence <u>facets are canonically oriented.</u>

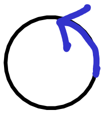
<u>Example</u> $\mathbb{R}^2 \supseteq$



in smooth world: $\mathbb{D}^2$

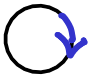


so $\partial\mathbb{D}^2 = S^1$



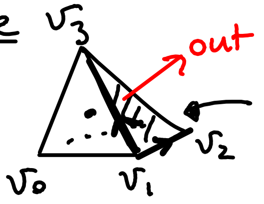
anticlockwise

Any <u>reflection of $\mathbb{R}^n$ will swap orientation</u> : after $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ get



clockwise

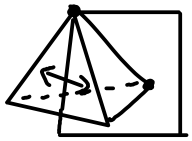
<u>Example</u>

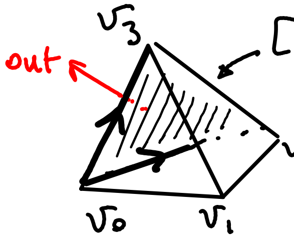


$[\hat{v}_0, v_1, v_2, v_3]$

out, v_2-v_1, v_3-v_1 positive $\mathbb{R}^3$-basis

$$\xrightarrow[v_0 \leftrightarrow v_1]{\text{reflect}}$$



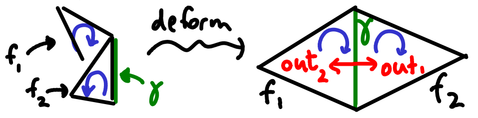


$[v_0, \hat{v}_1, v_2, v_3]$

out, v_2-v_0, v_3-v_0 negative $\mathbb{R}^3$-basis

<u>UPSHOT</u> $(-1)^i$ in $(-1)^i[v_0,\dots,\hat{v}_i,\dots,v_k]$ in definition of simplicial ∂ secretly keeps track of whether the orientation of the simplex agrees or not with the orientation induced geometrically by the above conventions. From this point of view, the equation $\partial\circ\partial=0$ holds because a codimension 2 face γ of a simplex σ arises as the facet of exactly two facets f_1, f_2 of σ, and the geometric orientations of f_1, f_2 induce <u>opposite</u> geometric orientations on γ (therefore if we keep track of orientation signs we count $+\gamma-\gamma=0$).

(Checking that they are opposite requires some thought, one approach is to say that we can deform f_1, f_2 until they make a flat angle, and then their outward normals will be opposite.) Picture:



get $\uparrow\gamma$ from ∂f_1
get $\uparrow\gamma$ from ∂f_2

Lemma $H_*^\Delta(X) \cong \bigoplus H_*^\Delta(X_i)$ where X_i are the path-components of X.

Pf $\bigoplus C_*^\Delta(X_i) \longrightarrow C_*^\Delta(X)$, $\oplus c_i \longmapsto \sum c_i$

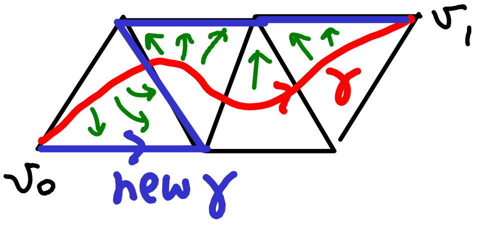
since Δ^k path-conn.

is chain isomorphism since any simplex $\sigma : \Delta^k \to X$ has path-connected image, so $\subseteq X_i$ some i. □

Theorem X has Δ-cx structure $\Longrightarrow$ $H_0^\Delta(X) \cong \bigoplus_{\text{path-conn. components}} \mathbb{Z}$

Pf By Lemma, WLOG X path-connected

- vertex v $\Longrightarrow$ $v \in C_0(X) \xrightarrow{\partial} C_{-1}(X) \equiv 0$ $\Rightarrow$ $[v] \in H_0(X)$
- vertices $v_0, v_1 \in X$ $\Longrightarrow$ $\exists$ path γ from v_0 to v_1



$\Rightarrow$ can homotope path so that go along edges (continuously deform)

$\Rightarrow$ γ is sum of 1-chains s.t. $\partial\gamma = v_1 - v_0$

$\Rightarrow$ $[v] \in H_0(X)$ independent of choice of v

$\Longrightarrow$ $H_0(X) = \langle [v] \rangle$

- $H_0(X) = \langle [v] \rangle \leftarrow \mathbb{Z}$ is injective?

$nv \leftarrow\!\shortmid n$ Suppose $nv = \partial c$ some $c \in C_1(X)$

consider the augmentation hom

$$C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$$

$$\underbrace{\sum n_i \sigma_i}_{\text{0-simplices}} \longmapsto \sum n_i$$

notice composite is 0 since $\partial\left(\overset{\text{1-simplex}}{\sigma_0 \to \sigma_1}\right) = \sigma_1 - \sigma_0 \xrightarrow{\varepsilon} 1 - 1 = 0$

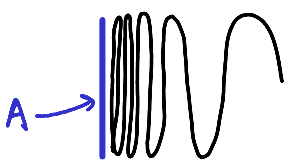
$\Rightarrow$ $n = \varepsilon(nv) = \varepsilon\partial c = 0$. □

Rmk X top. space $\Longrightarrow$ path conn. component $\subseteq$ connected component since path-conn. $\Rightarrow$ connected. For Δ-cx, these are same (since connected + locally path-conn. $\Longrightarrow$ path-connected).

However for top. space they need not be the same, e.g. Topologist's sine curve

$$\{(x, \sin\tfrac{1}{x}) : x \in (0,1]\} \cup \underbrace{0 \times [0,1]}_{A} \subseteq \mathbb{R}^2$$

2 path-conn. components

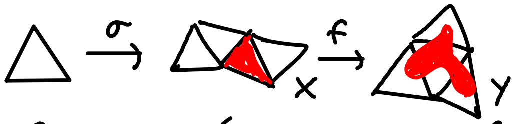


- connected
- not path-connected
- not locally path-connected

3. SINGULAR HOMOLOGY

Motivation: Not obvious that H_*^{Δ} is functorial: $\Delta^k \xrightarrow{\sigma} X \xrightarrow{f} Y$ ↖ continuous map

then $f \circ \sigma$ typically not a simplex:



Solution 1: only allow **simplicial maps** $f: X \to Y$ (so $f\circ\sigma$ simplex $\forall \sigma$)

Solution 2: show that any cts map $f: X \to Y$ can be approximated arbitrarily well by a simplicial map after having performed barycentric subdivision on X,Y enough times. Also any two such approximations induce the same map $H_*^{\Delta}(X) \to H_*^{\Delta}(Y)$

Solution 3: develop new homology $H_*(X)$ which allows any cts map $\Delta^n \to X$ and prove $H_*^{\Delta}(X) \cong H_*(X)$ for Δ-complexes X.

↑ WILL DO THIS.

X is any top. space

Def **Singular n-simplex** is any $\boxed{\text{continuous map } \sigma : \Delta^n \longrightarrow X}$

Singular n-chains $C_n(X)$ = free abelian group generated by σ

$$= \left\{ \sum_{\substack{\text{singular}\\ n\text{-simplices } \sigma}} c_\sigma \cdot \sigma \; : \; c_\sigma \in \mathbb{Z} \; \text{only finitely many } c_\sigma \neq 0 \right\}$$

$$\boxed{\partial_n \sigma = \sum (-1)^i \cdot \sigma|_{\Delta_i^{n-1}}}$$ ← i-th facet (and extend linearly)

Rmk Here $\Delta_i^{n-1} = [e_0, .., \hat{e}_i, .., e_n]$ is identified canonically with Δ^{n-1} (send $e_k \to e_k$ $k<i$, $e_k \to e_{k-1}$ for $k>i$)

Lemma $\partial \circ \partial = 0$

Proof $\partial_{n+1}(\partial_n \sigma) = \partial_{n+1}\left(\sum (-1)^i \sigma|_{\Delta_i^{n-1}}\right)$ ← $[e_0, .., \hat{e}_i, .., e_n]$

$$= \sum_{j<i} (-1)^i (-1)^j \sigma|_{[e_0, .., \hat{e}_j, .., \hat{e}_i, .., e_n]}$$

$$+ \sum_{j>i} (-1)^i \underline{\underline{(-1)^{j-1}}} \sigma|_{[e_0, .., \hat{e}_i, .., \hat{e}_j, .., e_n]}$$

antisymmetric if swap i,j

$= 0$. □

⟹ **singular homology**: $\boxed{H_*(X) = H_*(C_*, \partial_*)}$

For Δ-complex X have inclusion of subcomplex $C_*^{\Delta} \to C_*$

⟹ induces $H_*^{\Delta}(X) \longrightarrow H_*(X)$ Fact: isomorphism (proof later, see cellular $H_*^{CW} \cong H_*$)

Corollary $H_*^{\Delta}(X)$ is independent of choice of Δ-cx structure on X

Example $X = \text{point} \implies C_n(X) = \mathbb{Z}\cdot(\sigma_n : \Delta^n \to X \text{ the constant map})$

$$\partial\sigma_n = \sum(-1)^i \sigma_n|_{\Delta_i^{n-1}} = \underbrace{\sum(-1)^i}_{\begin{cases}1 & \text{if } n \text{ even}\\ 0 & \text{'' odd}\end{cases}} \sigma_{n-1} \quad \begin{array}{l} \implies \cdots \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \underset{\overset{\wr\wr}{C_1}}{\mathbb{Z}} \xrightarrow{0} \underset{\overset{\wr\wr}{C_0}}{\mathbb{Z}} \to 0 \\ \implies H_*(pt) \cong \begin{cases}\mathbb{Z} & *=0\\ 0 & \text{else}\end{cases}\end{array}$$

Lemma $\boxed{H_*(X) \cong \bigoplus H_*(X_i)}$ where X_i are path-components of X

Pf Image of cts map $\Delta^n \to X$ is path conn. so lies in some X_i. □

Cor $\boxed{H_0(X) = \bigoplus_{X_i} \mathbb{Z}}$

$\leftarrow$ generators of $C_0(X)$

Pf By Lemma, WLOG X path-connected. $\Delta^0 = pt \to X$ is cycle since $C_{-1}(X) = 0$. Given 2 points $x, y \in X$, a path $\Delta^1 = [0,1] \xrightarrow[\gamma]{} X$, $\gamma(0) = x$, $\gamma(1) = y$ is also a 1-chain! So $y - x = \partial\gamma$, so x, y are homologous. Finally if $n\cdot[x] = 0 \in H_0(X)$ then $nx = \partial c$ some $c \in C_1(X)$ = generated by paths. Now run the augmentation hom. trick like we did for H_0^Δ: $n = \varepsilon(nx) = \varepsilon\partial c = 0$ as $\varepsilon\circ\partial = 0$. □

Naturality (i.e. functoriality)

Lemma $f : X \to Y$ continuous

$\implies H_*(f) = f_* : H_*(X) \to H_*(Y)$ induced by chain map

$f_* : C_*(X) \to C_*(Y)$

induced map

$\boxed{f_*(\sigma) = f\circ\sigma}$ and extend linearly

$$\begin{tikzcd} \Delta^n \arrow[r, "\sigma"] \arrow[dr, "f_*\sigma"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

Pf $\partial_n(f_*\sigma) = \sum(-1)^n f\circ\sigma|_{\Delta_i^{n-1}} = f_*\left(\sum(-1)^n \sigma|_{\Delta_i^{n-1}}\right) = f_*(\partial_n\sigma)$ □

Properties
1) $X \xrightarrow{f} Y \xrightarrow{g} Z \implies (g\circ f)_* = g_*\circ f_*$
2) $id_* = id$

Pf
1) $(g\circ f)_*\sigma = g\circ f\circ\sigma = g_*(f\circ\sigma) = g_*(f_*\sigma)$ ✓
2) $id_*\sigma = id\circ\sigma = \sigma$ ✓ □

Cor $H_* : \left\{\begin{array}{c}\text{topological spaces}\\ \&\\ \text{cts maps}\end{array}\right\} \to \left\{\begin{array}{c}\text{graded abelian groups}\\ \&\\ \text{graded homs}\end{array}\right\}$ is a functor

Cor $X \cong Y \implies H_*(X) \cong H_*(Y)$

4. CHAIN HOMOTOPIES AND HOMOTOPY INVARIANCE

Algebra: chain homotopies

$f_*, g_* : (C_*, \partial_*) \longrightarrow (\tilde{C}_*, \tilde{\partial}_*)$ chain maps

Def f_*, g_* are **chain homotopic** if $\exists$ (degree +1) hom $h : C_* \longrightarrow \tilde{C}_*[1]$ s.t.

$$\boxed{\tilde{\partial} \circ h + h \circ \partial = f_* - g_*}$$

h is called a **chain homotopy**

Consequence $f_* = g_* : H_*(C_*, \partial_*) \longrightarrow H_*(\tilde{C}_*, \tilde{\partial}_*)$ on homology

Pf

$$\begin{tikzcd}
C_{n+1} \arrow[r] & C_n \arrow[r, "\partial_n"] \arrow[dl, "h_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1} \arrow[dl, "h_{n-1}"'] \\
\tilde{C}_{n+1} \arrow[r, "\tilde{\partial}_{n+1}"'] & \tilde{C}_n \arrow[r] & \tilde{C}_{n-1}
\end{tikzcd}$$

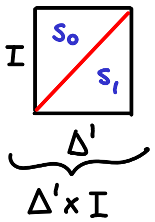
c cycle $\in C_n$

$$\Longrightarrow \quad f_n(c) - g_n(c) = \underbrace{\tilde{\partial}_{n+1} \circ h_n(c)}_{\text{boundary}} + h_{n-1} \circ \underbrace{\partial_n(c)}_{\overset{\shortparallel}{0}}$$

$$\Longrightarrow \quad f_n(c) = g_n(c) \text{ in } H_*(\tilde{C}) \qquad \square$$

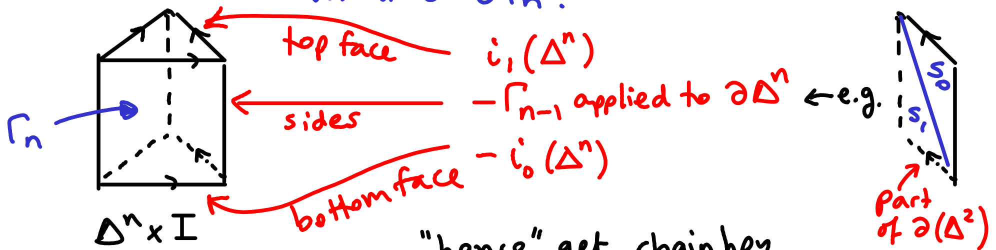
Theorem $i_0 : X \longrightarrow X \times I$, $i_0(x) = (x, 0)$ where $I = [0,1]$
$i_1 : X \longrightarrow X \times I$, $i_1(x) = (x, 1)$
$\Longrightarrow$ $i_{0*}, i_{1*} : C_*(X) \to C_*(X \times I)$ are chain hpic.

Key idea Need the "prism operator" which cuts $\Delta^n \times I$ into a sum Γ_n of $(n+1)$-simplices in $\Delta^n \times I$:



$\Gamma_1 = s_0 - s_1$

What is $\partial \Gamma_n$?



"hence" get chain hpy

Pf ← Non-examinable

$v_i = e_i \times 0$

bottom facet $\Delta^n \times 0 = [v_0, \ldots, v_n]$

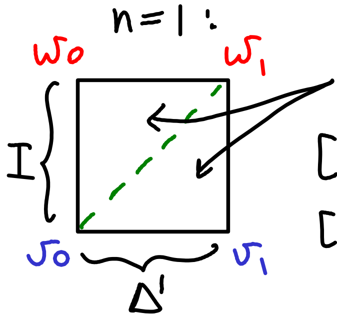
top facet $\Delta^n \times 1 = [w_0, \ldots, w_n]$

$\Big\} \subseteq \Delta^n \times [0,1] \subseteq \mathbb{R}^{n+1}$

$w_i = e_i \times 1$

Examples

$n=1$:

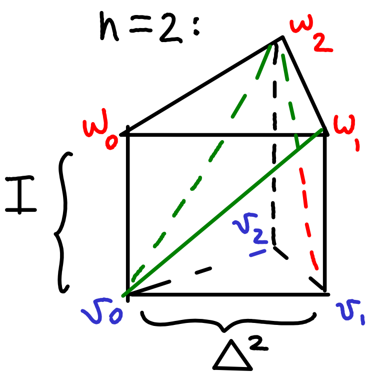


two 2-simplices

$[v_0, v_1, w_1]$

$[v_0, w_0, w_1]$

$n=2$:



three 3-simplices:

$[v_0, v_1, v_2, w_2]$

$[v_0, v_1, w_1, w_2]$

$[v_0, w_0, w_1, w_2]$

Let $s_i = [v_0, \ldots, v_i, w_i, \ldots, w_n]$

Claim The s_i cover $\Delta \times [0,1]$ and give Δ-cx structure on $\Delta^n \times I$

Pf $\sum_{k \le i} t_k v_k + \sum_{k \geq i} s_k w_k = (t_0, \ldots, t_{i-1}, \underline{t_i + s_i}, s_{i+1}, \ldots, s_n, s_i + \cdots + s_n)$

So given $(x_0, \ldots, x_n, a) \in \Delta^n \times I$, equate and solve:

$t_0 = x_0, \ldots, t_{i-1} = x_{i-1}, \; s_{i+1} = x_{i+1}, \ldots, s_n = x_n$, and $\begin{cases} s_i = a - x_{i+1} - \cdots - x_n \\ t_i = x_i - s_i \end{cases}$

Note $x_k \geq 0, \Sigma x_k = 1, a \in [0,1]$ hence $\Sigma t_k + \Sigma s_k = 1$ ✓ $\begin{cases} t_k \geq 0 \text{ for } k < i \; ✓ \\ s_k \geq 0 \text{ for } k > i \end{cases}$

but $\left.\begin{matrix} s_i \geq 0 \\ t_i \geq 0 \end{matrix}\right\} \Leftrightarrow \begin{cases} a \geq x_{i+1} + \cdots + x_n \\ x_i + x_{i+1} + \cdots + x_n \geq a \end{cases}$. Thus a solution exists if we pick $i = \min\{k : a \geq x_{i+1} + \cdots + x_n\}$

There are multiple solutions if $x_{i+1} = x_{i+2} = \cdots = x_j = 0$, but that is as expected: those points of $\Delta^n \times I$ belong to the faces of $s_i, s_{i+1}, \ldots, s_j$. □

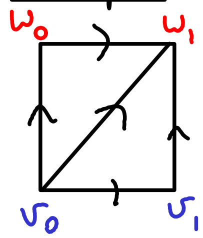
Def

$\Gamma_n = \sum (-1)^i s_i \in C_{n+1}(\Delta^n \times [0,1])$ ← geometrically this "represents" $\Delta^n \times I$ as a simplicial chain

$$\Rightarrow \partial \Gamma_n = \sum_i \sum_{j \le i} (-1)^i (-1)^j [v_0, \ldots, \hat{v}_j, \ldots, v_i, w_i, \ldots, w_n] + \sum_i \sum_{j \geq i} (-1)^i (-1)^{j+1} [v_0, \ldots, v_i, w_i, \ldots, \hat{w}_j, \ldots, w_n]$$

} geometrically this "represents" $\partial(\Delta^n \times I) = (\partial \Delta^n \times I) \sqcup (\Delta^n \times \partial I)$

Example



$\Gamma_1 = [v_0, w_0, w_1] - [v_0, v_1, w_1]$ "is the square"

$\partial \Gamma_1 = [w_0, w_1] - \cancel{[v_0, w_1]} + [v_0, w_0] - [v_1, w_1] + \cancel{[v_0, w_1]} - [v_0, v_1]$

"is ∂ of square"   "inside facets" cancel

Prism operator

$P : C_n(X) \longrightarrow C_{n+1}(X \times [0,1])$

$$\boxed{P(\sigma) = (\sigma \times id)_*(\Gamma_n)}$$

$\sigma : \Delta^n \to X$

$\sigma \times id : \Delta^n \times [0,1] \to X \times [0,1]$

$(\sigma \times id)(x,t) = (\sigma(x), t)$

$$\begin{aligned} \partial P(\sigma) &= \partial(\sigma \times id)_*(\Gamma_n) \\ &= (\sigma \times id)_*(\partial \Gamma_n) \\ &= \sum_i \sum_{j \le i} (-1)^i (-1)^j [i_0\sigma e_0, \ldots, \widehat{i_0\sigma e_j}, \ldots, i_0\sigma e_i, i_1\sigma e_i, \ldots, i_1\sigma e_n] \\ &\quad + \sum_i \sum_{j \ge i} (-1)^i (-1)^{j+1} [i_0\sigma e_0, \ldots, i_0\sigma e_i, i_1\sigma e_i, \ldots, \widehat{i_1\sigma e_j}, \ldots, i_1\sigma e_n] \\ &= \underset{\substack{\uparrow \\ i=j=0 \\ 1^{st}\text{ sum}}}{i_{1*}\sigma} - \underset{\substack{\uparrow \\ i=j=n \\ 2^{nd}\text{ sum}}}{i_{0*}\sigma} - \underbrace{P\partial\sigma}_{\substack{\uparrow \\ ((\partial\sigma)\times id)_* \Gamma_{n-1}}} . \quad \square \end{aligned}$$

this abbreviated notation means the map

$$\underset{\substack{\cap \\ \Delta^n}}{(t_0, \ldots, t_n)} \mapsto \left(\sigma\left(\begin{matrix} t_0 e_0 + \cdots + \widehat{t_j e_j} + t_j e_{j+1} + \cdots \\ + t_{i-1} e_i + t_i e_i + \cdots + t_n e_n \end{matrix}\right), t_i + \cdots + t_n\right) \in X \times I$$

⊛

$$\begin{aligned} &(\sigma \times id)(v_i) \\ &= (\sigma \times id)(e_i, 0) \\ &= (\sigma(e_i), 0) \\ &= i_0(\sigma)(e_i) \\ &(\sigma \times id)(w_i) \\ &= (\sigma(e_i), 1) \\ &= i_1(\sigma)(e_i) \end{aligned}$$

$\Gamma_{n-1} = \sum_i (-1)^i [v_0, \ldots, v_i, w_i, \ldots, w_{n-1}]$

now use ⊛ and

$\partial\sigma = \sum (-1)^k [\sigma e_0, \ldots, \widehat{\sigma e_k}, \ldots, \sigma e_n]$

(Note for $k < i$ the P operator on $(-1)^k [\sigma e_0, \ldots, \widehat{\sigma e_k}, \ldots, \sigma e_n]$ gives $\sum \underline{\underline{(-1)^{i-1}}} (-1)^k [i_0\sigma e_0, \ldots, \widehat{i_0\sigma e_k}, \ldots, i_0\sigma e_i, i_1\sigma e_i, \ldots, i_1\sigma e_n]$)

Homotopy invariance

Def $f_0, f_1 : X \longrightarrow Y$, **$f_0 \simeq f_1$** **homotopic** if $\exists$ continuous map $F : X \times [0,1] \longrightarrow Y$ called **homotopy** s.t. $\begin{cases} f_0 = F \circ i_0 \\ f_1 = F \circ i_1 \end{cases}$.

Idea Think of this as a continuous family of maps $f_t = F(\cdot, t) : X \to Y$ from f_0 to f_1.

Exercise $\simeq$ is an equivalence relation.

Homotopic **relative to $A \subseteq X$** if $F(a,t) = f_0(a) = f_1(a)$ all $a \in A$, all t.

write "$f \simeq g$ **rel A**"

Def $X \simeq Y$ **homotopy equivalent** spaces if $\exists$ maps

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd} \quad \text{with} \quad \begin{array}{l} g\circ f \simeq id \\ f \circ g \simeq id \end{array}$$

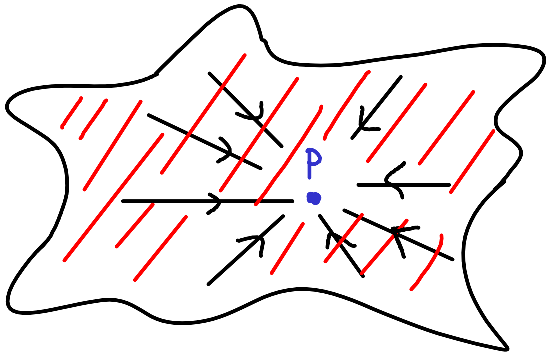
Rmk homeo $\Rightarrow$ hpy equivalent

Def X **contractible** if $X \simeq pt$

equivalently $(X \xrightarrow{id} X) \simeq (X \xrightarrow{const} \text{point} \in X)$

Example
- $\mathbb{R}^n \simeq pt$

  $F(x,t) = tx$ then $f_0 \equiv 0$, $f_1 = id$.
- (star-shaped subsets of $\mathbb{R}^n$) $\simeq pt$



contains line segments to a specific point p

WLOG $p=0$ & use same F

↑

translate

(examples: $\mathbb{D}^n$, convex sets, ...)

Theorem $f_0 \simeq f_1 \Longrightarrow f_{0*} = f_{1*} : H_*X \to H_*Y$

Pf

$$\begin{aligned} f_{1*} - f_{0*} &= F_* i_{1*} - F_* i_{0*} \\ &= F_* (i_{1*} - i_{0*}) \\ &\overset{\text{previous Thm}}{=} F_*(\partial P + P\partial) \\ &\overset{F_* \text{ chain map}}{=} \partial \circ (F_* \circ P) + (F_* \circ P)\circ \partial \end{aligned}$$

(where F = homotopy, i_0, i_1 as in previous Thm)

$\Longrightarrow F_* \circ P$ is chain hpy from f_{0*} to f_{1*} $\square$

Cor $X \simeq Y \Longrightarrow H_*X \cong H_*Y$

Pf $f_* g_* = id_*$, $g_* f_* = id_*$ $\square$

Example X contractible $\Longrightarrow H_*X \cong H_*(pt) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$

Cultural Rmk (Whitehead's theorem) For nice topological spaces ← (CW complexes – see later in course) if X, Y are simply connected and $\exists f : X \to Y$ inducing isomorphisms on H_* then $X \simeq Y$ are homotopy equivalent.

Relative homology

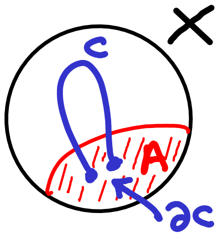
Def (X,A) *pair of spaces* if $A \subseteq X$ topological subspace

$\Rightarrow i = \mathrm{incl}: A \hookrightarrow X$ induces a subcx $i_*: C_*A \to C_*X$

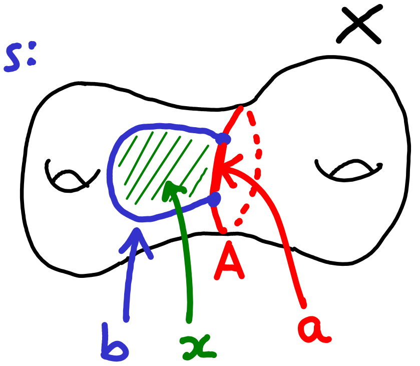
$\Rightarrow C_*X/C_*A$ quotient chain cx (recall $\partial[x] = [\partial x]$)

$$\boxed{H_*(X,A) = H_*(C_*X/C_*A)}$$

Idea: *relative cycles*: $c \in C_*X$ s.t. $\partial c \in C_*A$



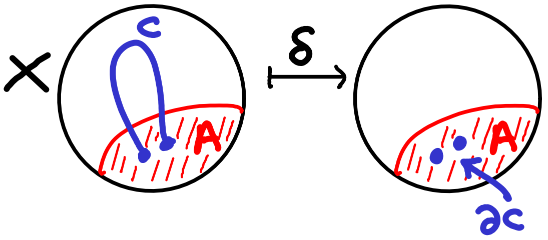
relative boundaries: $b \in C_*X$ s.t. $\exists\, x \in C_{*+1}X$ $\partial x = b + \underset{\in C_*A}{a}$



$\Rightarrow 0 \to C_*A \to C_*X \to C_*X/C_*A \to 0$ SES

Cor $$\boxed{\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \to H_n(X,A) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*} \cdots}$$

LES for the pair



$\delta c = \underset{\substack{\Uparrow \\ C_*A}}{\partial c}$

Need not be ∂a some $a \in C_*A$

Reduced homology

$\tilde{H}_*X = \ker\big(H_*X \longrightarrow H_*(pt)\big)$ induced by $X \to pt$

For $X \neq \emptyset$, $\tilde{H}_*X = H_*$ of *augmented chain complex*:

$$\cdots \to C_n(X) \to \cdots \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

augmentation $\varepsilon(\Sigma\, \underset{\in \mathbb{Z}}{n_i} \cdot \underset{\text{points } \in X}{p_i}) = \Sigma n_i$

can view $C_{-1}(X) = \mathbb{Z}\cdot(\text{map } \emptyset \to X)$ where allow the empty simplex $\emptyset$

Example $\tilde{H}_*(\mathbb{R}^n) \cong \tilde{H}_*(\mathbb{D}^n) \cong \tilde{H}_*(pt) = 0$

Check
- $H_*X = \tilde{H}_*X$ $* \neq 0$, and $H_0X \cong \tilde{H}_0X \oplus \mathbb{Z}$ for $X \neq \emptyset$
- $f: X \to Y \Rightarrow f_*: \tilde{H}_*X \longrightarrow \tilde{H}_*Y$

Lemma (X,A) pair $\Rightarrow \exists$ LES

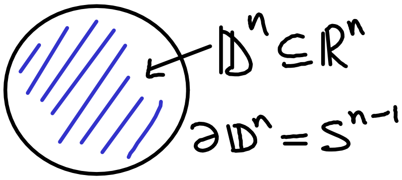
$$\boxed{\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \to H_n(X,A) \to \tilde{H}_{n-1}(A) \xrightarrow{i_*} \cdots \to H_0(X,A) \to 0}$$

Pf use augmented ch. cx. and $\mathbb{Z}/\mathbb{Z} = 0$ $\square$

Cor $H_*(X,pt) \cong \tilde{H}_*(X)$ for $X \neq \emptyset$

Pf $\tilde{H}_*(pt)=0$. $\square$

Example LES:



$$\begin{tikzcd}
\tilde{H}_*(S^{n-1}) \arrow[rr] & & \tilde{H}_*(\mathbb{D}^n)=0 \arrow[dl] \\
& H_*(\mathbb{D}^n,S^{n-1}) \arrow[ul, "{[-1]}"] &
\end{tikzcd}$$

$\Rightarrow \boxed{H_*(\mathbb{D}^n,S^{n-1}) \cong \tilde{H}_{*-1}(S^{n-1})}$

Naturality of the LES for pairs

Def A map of pairs of spaces $(X,A) \xrightarrow{f} (Y,B)$

means $f: X \to Y$ and $f(A) \subseteq B$.

Lemma

$$\begin{tikzcd}
\cdots \arrow[r] & H_*A \arrow[r] \arrow[d, "f_*"'] & H_*X \arrow[r] \arrow[d, "f_*"'] & H_*(X,A) \arrow[r] \arrow[d] & H_{*-1}A \arrow[r] \arrow[d, "f_*"'] & \cdots \\
\cdots \arrow[r] & H_*B \arrow[r] & H_*Y \arrow[r] & H_*(Y,B) \arrow[r] & H_{*-1}B \arrow[r] & \cdots
\end{tikzcd}$$

and similarly for $\tilde{H}_*$.

Pf

$$\begin{tikzcd}
0 \arrow[r] & C_*A \arrow[r] \arrow[d, "f_*"'] & C_*X \arrow[r] \arrow[d, "f_*"'] & C_*X/C_*A \arrow[r] \arrow[d, "f_*"'] & 0 \\
0 \arrow[r] & C_*B \arrow[r] & C_*Y \arrow[r] & C_*Y/C_*B \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow$ claim follows by naturality of LES induced by SESs of chain cxes. $\square$

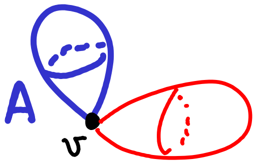
5. EXCISION THEOREM AND QUOTIENTS

(equivalently $r^2=r$ then define $A=\text{im}(r)$)

(X,A) pair

Def $r: X \to X$ **retraction** onto A if $\begin{cases} r(X)=A \\ r|_A = id_A \end{cases}$

Example



$X = \underbrace{S^2}_{=A} \vee S^2$ = two spheres glued at one point v (**wedge sum**)

$r: X \to A$ map second sphere to v

Example In Pf of Brower fixed pt thm we built a retraction r by contradiction

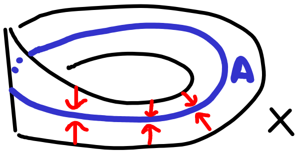
Cor r retraction $\Rightarrow$ $r_*: H_*X \to H_*A$ surjective

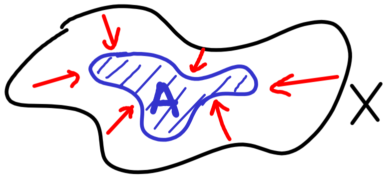
$\text{incl}_*: H_*A \to H_*X$ injective

Pf
$$\begin{tikzcd}
A \arrow[r, "\text{incl}"] \arrow[rr, bend right, "id_A"'] & X \arrow[r, "r"] & A
\end{tikzcd}$$
now use H_* functorial $\square$

Def $r : X \to X$ **deformation retraction** onto A if $\begin{cases} r \text{ retraction} \\ r \simeq \mathrm{id}_X \ \text{ rel } A \end{cases}$

Example $X =$ Möbius Strip
$A =$ equator

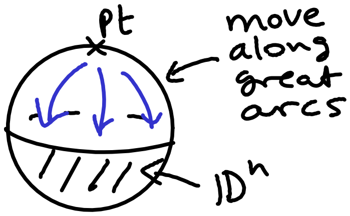




Lemma r def. retr. $\Rightarrow$ • $A \xrightarrow[\simeq]{\text{incl}} X$ is a homotopy equivalence.
• incl_* and r_* are isos on H_*, so $H_*A \cong H_*X$

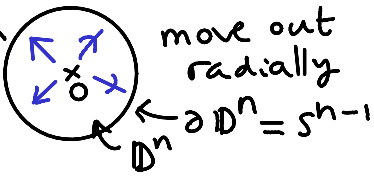
Pf $$\begin{tikzcd} A \arrow[r, bend left, "\text{incl}"] & X \arrow[l, bend left, "r"] \end{tikzcd}$$ $\mathrm{incl} \circ r = r \simeq \mathrm{id}_X$, $r \circ \mathrm{incl} = r|_A = \mathrm{id}_A$ □

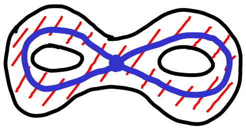
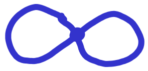
Example $S^n \setminus pt$ def. retracts to $\mathbb{D}^n \cong$ lower hemisphere:



$\Rightarrow S^n \setminus pt \simeq \mathbb{D}^n$

$\Rightarrow S^n \setminus [2 \text{ points}] \simeq \mathbb{D}^n \setminus pt \cong \mathbb{D}^n \setminus 0 \simeq S^{n-1}$



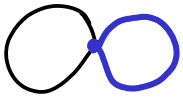
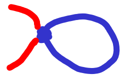
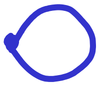
$\Rightarrow S^n \setminus \{3 \text{ points}\} \underset{\text{def.ret.}}{\simeq}$  $\underset{\text{ret.}}{\simeq}$  $S^{n-1} \vee S^{n-1}$

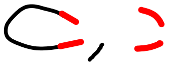
Excision theorem

$E \subseteq A \subseteq X$ subspaces with $\overline{E} \subseteq A^\circ$ $\Longrightarrow$ $(X \setminus E, A \setminus E) \longrightarrow (X, A)$ induces iso

$$\boxed{H_*(X \setminus E, A \setminus E) \xrightarrow{\cong} H_*(X, A)}$$

Proof Later.

Example $X = S^1 \vee S^1 =$  $\supseteq A =$  $\supseteq E =$  $\cong S^1$

$\Rightarrow H_1(X, A) \underset{\text{exc. thm.}}{\cong} H_1($  $) \underset{\text{hpy invce}}{\cong} H_1(\mathbb{D}^1, \underbrace{\partial \mathbb{D}^1}_{\simeq S^0}) \cong \tilde{H}_0(\underbrace{S^0}_{= 2 \text{ points}}) \cong \mathbb{Z}$

Example Invariance of dimension from chapter 0 also holds if replace $\mathbb{R}^n, \mathbb{R}^m$ by non-empty open sets $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ because for $p \in U$:

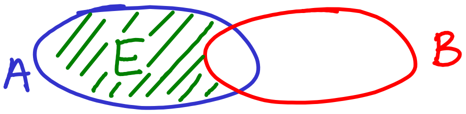
(statement becomes: $U \cong V \Rightarrow n = m$)

$$H_*(U, U-p) \underset{\text{excise } \mathbb{R}^n \setminus U}{\cong} H_*(\mathbb{R}^n, \mathbb{R}^n - p) \underset{\substack{\text{LES of pair} \\ \text{using } H_*(\mathbb{R}^n) = 0}}{\cong} H_{*-1}(\mathbb{R}^n - p) \underset{\text{deformation retract } \mathbb{R}^n - p \simeq S^{n-1}}{\cong} H_{*-1}(S^{n-1})$$

Rephrasing of Excision Thm

$X = A^\circ \cup B^\circ$ $\Longrightarrow$ $\boxed{H_*(X, A) \cong H_*(B, A \cap B)}$ induced by inclusion $(X, A) \leftarrow (B, A \cap B)$

$(A, B \subseteq X$ subspaces$)$



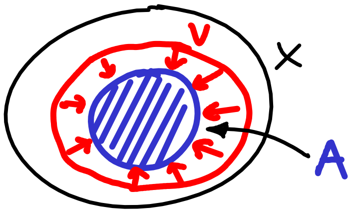
Pf Take $E = X \setminus B$ so $X \setminus E = B$ and $A \cap B = A \setminus E$. □

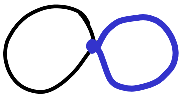
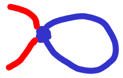
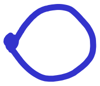
Idea why excision holds: $C_*(A) + C_*(B) \longrightarrow C_*(X)$ is a homotopy equivalence and $C_*(A) \cap C_*(B) = C_*(A \cap B)$. Idea: can subdivide chains in X many times, and small enough chains lie either in A or in B (or in both).

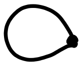
Good pairs and quotients

For (X,A) pair:

- **Quotient** $X/A = X/\sim$ ← equivalence relation $x \sim y \Leftrightarrow \begin{cases} x,y \in A \\ \text{or} \\ (x=y) \end{cases}$
- (X,A) **good pair** if $\begin{cases} A \neq \emptyset \\ A \text{ closed} \\ A \text{ deformation retract of nbhd } V \text{ of } A \end{cases}$

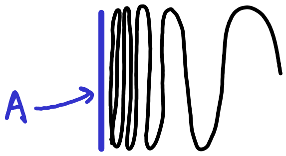


Example $X = S^1 \vee S^1 =$

$\supseteq V =$

$\supseteq A =$

$\cong S^1$

$X/A \cong$

← (all points of A are identified with the node)

Non-example **Topologist's sine curve**

$\{(x, \sin\frac{1}{x}) : x \in (0,1]\} \cup \underbrace{0 \times [0,1]}_{A} \subseteq \mathbb{R}^2$

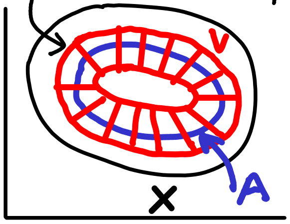


not a good pair.

connected
not path-connected
not locally connected
not locally path-connected

Cultural Rmk

Smooth submanifold ⊆ smooth manifold is a good pair (tubular neighbourhood theorem)



Cor (X,A) good $\Rightarrow (X,A) \longrightarrow (X/A, pt)$ induces iso

$$\boxed{H_*(X,A) \longrightarrow H_*(X/A, pt) = \widetilde{H}_*(X/A)}$$

Pf good $\Rightarrow \exists$ nbhd V of A, and $A \xrightarrow[\text{incl}]{\simeq} V$.

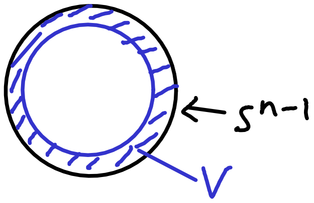
LES for pairs & 5-Lemma since $A \simeq V$, $A/A \simeq V/A$

excision

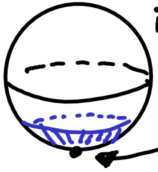
$$\begin{tikzcd} H_n(X,A) \arrow[r, "\cong"] \arrow[d, "\text{quot.}"'] & H_n(X,V) \arrow[d, "\text{quot.}"'] & H_n(X\setminus A, V\setminus A) \arrow[l, "\cong"'] \arrow[d, "="', "id_* = \text{identity}"] \\ H_n(X/A, \underbrace{A/A}_{\text{call this point } p}) \arrow[r, "\cong"] & H_n(X/A, V/A) & H_n(X/A \setminus p, V/A \setminus p) \arrow[l, "\cong"'] \end{tikzcd}$$

Hence all arrows are isos. □

Example $\mathbb{D}^n \supseteq S^{n-1}$ good:



$\xrightarrow{\text{quotient}}$



points of $A = S^{n-1}$ identified

$\Longrightarrow H_*(\mathbb{D}^n, S^{n-1}) \underset{\text{Cor}}{\cong} \widetilde{H}_*(\mathbb{D}^n/S^{n-1}) \cong \widetilde{H}_*(S^n)$ ← $\mathbb{D}^n/S^{n-1} \cong S^n$

Exercise Check that the iso in the Cor is natural (for a map $(X,A) \to (Y,B)$ of good pairs get comm. diagram...

Recall we proved $H_*(\mathbb{D}^n, S^{n-1}) \cong \tilde{H}_{*-1}(S^{n-1})$ (from LES & $\tilde{H}_*(\mathbb{D}^n)=0$)

$\Rightarrow$ inductively, using Example

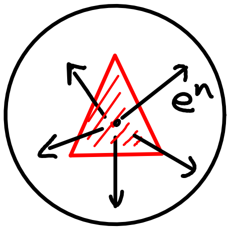
$$\tilde H_k(S^n) \cong \tilde H_{k-1}(S^{n-1}) \cong \dots \cong \tilde H_{k-n}(\underset{\underset{2\text{ points}}{\|}}{S^0}) \underset{\uparrow}{=} \begin{cases} \mathbb{Z} & n=k \\ 0 & \text{else} \end{cases}$$

$H_0(2\text{pts}) = \mathbb{Z}\oplus\mathbb{Z}$

Generator of $H_n(S^n) \cong \tilde H_n(\mathbb{D}^n/S^{n-1}) \cong H_n(\mathbb{D}^n, S^{n-1}) \cong \mathbb{Z}$

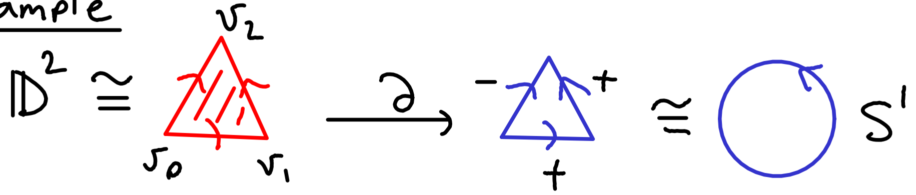
Observe $\exists$ homeo $e^n : \Delta^n \cong \mathbb{D}^n$ (homework) inducing Δ-cx structure on S^{n-1}:

$$\boxed{\partial\Delta^n \cong \partial\mathbb{D}^n = S^{n-1}}$$



stretch ctsly outwards from barycentre(Δ^n)

Example



Upshot ($n \geqslant 2$)

$$\boxed{\begin{aligned} H_n(\mathbb{D}^n, S^{n-1}) &= \mathbb{Z}\cdot e^n \\ H_{n-1}(S^{n-1}) &= \mathbb{Z}\cdot \partial e^n \\ \tilde H_n(\mathbb{D}^n/S^{n-1}) &= \mathbb{Z}\cdot [e^n] \end{aligned}}$$

LES

for $n-1 \geqslant 1$, so $n \geqslant 2$

by Cor

($[e^n]$ really lives in $H_n(\mathbb{D}^n, S^{n-1}) \cong H_n(\mathbb{D}^n/S^{n-1}, S^{n-1}/S^{n-1})$)

Exercise Recall another Δ-cx structure on S^n:

$$\boxed{S^n = \underset{\text{call this } \Delta_1}{\Delta^n} \sqcup \underset{\text{this } \Delta_0}{\Delta^n} / \text{glue along } \partial\Delta^n}$$

then $\boxed{H_n S^n = \mathbb{Z}\cdot(\Delta_1 - \Delta_0)}$ because $\Delta_1 - \Delta_0$ is a cycle and

$$\begin{array}{ccccccc} H_n(S^n) & \cong & H_n(S^n, \Delta_0) & \overset{\text{exc.}}{\cong} & H_n(\Delta_1, \partial\Delta_1) & \cong & \overset{\overset{H_n(S^n)}{\|}}{H_n(\mathbb{D}^n, \partial\mathbb{D}^n)} \\ \Delta_1 - \Delta_0 & \longmapsto & & & \Delta_1 & \longmapsto & e^n \end{array}$$

(by LES for (S^n, Δ_0) using $n \geqslant 1$)

Another remark about orientations

Fact $\{\text{homeos } \Delta^n \to \mathbb{D}^n\}$ has 2 path-components

Above we chose a path-component by constructing e^n.

If r is any reflection in $\mathbb{R}^{n+1}$ then $e^n \circ r$ is in the other path-component

e.g. swap 2 coordinates in Δ^n

$$\begin{array}{rcl} H_n(S^n) \cong H_n(\mathbb{D}^n, S^{n-1}) & \xrightarrow{\cong} & \mathbb{Z} \\ e^n & \longmapsto & +1 \\ e^n \circ r & \longmapsto & -1 \end{array}$$

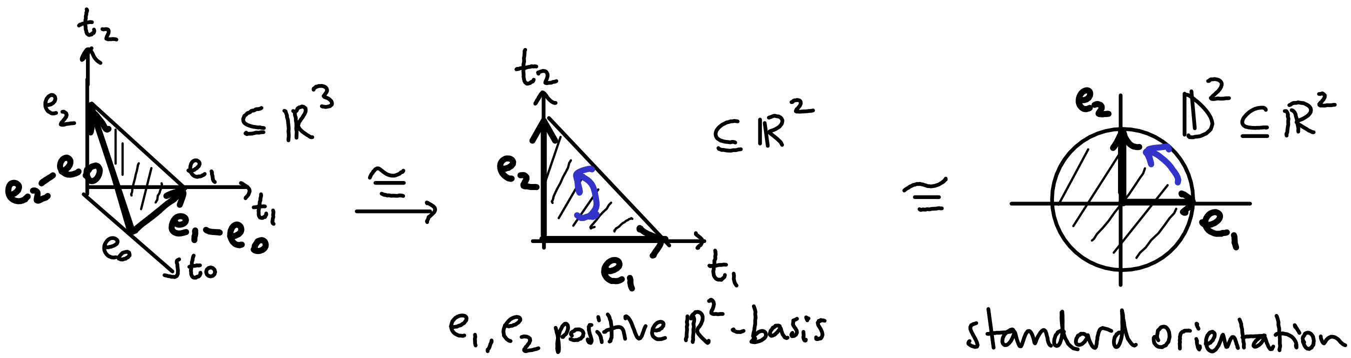
We will see later in the course that this corresponds to a choice of orientation of $\mathbb{D}^n$ and S^n.

Our choice is consistent with the inclusion $\mathbb{D}^n \subseteq \mathbb{R}^n$ (with the positive (canonical) orientation of $\mathbb{R}^n$) and the inclusion

$$(\Delta^n \subseteq \mathbb{R}^{\underline{\underline{n+1}}}) \xrightarrow{\cong} \{(t_1,\dots,t_n) \in \mathbb{R}^{\underline{\underline{n}}} : t_i \geq 0, \sum_{i=1}^{n} t_i \leq 1\}$$
$$(\underline{\underline{t_0}},\dots,t_n) \longmapsto (\underline{\underline{t_1}},\dots,t_n)$$
$$t_i \geq 0, \sum t_i = 1$$

Example

$\Delta^2 = [e_0,e_1,e_2]$

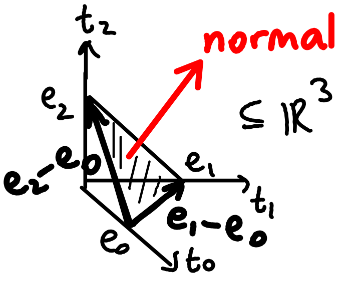


Our choice is also consistent with the "normal first" convention for orienting hyperplanes with a given choice of normal:

$\Delta^n \subseteq$ hyperplane $\{(t_0,\dots,t_n) : \sum t_j = 1\} \subseteq \mathbb{R}^{\underline{\underline{n+1}}}$ normal $(1,1,\dots,1)$ (so pointing to ∞ in positive quadrant)

Example

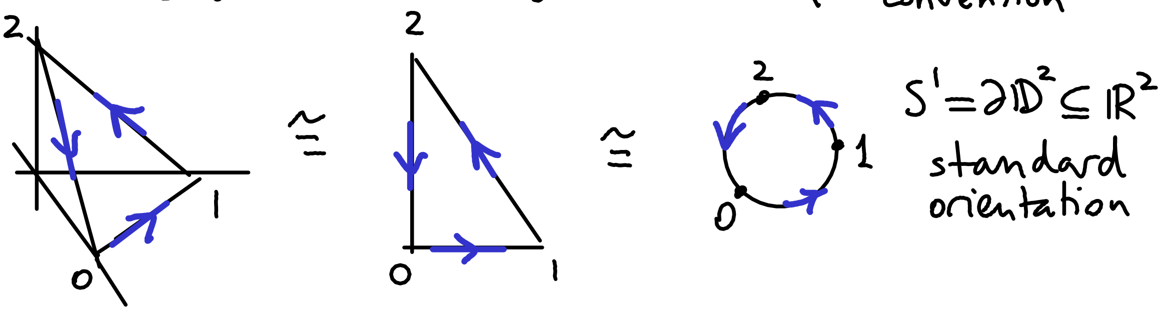
$\Delta^2 = [e_0,e_1,e_2]$



normal, e_1-e_0, e_2-e_0 positive $\mathbb{R}^3$-basis

Consistent also with the geometric boundary orientation (outward normal first convention)

$\partial_{\text{geometric}}\Delta^2 =$



Compare $\partial\Delta = +[\hat{e}_0, e_1, e_2] - [e_0, \hat{e}_1, e_2] + [e_0, e_1, \hat{e}_2]$

This $-[e_0, e_2]$ is not equal to singular chain $[e_2, e_0]$ since they are different maps and we take free abelian group generated by maps. But $[e_0, e_2] + [e_2, e_0]$ is homologous to 0 (Homework).

Locality (or "small simplices theorem")

$\mathcal{U} = \{$ subspaces $U_i \subseteq X \}$ whose interiors cover X:

$$X = \bigcup \mathcal{U}_i^{\circ}$$

Def $C_*^{\mathcal{U}}(X) \subseteq C_*(X)$ subcx generated by n-simplices σ with $\sigma(\Delta^n) \subseteq U_i$ some i

Theorem

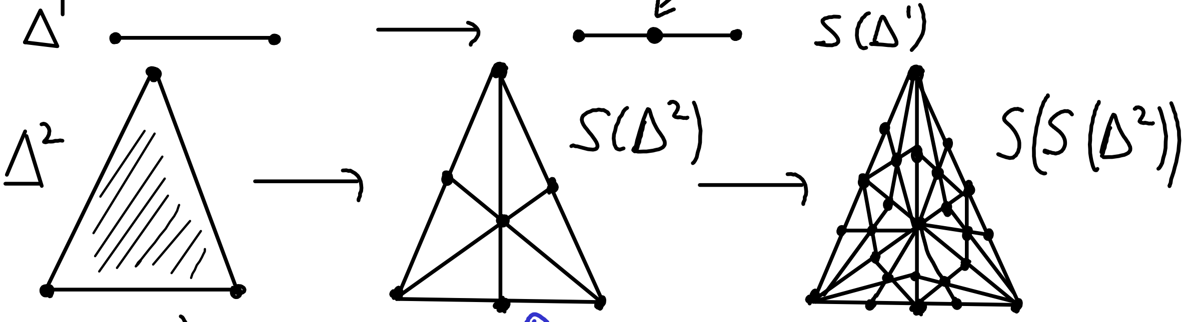
$$\boxed{H_*(C_*^{\mathcal{U}}(X)) \cong H_*(C_*(X)) = H_* X}$$

barycentre of $[v_0, \dots, v_n]$ is $\frac{1}{n+1}(v_0 + \dots + v_n)$

Sketch Pf ① Barycentric subdivision

Non-examinable

barycentre divides edge in 2



$\Rightarrow$ chain map $S : C_*(X) \longrightarrow C_*(X)$

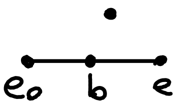
$$\sigma \longmapsto \text{"}\sigma \circ S\text{"}$$

and $S(C_*^{\mathcal{U}}) \subseteq C_*^{\mathcal{U}}$

subdivide the boundary (inductively by dimension) then draw the new faces obtained by convex combinations involving the new vertices and the barycentre

Construction of "$\sigma \circ S$" is inductive:

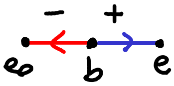
On linear simplices (then for maps σ you restrict $\sigma|_{\dots}$)

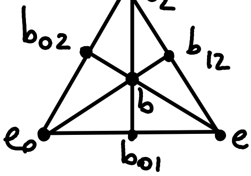


$S[e_0] = [e_0]$

$S[e_0, e_1] = [b, e_1] - [b, e_0]$ $\quad$ (= "$[b, S\partial[e_0, e_1]]$")

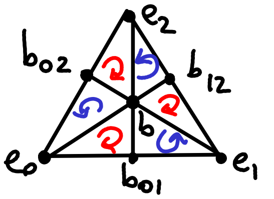
geometrically





$$\begin{aligned} S[e_0, e_1, e_2] &= \text{"}[b, S\partial[e_0, e_1, e_2]]\text{"} \\ &= \text{"}[b, S[e_1, e_2]] - [b, S[e_0, e_2]] + [b, S[e_0, e_1]]\text{"} \\ &= ([b, b_{12}, e_2] - [b, b_{12}, e_1]) - ([b, b_{02}, e_2] - [b, b_{02}, e_0]) \\ &\quad + ([b, b_{01}, e_1] - [b, b_{01}, e_0]) \end{aligned}$$

geometrically:

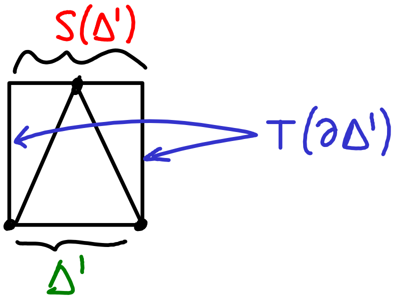


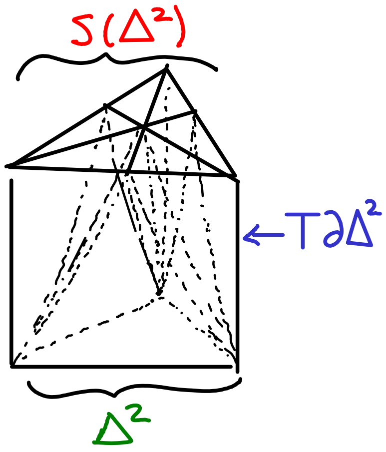
so for $\sigma : \Delta^2 \to X$ you take $S(\sigma) = \sigma|_{[b, b_{12}, e_2]} - \sigma|_{[b, b_{12}, e_1]} - \cdots$

② S chain hpic to id:

$$\left.\begin{array}{l} T : C_n(X) \longrightarrow C_{n+1}(X) \\ T(\sigma) : \Delta^n \times I \xrightarrow{\text{project}} \Delta^n \xrightarrow{\sigma} X \\ \underline{\text{exercise}}: \ \partial T + T\partial = S - id \end{array}\right\} \Rightarrow \boxed{S_* : H_*(X) \xrightarrow{id} H_*(X)}$$

Idea:

$\Delta^1 \times I =$ 

$\Delta^2 \times I =$ 

③ ∀ n-simplex $\sigma : \Delta^n \to X$, apply $S(.)$ enough times until σ(each n-simplex of subdivision) $\subseteq \mathcal{U}_i$ some i

③ $\Rightarrow$ ∀ cycle c, ∃ n s.t. $S^n(c) \in C^{\mathcal{U}}_*(X)$ cycle

$\Rightarrow H^{\mathcal{U}}_*(C) \longrightarrow H_*(X)$ surjective

$[S^n(c)] \longmapsto S^n_*[c] = [c]$ by ②

③ $\Rightarrow$ ∀ bdry $c = \partial b$, ∃ n s.t. $S^n(b) \in C^{\mathcal{U}}_*(X)$

<u>claim:</u> $H^{\mathcal{U}}_*(C) \longrightarrow H_*(X)$ injective

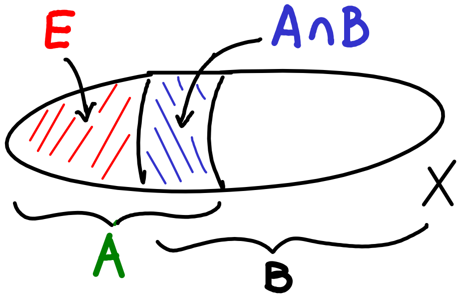
suppose $[c] \longmapsto 0$ then $c = \partial b$ for $b \in C_*(X)$

now $S^n c, S^n b \in C^{\mathcal{U}}_*(X)$ for large n

$\Rightarrow \partial S^n b = S^n \partial b = S^n c$ in $C^{\mathcal{U}}_*(X)$

$\Rightarrow [c] \underset{②}{=} S^n_*[c] = [S^n c] = [\partial S^n b] = 0$ in $H^{\mathcal{U}}_*(X)$ ✓ □

Proof of excision theorem



Let $B = X \setminus E$

use $\mathcal{U} = \{A, B\}$

so $C^{\mathcal{U}}_*(X) = C_*(A) + C_*(B) \subseteq C_*(X)$

$$\Rightarrow \frac{C_*(X \setminus E)}{C_*(A \setminus E)} = \frac{C_*(B)}{C_*(A \cap B)} \cong \frac{C_*(B)}{C_*(A) \cap C_*(B)} \underset{\text{2nd isomorphism theorem for groups}}{\cong} \frac{C^{\mathcal{U}}_*(X)}{C_*(A)}$$

$\Rightarrow$ Compare LES's:

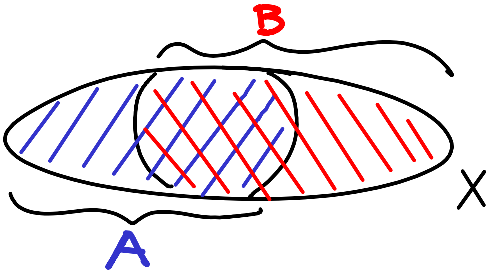
$$\begin{tikzcd}
 & & H_*(X \setminus E, A \setminus E) \arrow[d, phantom, "\cong \leftarrow \text{by above isos}"] & & \\
H_*(A) \arrow[r] \arrow[d, equal] & H_*(C^{\mathcal{U}}_* X) \arrow[r] \arrow[d, "\text{locality}"', "\cong"] & H_*(C^{\mathcal{U}}_* X / C_* A) \arrow[r] \arrow[d, "\text{iso by 5-lemma}"] & H_{*-1}(A) \arrow[r] \arrow[d, equal] & H_{*-1}(C^{\mathcal{U}} X) \arrow[d, "\text{locality}"', "\cong"] \\
H_*(A) \arrow[r] & H_*(X) \arrow[r] & H_*(C_* X / C_* A) \arrow[r] \arrow[d, equal] & H_{*-1}(A) \arrow[r] & H_{*-1}(X) \\
 & & H_*(X, A) & &
\end{tikzcd}$$

(we are using naturality of LES's induced by SES's)

□

6. MAYER-VIETORIS SEQUENCE ← Key computational tool

$X = \underbrace{A \cup B}_{\text{any subspaces}}$ s.t. $X = A^\circ \cup B^\circ$



MV Theorem ∃ LES:

$$\boxed{\cdots \longrightarrow H_*(A\cap B) \xrightarrow{i_*} H_*(A)\oplus H_*(B) \longrightarrow H_*(X) \longrightarrow H_{*-1}(A\cap B) \xrightarrow{i_*[-1]} \cdots}$$

& same holds for $\tilde{H}_*$ provided $A\cap B\neq\emptyset$.

Pf SES $0 \longrightarrow C_*(A\cap B) \longrightarrow C_*(A)\oplus C_*(B) \longrightarrow C_*^{\mathcal{U}}(X) \longrightarrow 0$ $\quad \mathcal{U}=\{A,B\}$

$\sigma \longmapsto (\sigma, -\sigma)$

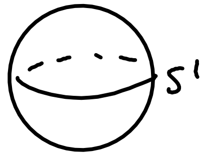
$(\alpha, \beta) \longmapsto \alpha+\beta$

⇒ induces the LES (using locality $H_*^{\mathcal{U}}X \cong H_*X$). □

Exercise connecting map is $\delta: H_*(X) \longrightarrow H_{*-1}(A\cap B)$

$[\alpha+\beta] \longmapsto [\partial\alpha] = -[\partial\beta]$

Example S^2



$A \simeq pt$

$B \simeq pt$

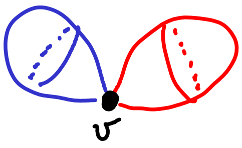
$A\cap B \simeq S^1$

$$\cdots \to \underset{0}{\underset{\parallel}{H_2(pt)\oplus H_2(pt)}} \to \underset{\text{hence } \mathbb{Z}}{\underset{\uparrow}{H_2S^2}} \longrightarrow \underset{\mathbb{Z}}{\underset{\wr\parallel}{H_1(S^1)}} \longrightarrow \underset{0}{\underset{\parallel}{H_1(pt)\oplus H_1(pt)}} \to \cdots$$

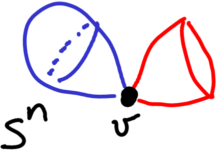
Exercise Compute $H_*S^n = \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z} & *=n \\ 0 & \text{else} \end{cases}$ using MV

Example **wedge sum** of X, Y with basepoints $x \in X$, $y \in Y$ $\qquad X\vee Y = \dfrac{X\sqcup Y}{x\sim y}$

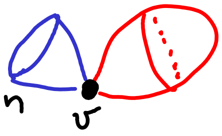
$X = S^n \vee S^n =$



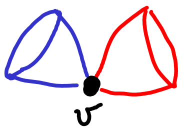
$A =$ $\simeq S^n$



$B =$ $\simeq S^n$



$A\cap B =$ $\simeq pt$

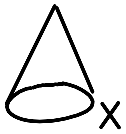


$$0 \longrightarrow \mathbb{Z}\oplus\mathbb{Z} \xrightarrow[\cong]{} H_n(X) \longrightarrow 0 \to \cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}\oplus\mathbb{Z} \longrightarrow \underset{\mathbb{Z}}{\underset{\wr\parallel}{H_0(X)}} \longrightarrow 0$$

$1 \longmapsto (1,-1)$

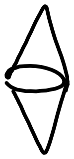
Similarly $\boxed{H_*(X\vee Y) \cong H_*(X)\oplus H_*(Y) \text{ for } *\neq 0}$ if ∃ contractible nbhds of $x\in X$, of $y \in Y$.

Cones and suspensions



$CX = (X \times [0,1]) \,/\, (x,s)\sim(y,t)$ iff equal or $s=t=1$

$\simeq pt$

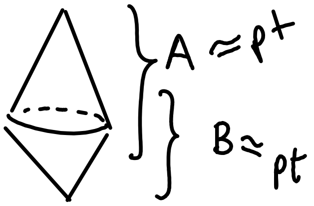


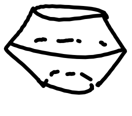
$\Sigma X = (X \times [0,1]) \,/\, (x,s)\sim(y,t)$ iff equal or $s=t=0$ or $s=t=1$

Example $CS^n \cong D^{n+1}$, $\Sigma S^n \cong S^{n+1}$.

Lemma $\boxed{\tilde{H}_*(\Sigma X) \cong \tilde{H}_{*-1}(X)}$

Pf





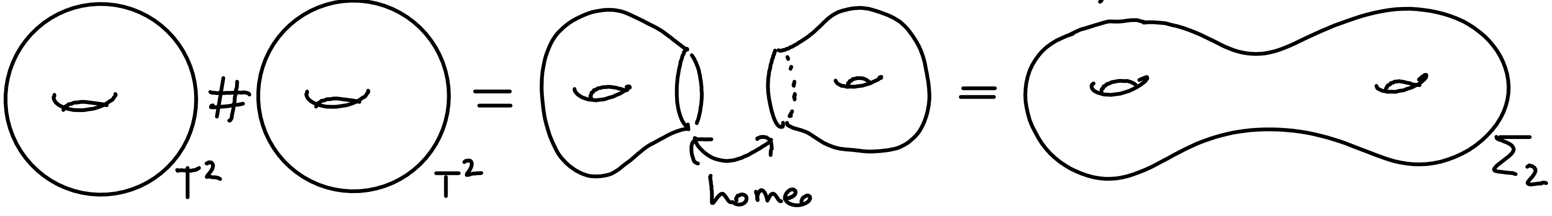
$A \cap B \simeq X$ now apply MV. □

Rmk $\emptyset \neq A \subseteq X \Rightarrow \tilde{H}_*(X \cup_A CA) \overset{LES}{\cong} H_*(X \cup_A CA, CA) \overset{exc.}{\cong} H_*(X,A)$

identify $a \in A \subseteq X$ with $(a,0) \in CA$

Connected sum

M, N connected n-manifolds $\Rightarrow$ $M \# N = \underbrace{(M \setminus \text{open } n\text{-ball}) \cup (N \setminus \text{open } n\text{-ball})}_{\text{identify } \partial \text{ balls via a homeo}}$



Fact compact connected orientable surfaces are homeo to S^2 or $\underbrace{T^2 \# \dots \# T^2}_{g = \# \text{ copies}}$

and " " non-orientable ones : $\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2$.

genus $g=0$

called Σ_g

Exercise (Homework) For M,N compact connected n-mfds:

By MV, $\boxed{H_*(M\#N) \cong H_*(M) \oplus H_*(N) \quad \text{for } 1 \leq * \leq n-2}$

If M or N orientable: $*=n-1$ also works

If both non-orientable: $*=n-1$ one of $\mathbb{Z}/2$ summands becomes $\mathbb{Z}$

$H_0(M\#N) \cong \mathbb{Z}$ since connected

fact: $H_n(M\#N)$ is $\mathbb{Z}$ or 0
if M,N both orientable
else
(see later in course)

Cor 1) $\chi(M\#N) = \chi(M) + \chi(N) - \underbrace{\begin{cases} 2 & \text{for } n \text{ even} \\ 0 & \text{" " odd} \end{cases}}_{\chi(S^n)}$

2) $H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}^{2g} & *=1 \\ \mathbb{Z} & *=2 \end{cases}$ ← genus g

7. DEGREE OF MAPS OF SPHERES

$$\begin{tikzcd} f: S^n \arrow[r] & S^n & \Longrightarrow & f_n: H_nS^n \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H_nS^n \arrow[d, phantom, "\cong" rotate=90] \\ & & & \mathbb{Z} \arrow[r] & \mathbb{Z} \\ & & & 1 \arrow[r, mapsto] & \underline{\deg(f)} \in \mathbb{Z} \end{tikzcd}$$

$\Rightarrow f_*: \tilde{H}_*S^n \to \tilde{H}_*S^n$ is $\deg(f)\cdot id$

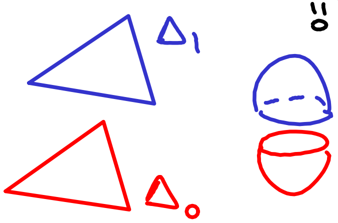
Properties
1) $\deg(id) = 1$
2) $\deg(f\circ g) = \deg f \cdot \deg g$
3) $f \simeq g \Rightarrow \deg f = \deg g$
4) $f \simeq const \Rightarrow \deg f = 0$
5) f homeomorphism $\Rightarrow \deg f = \pm 1$ ← sign depends on whether f is orientation-preserving or reversing

Pf
$id_* = id$, $(f\circ g)_* = f_* \circ g_*$, $f \simeq g \Rightarrow f_* = g_*$, $const_* = 0$, f homeo $\Rightarrow f_n$ iso. □
since $S^n \to pt \to S^n$ factors so $H^nS^n \to \underset{0}{H^n(pt)} \to H^nS^n$

Examples

1) $S^n = \dfrac{\Delta^n \times 1 \cup \Delta^n \times 0}{(b,1)\sim(b,0) \text{ if } b\in\partial\Delta}$

call this Δ_1 ($\Delta^n \times 1$); call this Δ_0 ($\Delta^n \times 0$)



recall $H_nS^n = \mathbb{Z}\cdot(\Delta_1 - \Delta_0)$

reflection: $r: S^n \to S^n$, $r(x,t) = (x, 1-t)$

so $\Delta_0 \leftrightarrow \Delta_1$ swapped by r, so $r_*(\Delta_1 - \Delta_0) = -(\Delta_1 - \Delta_0)$

$\Rightarrow \deg(r) = -1$

2) antipodal map $-id: S^n \to S^n$ viewing $S^n \subseteq \mathbb{R}^{n+1}$

$\Rightarrow \boxed{\deg(-id) = (-1)^{n+1}}$

Pf $-id = \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots \end{pmatrix} \circ \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} \circ \cdots \circ \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$ composition of $n+1$ reflections each homotopic to r. □

3) $A \in O(n) \Rightarrow A: S^{n-1} \to S^{n-1} \Rightarrow \deg A = \det A \in \{\pm 1\}$

Pf **fact** $SO(n)$ is path-connected so $A \in SO(n)$ is $\simeq id$ so $\deg A = \det A = +1$
The other path-component of $O(n)$ is $r \circ SO(n)$ where r is any reflection. □

4) $\boxed{f \text{ not surjective} \Longrightarrow \deg f = 0}$

Pf If $y \notin \operatorname{Im} f \Rightarrow$

$$\begin{tikzcd} H_n(S^n) \arrow[rr, "f_*"] \arrow[dr, "f_*"'] & & H_n(S^n) \\ & H_n(S^n\setminus y) \cong H_n(\mathbb{R}^n) = 0 \arrow[ur] & \end{tikzcd}$$

□

Application to vector fields on S^n

$v: S^n \longrightarrow \mathbb{R}^{n+1}$ tangent vector field on S^n

so $v(x) \perp x$

Cor **Hairy ball theorem** $\boxed{\exists \text{ nowhere zero v.f. on } S^n \iff n \text{ odd}}$

(case $n=2$: "you cannot comb a ball of hair without creating a tuft")

Pf Suppose $v(x) \neq 0 \quad \forall x$

$\Rightarrow$ hpy $F: S^n \times [0,1] \longrightarrow S^n$

$$F(x,t) = \cos(\pi t)\, x + \sin(\pi t) \frac{v(x)}{\|v(x)\|}$$

$\Rightarrow F_0 = id, \quad F_1 = -id$

$\Rightarrow 1 = \deg F_0 = \deg F_1 = (-1)^{n+1}$

$\Rightarrow n$ odd

For n odd $\exists$ $v(x) = (-x_2, x_1, \dots, -x_{2k}, x_{2k-1}) \in \mathbb{R}^{\overset{n=2k-1}{2k}}$. $\square$

Cultural Remark Adams in 1962 proved using alg. topology:

(max # pointwise linearly independent vector fields on S^n) $= 2^b + 8a - 1$

where $n+1 = 2^{4a+b} \cdot$ (odd number), $0 \leq b \leq 3$, $a,b \in \mathbb{N}$, $n \geq 1$.

get 0 if n even $\Rightarrow$ Cor ✓

Local degree

$$\begin{tikzcd} f: S^n \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & S^n \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\ x \arrow[r] & y = f(x) \end{tikzcd}$$

(★) Suppose points $\neq x$ near x do not map to y:

$\exists$ nbhds $x \in \underset{D^n}{\underset{\cong}{U}}$, $y \in \underset{D^n}{\underset{\cong}{V}}$ s.t. $\boxed{(U, U\setminus x) \xrightarrow{f} (V, V \setminus y)}$

call this $f|_x$ **local map at x**

$$\begin{tikzcd} \Rightarrow (f|_x)_*: H_n(U, U\setminus x) \arrow[r, "f_*"] & H_n(V, V\setminus y) \arrow[ddd, equal] \\ H_n(S^n, \underbrace{S^n \setminus x}_{\simeq \mathbb{R}^n \simeq \text{point}}) \arrow[u, "\cong", "\text{excise } S^n\setminus U"'] & \\ \tilde{H}_n S^n \arrow[u, "\cong"] & \\ \mathbb{Z} \arrow[u, "\cong"] \arrow[r] & \mathbb{Z} \\ 1 \arrow[r, mapsto] & \deg_x f \end{tikzcd}$$

will use this again later:

$\tilde{H}_n(S^n) \underset{\text{quotient}}{\cong} H_n(S^n, S^n \setminus pt)$

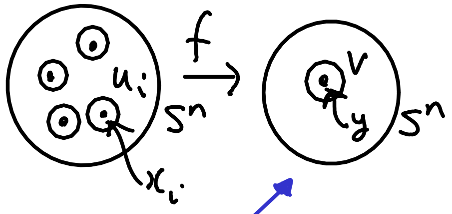
Lemma

$f: S^n \to S^n$, $f^{-1}(y) = \{x_1, -, x_k\}$

$$\Rightarrow \boxed{\deg f = \sum \deg_{x_i} f}$$

Pf

$$\begin{tikzcd}
\tilde{H}_n S^n \arrow[r, "f_*"] \arrow[d, "\text{quotient}"'] & \tilde{H}_n(S^n) \arrow[d, "\text{quotient}", "\cong"'] \\
H_n(S^n, S^n \setminus \{x_1, \to x_k\}) \arrow[r, "f_*"] \arrow[d, "\text{exc. } S^n \setminus \cup U_i"', "\cong"] & H_n(S^n, S^n \setminus y) \arrow[d, "\text{exc. } S^n \setminus V", "\cong"'] \\
\bigoplus H_n(U_i, U_i \setminus x_i) \arrow[r, "\bigoplus (f|_{x_i})_*"] & H_n(V, V \setminus y)
\end{tikzcd}$$



Rmk
can use same V for all i by taking
$\tilde{V} = \cap V_i$
$\tilde{u}_i = f^{-1}(V) \cap U_i$

is:

$$\begin{tikzcd}
1 \arrow[d] & \in & \mathbb{Z} \arrow[r, "\deg f"] \arrow[d] & \mathbb{Z} \arrow[d] \\
(1, -, 1) & \in & \bigoplus_i \mathbb{Z} \arrow[r, "\oplus \deg_{x_i} f"'] & \mathbb{Z}
\end{tikzcd} \qquad \square$$

map to each summand is exc. of $S^n \setminus U_i$ so iso.

(the 2 squares commute:
1^{st}: quotient is natural
2^{nd}: excision is natural)

Example $p: \mathbb{C} \to \mathbb{C}$ polynomial $\overset{a_n \neq 0}{a_n} z^n + a_{n-1} z^{n-1} + \dots + a_0$, $a_i \in \mathbb{C}$

$$\Rightarrow f: S^2 = \mathbb{C}P^1 \to \mathbb{C}P^1 = S^2$$
$$z \mapsto p(z)$$
$$\infty \mapsto \infty$$

(where view $\mathbb{C}P^1 = \mathbb{C} \cup \infty \underset{\text{stereographic projection}}{\cong} S^2$)

$\Rightarrow$ hpy $F(z,t) = a_n z^n + t(a_{n-1} z^{n-1} + \dots + a_0)$

$F_0 = a_n z^n$ and $F_1 = f$

hpy is continuous at ∞ since $a_n z^n$ dominates other terms: $F^{-1}(\mathbb{C}P^1 \setminus K) = \mathbb{C}P^1 \setminus$(some compact set) $\forall$ compact K.

this would fail if you tried to homotope $t(a_n z^n) + a_{n-1} z^{n-1} + \dots$

$$\Rightarrow \deg f = \deg(a_n z^n) \overset{\text{Lemma}}{=} \sum_{k=0}^{n-1} \underbrace{\deg_{w_k} a_n z^n}_{=1} \leftarrow w_k = e^{\frac{2\pi i k}{n}}$$
$$= n$$
$$= \text{degree of the poly } p.$$

orient^n preserving homeo near w_k

holomorphic maps are always orientation preserving

Cor (Fundamental Thm of Algebra) $n \geq 1 \Rightarrow p$ has a root

Pf $p^{-1}(0) = \emptyset \Rightarrow f^{-1}(0) = \emptyset \Rightarrow \deg f = 0$ ↯ $\square$

Cultural Rmk For smooth $f: S^n \to S^n$

$\deg f =$ (the number of preimages of a generic point)

counted with orientation signs

(i.e. almost any point works)

Example $S^2 \to S^2$

$S^2 \setminus$ North Pole $\cong \mathbb{C}$

map $\mathbb{C} \to \mathbb{C}$, $z \mapsto z^d$ and North $\mapsto$ North

$\Rightarrow \deg = d = \#$ preimages of a point

except if pick North/South pole
$(z = \infty)$ $(z = 0)$

8. CELLULAR HOMOLOGY

Def **CW complex** X is sequence $\emptyset = X^{-1}\subseteq X^{0}\subseteq X^{1}\subseteq X^{2}\subseteq\cdots$

s.t. X^0 is any set

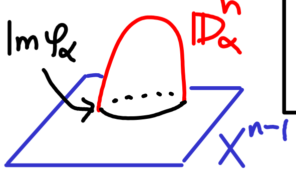
n-skeleton

$$\boxed{X^n = X^{n-1}\sqcup \bigsqcup_{\alpha\in I_n}\mathbb{D}^n_\alpha \Big/ x\sim\varphi_\alpha(x)}$$

n-discs labelled by some index set I_n

$\varphi_\alpha:\partial\mathbb{D}^n\longrightarrow X^{n-1}$ **attaching map**

(any continuous map, often not injective)



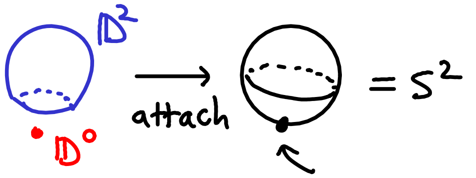
$\Longrightarrow X=\bigcup_{n\geq 0}X^n$ top. space with **weak topology**:

$U\subseteq X$ open $\iff U\cap X^n\subseteq X^n$ open $\forall n$.

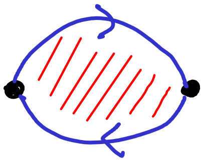
$(\iff U\cap\mathbb{D}^n_\alpha\subseteq\mathbb{D}^n_\alpha$ open $\forall n,\alpha)$

Call X **n-dimensional** if $X=X^n$ and this is the least such n.

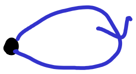
Example $S^n=(\mathbb{D}^0\sqcup\mathbb{D}^n)\,/\,(\mathbb{D}^0\sim\partial\mathbb{D}^n)$

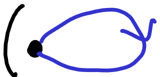
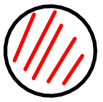
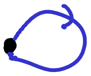


boundary $S^1=\partial\mathbb{D}^2$ identified with •

Example $X=\mathbb{R}P^2=$ 

$X^0=\bullet=\mathbb{D}^0$

$X^1=$  $=S^1=(\mathbb{D}^0\sqcup\mathbb{D}^1)/(\bullet\sim\varphi_1(x))$, $\partial\mathbb{D}^1=S^0=\{0,1\}\xrightarrow[\varphi_1]{}X^0=\bullet$

$X^2=($  $\sqcup$  $)\,/\,($wrap ∂ of  twice around $)$

$=(X^1\sqcup\mathbb{D}^2)\Big/\Big(\underset{z}{\overset{\partial\mathbb{D}^2=S^1}{\Downarrow}}\sim z^2\in X^1=S^1\Big)$ $\quad\partial\mathbb{D}^2=S^1\xrightarrow{\varphi_2}S^1$, $\varphi_2(z)=z^2$

Fact If we homotope φ_α, we get a homotopy equivalent space

Example If use another degree 2 map φ_2 above, get $X\simeq\mathbb{R}P^2$.

Cultural Rmk Every CW-cx X is hpy equivalent to a simplicial complex Y (so in particular a Δ-cx). [If X finite/n-dim then can ensure Y is finite/n-dim]

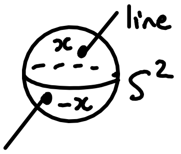
X is partitioned as a set by interiors of **n-cells** $\boxed{e^n_\alpha=\mathrm{Image}(\mathbb{D}^n_\alpha\to X)}$

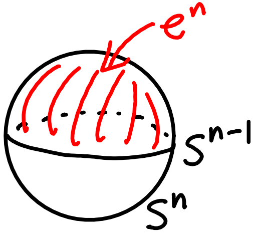
$$X^n=X^{n-1}\sqcup\bigsqcup_{\alpha\in I_\alpha}\mathring{e}^n_\alpha$$

$$=\Big(\bigsqcup_{\alpha\in I_0}e^0_\alpha\Big)\sqcup\Big(\bigsqcup_{\alpha\in I_1}\mathring{e}^1_\alpha\Big)\sqcup\Big(\bigsqcup_{\alpha\in I_2}\mathring{e}^2_\alpha\Big)\sqcup\cdots$$

Rmk interior $\mathbb{D}^0=\mathbb{D}^0$ so $\mathring{e}^0_\alpha=e^0_\alpha$

Examples real projective space $\mathbb{RP}^n = S^n / (\mathbb{Z}/2\text{- action by } \pm id)$ ← so $x \sim -x$





$X^k = \mathbb{RP}^k$ inductively

$X^n = X^{n-1} \cup e^n$ with $\varphi : S^{n-1} \to X^{n-1} = \mathbb{RP}^{n-1} = S^{n-1}/\pm id$

$x \longmapsto [x] = [-x]$

complex projective space

$\mathbb{CP}^n = (S^{2n+1} \subseteq \mathbb{C}^{n+1}) / (S^1\text{-action by } \lambda \cdot Id)$ ← $x \sim \lambda x$ for $\lambda \in S^1 \subseteq \mathbb{C}^*$

$X^0 = X^1 = pt = \mathbb{CP}^0$

$X^2 = X^3 = X^0 \cup e^2 = \mathbb{CP}^1$, $\varphi : S^1 \to pt$ ← $\mathbb{CP}^1 \cong$

$\cong S^2$

$X^4 = X^5 = X^2 \cup e^4 = \mathbb{CP}^2$, $\varphi : S^3 \to \mathbb{CP}^1 = S^3/(S^1\text{-action})$

$x \longmapsto [x] = [\lambda x], \forall \lambda \in S^1$

$X^{2n} = X^{2n-2} \cup e^{2n} = \mathbb{CP}^n$, $\varphi : S^{2n-1} \to X^{2n-2} = \mathbb{CP}^{n-1}$

$x \longmapsto [x]$

In coordinates: $\mathbb{CP}^n = \{[z_0 : \dots : z_n] : \text{not all } z_j \in \mathbb{C} \text{ are } 0\}$ and $[z] \sim [\lambda z], \forall \lambda \in \mathbb{C}^*$

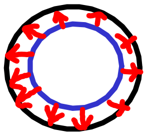
Can rescale so that $\sum |z_j|^2 = 1$ so $z \in S^{2n-1}$ and left with rescaling by $\lambda \in S^1 \subseteq \mathbb{C}^*$.

$\mathbb{CP}^{n-1} \cong X^{2n-2} = \{[z_0 : \dots : z_{n-1} : 0]\} \subseteq \mathbb{CP}^n = X^{2n}$ and

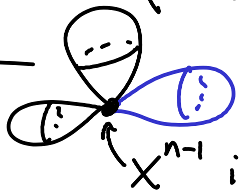
$e^{2n} : \mathbb{D}^{2n} = \{(w_0, \dots, w_{n-1}) : \sum |w_j|^2 \leq 1\} \longrightarrow X^{2n}$ via $[w_0 : \dots : w_{n-1} : \sqrt{1 - \sum |w_j|^2}]$ ← notice this $= 0$ if $w \in S^{2n-1} \cong \partial \mathbb{D}^{2n}$

Observe: For X CW complex, for $n \geq 1$: ← (For $n = 0$: $(X^0, X^{-1}) = (X^0, \emptyset)$
$X^0 / X^{-1} = X^0$)

- (X^n, X^{n-1}) is a good pair ← (since $\exists$ nbhd of $\partial \mathbb{D}^n$ that deformation retracts to $\partial \mathbb{D}^n$)



- $$\boxed{X^n / X^{n-1} \cong \bigvee_{\alpha \in I_n} S^n}$$ ←



$S^n_\alpha = \mathbb{D}^n_\alpha / \partial \mathbb{D}^n_\alpha$

X^{n-1} identified to a point

Def Cellular complex for X a CW cx,

$$\boxed{\begin{array}{l} C_n^{CW}(X) = H_n(X^n, X^{n-1}) \cong H_n(X^n / X^{n-1}) \\ \qquad = \text{free abelian gp gen. by the } n\text{-cells } e^n_\alpha \end{array}}$$

since $\Delta^n \cong \mathbb{D}^n \longrightarrow (e^n \subseteq X^n) \longrightarrow \mathbb{D}^n_\alpha / \partial \mathbb{D}^n_\alpha = S^n_\alpha$ generate ← (as usual we use the standard orientations of $\Delta^n, \mathbb{D}^n, S^n$.)

Will build cellular differential $d : C_*^{CW} \to C_{*-1}^{CW}$, prove $d \circ d = 0$

$\Rightarrow$ get $$\boxed{H_*^{CW}(X) = H_*(C_*^{CW}(X), d)}$$

Example $C_k^{CW}(\mathbb{CP}^n) = \begin{cases} \mathbb{Z}\cdot e^k & \text{for } k=0,2,4,\dots,2n \\ 0 & \text{else} \end{cases}$ hence $d=0$ so $H_*^{CW}(\mathbb{CP}^n) = C_*^{CW}(\mathbb{CP}^n)$ $\cong \begin{cases} \mathbb{Z} & 0 \le * \le 2n \text{ even} \\ 0 & \text{else} \end{cases}$

Cellular differential:

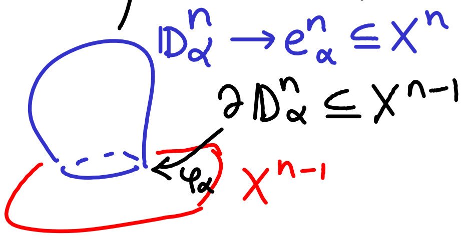
$$\boxed{d\, e^n_\alpha = \sum_{\beta \in I_{n-1}} d^n_{\alpha\beta} \cdot e^{n-1}_\beta}$$

now describe the coefficients $d^n_{\alpha\beta} \in \mathbb{Z}$ and why that is a finite sum.

$$\begin{tikzcd}
C_n^{CW} \arrow[rr, "d"] \arrow[d, equal] & & C_{n-1}^{CW} \arrow[d, equal] \\
H_n(X^n, X^{n-1}) \arrow[r, "\delta^n_*", "\text{connecting map of LES}"'] & H_{n-1}(X^{n-1}) \arrow[r, "q^{n-1}_*", "\text{quotient map}"'] & H_{n-1}(X^{n-1}, X^{n-2}) \\
e^n_\alpha \arrow[r, mapsto] & \left(\partial\mathbb{D}^n_\alpha \to \varphi_\alpha(S^{m-1})\right) \arrow[r, mapsto] & \sum d_{\alpha\beta}\, e^{n-1}_\beta
\end{tikzcd}$$

Recall LES

$$\begin{tikzcd}
H_*(X^{n-1}) \arrow[rr] & & H_*(X^n) \arrow[dl, "q^n_*"] \\
& H_*(X^n, X^{n-1}) \arrow[ul, "\delta^n_*"', "{[-1]}"] &
\end{tikzcd}$$



here it is important that we chose identifications $\Delta^n \cong \mathbb{D}^n$, $S^n \cong \mathbb{D}^n/\partial\mathbb{D}^n$ compatibly with orientations.

Therefore:

$$\boxed{d^n_{\alpha\beta} = \deg\Big(\underset{\substack{\| \\ \partial\mathbb{D}^n_\alpha}}{S^{n-1}} \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q} X^{n-1}/X^{n-2} \cong \bigvee_{I_{n-1}} S^{n-1} \xrightarrow{\text{Quotient by } \bigvee_{I_{n-1}\setminus\beta} S^{n-1}} \underset{\substack{\| \\ \mathbb{D}^{n-1}_\beta/\partial\mathbb{D}^{n-1}_\beta}}{S^{n-1}}\Big)}$$

Rmk Only finitely many $d^n_{\alpha\beta} \neq 0$ (for fixed α) because φ_α, q are continuous and S^{n-1} compact, so get a compact image in $\bigvee_\beta S^{n-1}$, therefore cannot surject onto ∞ many S^{n-1}_β.

recall if don't surject then deg=0

Lemma $d \circ d = 0$

Pf $d_n = q^{n-1}_{n-1} \circ \delta^n_n$

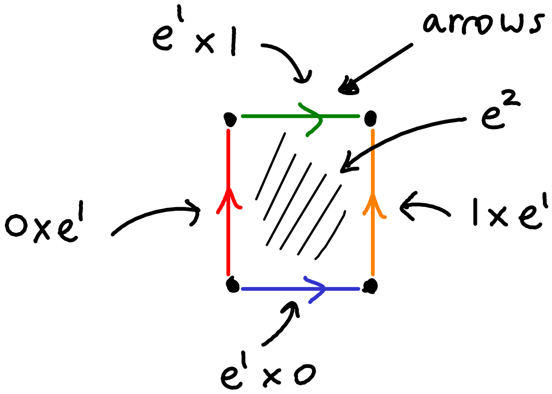
$$d_{n-1} \circ d_n = q^{n-2}_{n-2} \circ \overbrace{\delta^{n-1}_{n-1} \circ q^{n-1}_{n-1}}^{=0 \text{ by LES}} \circ \delta^n_n \qquad \square$$

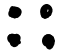
Cor $\boxed{\operatorname{rank} H^{CW}_n(X) \le \# n\text{-cells}}$

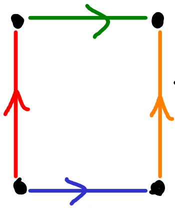
Pf $\# n\text{-cells} = \operatorname{rank} C^{CW}_n(X) \geq \operatorname{rank} \operatorname{Ker} d^{CW}_n \geq \operatorname{rank} H^{CW}_n(X)$ $\square$

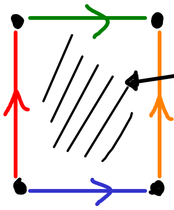
Example $I \times I$ $\quad I = [0,1]$ $\quad \mathbb{D}^1 = [-1,1]$

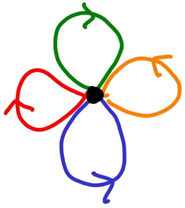
arrows here tell us how we map $[-1,1] \to$ edge (so orientation)

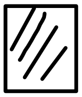


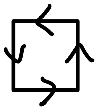
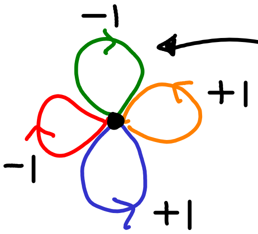
$X^0 =$  $= 4$ 0-cells

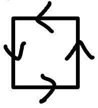
$X^1 =$  $\leftarrow$ 4 1-cells $e^1\times 0$, $1 \times e^1$, $e^1 \times 1$, $0 \times e^1$

$X^2 =$  $\leftarrow$ 2-cell e^2

$X^1/X^0 =$  orientations of cells tell us how to orient the circles

$e^2 : \mathbb{D}^2 \cong$  $\longrightarrow X^1$

$\partial e^2 : S^1 \cong$  $\longrightarrow X^1/X^0 =$ 

degree -1 because top edge of  maps to  by an orientation-reversing homeomorphism.

$$\Rightarrow \partial e^2 = +e^1 \times 0 + 1 \times e^1 - e^1 \times 1 - 0 \times e^1$$

$\left(= (\partial e^1) \times e^1 - e^1 \times (\partial e^1) \right.$ $\leftarrow$ we come back to this later$)$

Example $\mathbb{R}P^n$

recall: 1 cell in each dim, $\varphi : S^k \longrightarrow X^k = \mathbb{R}P^k$
$x \longmapsto [\pm x]$

$S^{k-1} =$ 

$$\Delta_1 \xrightarrow{\varphi} X^{k-1}/X^{k-2} = \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} \cong S^{k-1} \qquad \deg = +1$$

$$\Delta_2 \longrightarrow -\mathrm{id}(\Delta_1) \qquad \deg = (-1)^k$$

$$\Rightarrow d^k_{\alpha\beta} = \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

$C^{CW}_*(\mathbb{R}P^n)$

$$0 \to \underset{k=n}{\mathbb{Z}} \longrightarrow \mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z} \xrightarrow{2} \underset{k=1}{\mathbb{Z}} \xrightarrow{0} \underset{0}{\mathbb{Z}} \longrightarrow \underset{-1}{0}$$

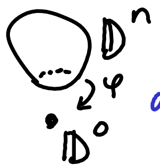
2 if n even
0 " odd

0
2 resp.

$$H^{CW}_*(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & * = 0, \text{ also } * = n \text{ if odd} \\ \mathbb{Z}/2 & * = 1,3,5,\dots < n \\ 0 & \text{else} \end{cases}$$

Example S^n: $C_*^{CW}(S^n)$: $n\geq 2$: $0 \to \mathbb{Z}\cdot\mathbb{D}^n \xrightarrow{0} 0 \to \cdots \to 0 \to \mathbb{Z}\cdot\mathbb{D}^0 \to 0$

$n=1$: $0 \to \mathbb{Z}\cdot\mathbb{D}^1 \xrightarrow{0} \mathbb{Z}\mathbb{D}^0 \to 0$



$\deg\varphi = 0$ $\Rightarrow H_*^{CW}(S^n) \cong \begin{cases} \mathbb{Z} & *=0,n \\ 0 & \text{else} \end{cases}$

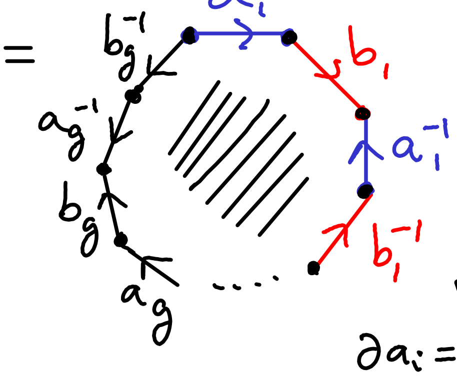
$H_1(S^1,pt) \xrightarrow{\delta} H_0(pt) \xrightarrow{q} H_0(pt,\emptyset)$

$(\Delta^1 \cong [0,1] \to S^1) \longmapsto \partial\sigma$
$\sigma = \text{quotient } 0\sim 1$ $\quad = pt - pt$
$= 0$

if you work with degrees, need to remember orientations:
$\partial\mathbb{D}^1 \cong \partial[0,1] = [1]-[0] \to \text{point}$
so degree $= +1-1=0$

Example $\Sigma_g =$

genus g surface



boundary identifications
$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$

← Notice all vertices are identified, call vertex v

$\partial a_i = v - v = 0$
$\partial b_i = v - v = 0$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\partial} \mathbb{Z}^{2g} \xrightarrow{\partial=0} \mathbb{Z} \longrightarrow 0$$

$\mathbb{Z} \cong \mathbb{Z}\cdot\mathbb{D}$, $\mathbb{Z}^{2g} \cong \mathbb{Z}\langle a_1, b_1, \ldots, a_g, b_g\rangle$, $\mathbb{Z} \cong \mathbb{Z}\cdot v$

$\mathbb{D} \longmapsto -a_1 - b_1 + a_1 + b_1 + \cdots - a_g - b_g + a_g + b_g = 0$

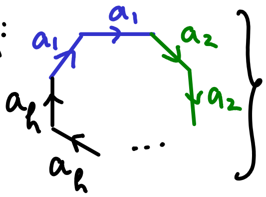
(signs: compare edge orientation with anticlockwise orientation of $\partial\mathbb{D}$)

$\Rightarrow H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z} & *=0,2 \\ \mathbb{Z}^{2g} & *=1 \\ 0 & \text{else} \end{cases}$

Example Non-orientable surface N_h:

$0 \to \mathbb{Z} \to \mathbb{Z}^h \xrightarrow{0} \mathbb{Z} \to 0$
$1 \mapsto (-2,\ldots,-2)$

(Since $\mathbb{D} \mapsto -a_1 - a_1 - a_2 - a_2 - \cdots - a_h - a_h$)



$\Rightarrow H_*(N_h) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}^{h-1} \oplus \mathbb{Z}/2 & *=1 \\ 0 & \text{else} \end{cases}$

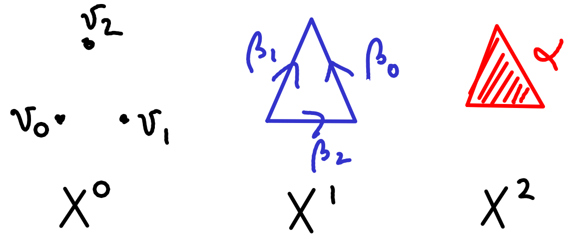
(use the standard basis for $\mathbb{Z}^h$ except replace $(0,\ldots,0,1)$ by $(1,1,\ldots,1)$.)

Lemma X Δ-cx structure $\Longrightarrow$ induces CW-cx structure on X and
$(C_*^{CW}(X), d^{CW}) \cong (C_*^{\Delta}(X), d^{\Delta})$

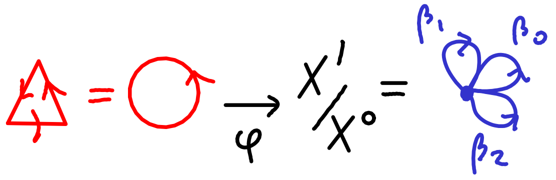
$$\Longrightarrow \boxed{H_*^{CW}(X) \cong H_*^{\Delta}(X)}$$

Pf $X^n = \bigcup n\text{-simplices of } X$ and degrees are ± 1 depending on orient^n
$\cong \mathbb{D}^n$ so can identify d^{CW} and d^{Δ}. ☐

Example $X = \text{triangle} = \Delta^2$



$\Rightarrow d^{\Delta}\alpha = \beta_0 - \beta_1 + \beta_2$



$d_{\alpha\beta_2} = d_{\alpha\beta_0} = +1$, $d_{\alpha\beta_1} = -1$

$\Longrightarrow d^{CW}\alpha = d^{\Delta}\alpha$ ✓ ☐

Theorem X CW cx (or Δ-cx) $\Longrightarrow$ $\boxed{H_*^{CW}(X) \cong H_*(X)}$

$\Longrightarrow$ H_*^{Δ}, H_*^{CW} independent of choice of CW-cx/Δ-cx structure.

Pf ① $H_*(X^n, X^{n-1}) \cong \widetilde{H}_*(X^n/X^{n-1}) \cong \widetilde{H}_*(\bigvee_\alpha S^n) \cong \bigoplus_\alpha \widetilde{H}_* S^n$

$= 0 \iff * \neq n$ $\qquad$ lives in degree n

LES for (X^n, X^{n-1}) $\Longrightarrow$ $H_*(X^{n-1}) \to H_*(X^n)$ iso for $* < n-1$, $* > n$

② for $* < n$: $H_*(X^n) \cong H_*(X^{n+1}) \cong H_*(X^{n+2}) \cong \dots \cong H_*(X)$

by ① $\qquad$ by compactness each sing. chain lands in X^N, some N

③ for $* > n$: $H_*(X^n) \cong H_*(X^{n-1}) \cong H_*(X^{n-2}) \cong \dots \cong H_*(X^{-1}) = 0$

④ LES: $\quad \dots \to \underset{\overset{\|}{0}}{H_n(X^{n-1})} \longrightarrow H_n(X^n) \xrightarrow[q_n^n]{} H_n(X^n, X^{n-1}) \longrightarrow \dots$

by ③

$\Longrightarrow$ $\underline{q_n^n \text{ injective } \forall n}$

⑤ LES: $\dots \xrightarrow{q_{n+1}^{n+1}} H_{n+1}(X^{n+1}, X^n) \xrightarrow{\delta_{n+1}^{n+1}} H_n(X^n) \longrightarrow H_n(X^{n+1}) \to \overset{①}{0}$

UPSHOT

$$
\begin{aligned}
H_n(X) &\overset{②}{\cong} H_n(X^{n+1}) \\
&\overset{⑤}{\cong} H_n(X^n) / \operatorname{im} \delta_{n+1}^{n+1} \qquad \text{1st iso thm} \\
&\overset{④}{\cong} \underbrace{(q_n^n H_n(X^n))}_{\operatorname{im} q_n^n} / \operatorname{im} \underbrace{q_n^n \circ \delta_{n+1}^{n+1}}_{d_{n+1}^{CW}} \cong H_n^{CW}(X)
\end{aligned}
$$

$$\operatorname{im} q_n^n \overset{\text{exactness LES}}{=} \operatorname{Ker} \delta_n^n \overset{④}{=} \operatorname{Ker} \underbrace{q_{n-1}^{n-1} \circ \delta_n^n}_{d_n^{CW}}$$

□

Rmk by ① $\boxed{H_k \text{ } \underline{\text{not}} \text{ affected if attach } (k+2)\text{-cells or higher}}$

by ② $\boxed{\text{Inclusion } X^n \to X \text{ induces iso } H_*(X^n) \to H_*(X) \text{ for } * < n}$

Cor X n-dimensional cell cx $\Longrightarrow$ $H_*(X) = 0$ for $* > n$

Axioms for homology: Eilenberg–Steenrod axioms

It was no accident that H_*^{Δ}, H_*^{CW}, H_* all agreed.

Def A **generalised homology theory (GHT)**

is a **functor** $F : \text{TopPairs} = \begin{pmatrix}\text{Category of pairs} \\ \text{of spaces, and} \\ \text{maps of pairs}\end{pmatrix} \longrightarrow \text{Graded Abelian Gps}$

with a **natural transformation** $\delta : F_*(X,A) \longrightarrow \underbrace{F_{*-1}(X,\emptyset)}_{\text{abbreviated: } F_{*-1}(X)}$ satisfying:

1) **homotopy invariance**: $f \simeq g \Rightarrow F(f) = F(g)$

2) **exactness**: $\exists$ LES $\boxed{\cdots \to F_*(A) \xrightarrow[F(\text{incl}: A \to X)]{} F_*(X) \xrightarrow[F(\text{incl}: (X,\emptyset) \to (X,A))]{} F_*(X,A) \xrightarrow{\delta} F_{*-1}(A) \to \cdots}$

3) **additivity**: $(X,A) = \bigsqcup (X_i, A_i)$, $\text{incl}_i : (X_i, A_i) \longrightarrow (X,A)$

then $\boxed{\sum F(\text{incl}) : \bigoplus F(X_i, A_i) \xrightarrow{\cong} F(X,A)}$

4) **excision**: $\boxed{\overline{E} \subseteq A^\circ \subseteq X \implies F(X \setminus E, A \setminus E) \xrightarrow[F(\text{incl})]{\cong} F(X,A)}$

Remark (4) $\Longleftrightarrow$ $X = A^\circ \cup B^\circ$, $\text{incl} : (B, A \cap B) \longrightarrow (X,A)$

then $\boxed{F(\text{incl}) : F(B, A \cap B) \xrightarrow{\cong} F(X,A)}$

Pf $B = X \setminus E$, $E = X \setminus B$ noticing that $(X \setminus E)^\circ \cup A^\circ = X$

$E = A \setminus B$ noticing that $\overline{E} \subseteq \overline{A} \setminus B^\circ \subseteq A^\circ \setminus B^\circ \subseteq A^\circ$. □

$X = A^\circ \cup B^\circ$
So $\partial B \subseteq A^\circ$

Rmk In (3), the topology on the disjoint union $\bigsqcup (X_i, A_i)$ is defined by: $U \subseteq \bigsqcup (X_i, A_i)$ open $\Longleftrightarrow$ $U \cap X_i \subseteq X_i$ open $\forall i$

FACT **Theorem**

a) $(F, \delta_F), (G, \delta_G)$ GHTs, $\alpha : F \to G$ a natural transformation commuting with δ_F, δ_G such that $\alpha_{\text{point}} : F(\text{point}) \to G(\text{point})$ is an iso, then α is an iso.

b) If (F, δ_F) GHT satisfies (5) **dimension**: $\boxed{F_*(\text{point}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}}$

Then $\exists$ natural iso $F \cong H_*$ (such an F is called a **homology theory**)

Rmk In (b) if require $F_0(\text{point}) = G$ an abelian group (instead of $\mathbb{Z}$)

$\Longrightarrow F(X,A) \cong H_*(X,A;G) =$ (homology with coefficients in G) ← later in course

9. COHOMOLOGY

(C_*, ∂_*) chain cx s.t. C_* free $\mathbb{Z}$-module ← $C_* \cong \bigoplus_\alpha \mathbb{Z}$

Def n-cochains $\boxed{C^n = \mathrm{Hom}(C_n, \mathbb{Z})}$

coboundary map (this is the dual of ∂) $\partial^n : C^n \longrightarrow C^{n+1}$

$$\partial^n(\phi) = \phi \circ \partial_{n+1}$$

$$\begin{tikzcd}
C_n \arrow[d, "\phi"'] & C_{n+1} \arrow[l, "\partial_{n+1}"'] \arrow[dl, "\partial^*\phi = \phi\circ\partial"] \\
\mathbb{Z} &
\end{tikzcd}$$

Notice ∂^* is degree +1 map (not −1)

$$\boxed{H^m(C_*, \partial_*) = \frac{\ker \partial^m \leftarrow \text{cocycles}}{\operatorname{Im}\partial^{m-1} \leftarrow \text{coboundaries}}}$$

← (Note $\partial^* \circ \partial^* = 0$: $\partial^*\partial^*\phi = \phi \circ \underbrace{\partial \circ \partial}_{=0} = 0$)

Rmk If use negative grading, (C^{-*}, ∂^{-*}) is a chain complex with homology so many results from H_* carry over to H^*. So abusively write

$$\boxed{H^*(C_*, \partial_*) = H_*(\mathrm{Hom}(C_*, \mathbb{Z}), \partial^*)}$$

Warning A cochain $\varphi \in C^*$ takes values $\varphi(c) \in \mathbb{Z}$ on chains $c \in C_*$. However the cohomology class $\alpha = [\varphi] \in H^*$ does not have a well-defined value on c: $[\varphi] = [\varphi + \partial^*(\psi)]$ and $(\varphi + \partial^*\psi)(c) = \varphi(c) + \psi(\partial_* c)$. If c is a cycle, so $\partial_* c = 0$ then $\alpha(c) = \varphi(c)$ is well-defined, so $\exists$ pairing $H^* \times H_* \to \mathbb{Z}$

Remarks about dualisation

$\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) \cong \mathbb{Z}^n$ generated by projection maps $\pi_i(x_1, \dots, x_n) = x_i$

← this is the dual of the standard basis: $\pi_i = e_i^*: e_i \to 1$, $e_k \to 0,\ k \neq i$

$$\begin{tikzcd}
\alpha : \mathbb{Z}^n \arrow[r] & \mathbb{Z}^m \\
x \arrow[r, mapsto] & A \cdot x
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Hom}(\mathbb{Z}^m, \mathbb{Z}) \arrow[l, "\text{dual}"', "\alpha^*"] \arrow[d, phantom, "\cong" rotate=90] \\
\mathbb{Z}^n & \mathbb{Z}^m \arrow[l, "\text{transpose}(A)"]
\end{tikzcd}
\qquad \alpha^*\phi = \phi \circ \alpha$$

↑ mxn matrix

Def X space ⟹ singular cohomology $\boxed{H^*(X) = H^*(C^*(X), \partial^*)}$

similarly define H^*_Δ, H^*_{CW}

↑ dualise $C_* = C_*(X)$

Example $\mathbb{RP}^3$: $C_*^{CW}(\mathbb{RP}^3)$: $0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$

dualise: $C^*_{CW}(\mathbb{RP}^3)$: $0 \longleftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \longrightarrow 0$

$$H^*(\mathbb{RP}^3) \cong H^*_{CW}(\mathbb{RP}^3) = \begin{cases} \mathbb{Z} & * = 0, 3 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$$

← notice $H_1(\mathbb{RP}^3) \cong \mathbb{Z}/2$ has moved to grading 2.

Functoriality

$f : X \longrightarrow Y \quad \Longrightarrow \quad f_* : C_* X \longrightarrow C_* Y$

$\Longrightarrow \quad C^*X \xleftarrow[f^*]{} C^*Y$ is dual : $\boxed{f^*\phi = \phi \circ f_*}$ ↙ called **pull-back**

Lemma f^* is a **cochain map** (meaning $\partial^* \circ f^* = f^* \circ \partial^*$)

$\Longrightarrow \quad \boxed{f^* : H^*Y \longrightarrow H^*X}$

Pf
$$\begin{aligned}
(\partial^* \circ f^*)(\phi) &= \partial^*(\phi \circ f_*) \\
&= (\phi \circ f_*) \circ \partial \\
&= (\phi \circ \partial) \circ f_* \qquad \text{as } f_* \text{ chain map} \\
&= f^* \circ (\phi \circ \partial) \\
&= f^* \circ (\partial^* \phi) \\
&= (f^* \circ \partial^*)(\phi)
\end{aligned}$$

Properties
- $id^* = id$
- $(f \circ g)^* = g^* \circ f^*$ notice order!

$\Longrightarrow \quad \boxed{H^* : \text{Top} \longrightarrow \text{GradedAbGps}}$ contravariant functor

Exercise $H^0(X) = \prod_{\pi_0 X} \mathbb{Z}$ where $\pi_0 X = \{\text{path-components of } X\}$

Homotopy invariance

Lemma $f_*, g_* : \underbrace{C_* \longrightarrow \tilde{C}_*}_{\text{free}}$ chain hpic $\Longrightarrow$ $f^* = g^* : H^*\tilde{C} \longrightarrow H^*C$

Pf $f_* - g_* = \tilde{\partial} \circ h + h \circ \partial$ some $h : C_* \longrightarrow \tilde{C}_*[1]$

$f^* - g^* = h^* \circ \tilde{\partial}^* + \partial^* \circ h^*$ for dual $h^* : \tilde{C}^* \longrightarrow C^*[-1]$.

(notice degree -1, not $+1$) □

Def h^* called **cochain homotopy**

Cor $f \simeq g : X \longrightarrow Y \quad \Longrightarrow \quad f^* = g^* : H^*Y \longrightarrow H^*X$ □

Algebra: dual of SES

Lemma $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ exact, A,B,C free

$\Longrightarrow \quad 0 \leftarrow A^* \xleftarrow{i^*} B^* \xleftarrow{j^*} C^* \leftarrow 0$ exact ←

Cultural Remark
$\left(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}\right)^* = \prod_{n\in\mathbb{N}} \mathbb{Z}^*$ is not free.
(Baer 1937)
so A^*, B^*, C^* are not free unless A,B,C have finite ranks

Pf C free $\Rightarrow$ $\exists$ splitting
$$\begin{tikzcd} B \arrow[r, "j"] & C \arrow[l, bend left, "s"] \end{tikzcd}$$
$j \circ s = id$

↑ Pick preimages b_i for basis e_i of C, then $s(e_i) = b_i$

$\Rightarrow A \oplus C \xrightarrow[i\oplus s]{\cong} B$

dual $\Rightarrow$ $A^* \oplus C^* \xleftarrow[i^*\oplus s^*]{\cong} B^*$ and $s^* \circ j^* = id$

→ so i^* surj → so j^* inj

Rmk inverse is
$B \cong A \oplus C$
$b \longmapsto i^{-1}(b - s(b)) \oplus j(b)$

$$\Rightarrow \begin{tikzcd} 0 & A^* \arrow[l] & B^* \arrow[l, "i^*"'] \arrow[r, bend right, "s^*"'] & C^* \arrow[l, "j^*"'] & 0 \arrow[l] \end{tikzcd}$$

where $0 = (j \circ i)^* = i^* \circ j^*$ so $\text{Im}\, j^* \subseteq \text{Ker}\, i^*$

prove $\supseteq$: $i^* b = 0 \Rightarrow b - j^* s^* b \in \text{Ker}\, i^* \cap \text{Ker}\, s^* = \{0\}$

$\Rightarrow b = j^* s^* b \in \text{Im}\, j^*$

↑ since $s^* j^* = id$ ← since $i^* \oplus s^*$ is iso.

$\Rightarrow \text{Ker}\, i^* = \text{Im}\, j^*$ □

Relative cohomology

$H^*(X,A) = H^*(\text{Hom}(C_*(X,A), \mathbb{Z}))$ ←

recall $C_*(X,A) = C_*(X)/C_*(A)$
and homs $C_*(X)/C_*(A) \longrightarrow \mathbb{Z}$
correspond precisely to homs $C_*(X) \to \mathbb{Z}$
which vanish on $C_*(A)$.
So relative cocycles are cocycles on X
which vanish on chains in A.

Excision, LES, Mayer-Vietoris

By previous Lemma get dual results:

Excision $\overline{E} \subseteq A^\circ \subseteq X \Rightarrow H^*(X \setminus E, A \setminus E) \xleftarrow[i^*]{\cong} H^*(X,A)$

LES for pair (X,A) $\cdots \xleftarrow{q^*[+1]} H^{*+1}(X,A) \xleftarrow{\delta} H^*(A) \xleftarrow{i^*} H^*(X) \xleftarrow{q^*} H^*(X,A) \leftarrow \cdots$

M.V. $X = A^\circ \cup B^\circ \Rightarrow \cdots \leftarrow H^{*+1}(X) \leftarrow H^*(A\cap B) \xleftarrow[i_A^* \oplus -i_B^*]{} H^*(A) \oplus H^*(B) \xleftarrow[j_A^* \oplus j_B^*]{} H^*(X) \leftarrow \cdots$

where
$$\begin{tikzcd} & A \arrow[dr, "j_A"] & \\ A \cap B \arrow[ur, "i_A"] \arrow[dr, "i_B"'] & & X \\ & B \arrow[ur, "j_B"'] & \end{tikzcd}$$
are the obvious maps

Axioms for cohomology

These are analogous to the axioms for homology except we reverse all arrows, and we change axiom (3): $\prod$ instead of $\bigoplus$

additivity: $(X,A) = \bigsqcup (X_i, A_i)$, $\text{incl}_i : (X_i, A_i) \longrightarrow (X,A)$

then $\boxed{\prod F(\text{incl}_i) : \prod F(X_i, A_i) \xleftarrow{\cong} F(X,A)}$

10. CUP PRODUCT

Theorem $\overset{\text{space}}{H^*(X)}$ is unital ① graded-commutative ② ring via

$\cup : H^k(X) \times H^\ell(X) \longrightarrow H^{k+\ell}(X)$ determined by

$$\boxed{\begin{gathered}\cup : C^k(X) \times C^\ell(X) \longrightarrow C^{k+\ell}(X) \\ (\phi \cup \psi)(\sigma) = \phi(\sigma|_{[e_0,\dots,\underline{\underline{e_k}}]}) \cdot \psi(\sigma|_{[\underline{\underline{e_k}},\dots,e_{k+\ell}]})\end{gathered}}$$

① $1 \in C^0(X)$ constant function $\Rightarrow$ $1 \cup \phi = \phi \cup 1 = \phi$

② $\phi \cup \psi = (-1)^{\deg\phi \cdot \deg\psi} \psi \cup \phi$

Useful trick If X is Δ-cx, $C_*^\Delta(X) \xrightarrow[\simeq]{\text{inclusion}} C_*(X)$, so $C^*_\Delta(X) \xleftarrow[\simeq]{\text{restriction}} C^*(X)$

$\phi \circ \text{incl} \longmapsfrom \phi$

and can define cup product on $C^*_\Delta(X)$ so that:

$$\begin{tikzcd} H^*_\Delta(X) \times H^*_\Delta(X) \arrow[r, "\cup"] & H^*_\Delta(X) \\ H^*(X) \times H^*(X) \arrow[u, "\cong"] \arrow[r, "\cup"] & H^*(X) \arrow[u, "\cong"'] \end{tikzcd}$$

$\leftarrow$ at chain level

$$(\phi \cup \psi)([v_0,\dots,\underset{n=\ell+k}{v_n}]) = \phi([v_0,\dots,v_k]) \cdot \psi([v_k,\dots,v_n])$$

So you can compute cup products on $H^*(X)$ by picking simplicial cocycle representatives: so define values on the simplicial chains defining the Δ-cx structure, and use

(cannot do this for $C_*^{CW}(X)$ because there is no meaningful analogue for D^n of the "bottom face" $[e_0,\dots,e_k]$ and "top face" $[e_k,\dots,e_n]$)

Proof of Theorem

$$\begin{aligned}\partial^*(\phi \cup \psi)(\sigma) &= (\phi \cup \psi)(\partial\sigma) \\ &= (\phi \cup \psi) \sum (-1)^i \sigma|_{[e_0,\dots,\hat{e}_i,\dots,e_n]} \qquad n = k+\ell \\ &= \sum_{i \leq k} (-1)^i \phi(\sigma|_{[e_0,\dots,\hat{e}_i,\dots,\underline{\underline{e_{k+1}}}]}) \cdot \psi(\sigma|_{[\underline{\underline{e_{k+1}}},\dots,e_n]}) \\ &\quad + \sum_{i > k} (-1)^i \phi(\sigma|_{[e_0,\dots,e_k]}) \cdot \psi(\sigma|_{[e_k,\dots,\hat{e}_i,\dots,e_n]}) \cdot \underbrace{(-1)^{i-k}(-1)^{k-i}}_{1} \\ &= ((\partial^*\phi) \cup \psi(\sigma)) + (-1)^k \phi \cup \partial^*\psi\end{aligned}$$

induces $[\phi] \cup [\psi] = [\phi \cup \psi]$:

well-defined:
- cycles→cycle: $\partial(\phi \cup \psi) = \overbrace{(\partial\phi)}^{=0} \cup \psi \pm \phi \cup \overbrace{(\partial\psi)}^{=0} = 0$ ✓
- $[\phi] = [\phi + \partial\alpha]$ so need $[(\partial\alpha) \cup \psi] = 0$

  $(\partial\alpha) \cup \psi = \partial(\alpha \cup \psi)$ ✓ (using $\partial\psi = 0$)
- Similarly $[\phi] \cup [\partial\beta] = 0$

bilinear, associative, distributive: true at chain level unital: $(\partial 1)(\sigma) = 1(\sigma|_{[e_1]}) - 1(\sigma|_{[e_0]}) = 1-1=0$

$(1\cup\psi)(\sigma) = 1(\sigma|_{[e_0]})\cdot\psi(\sigma|_{[e_0,\to e_n]}) = \psi(\sigma)$ $\quad$ ($\phi\cup 1=\phi$ similar)

graded-comm. sketch proof: ⟵ non-examinable

Let $r : C_n(X)\longrightarrow C_n(X)$, $r(\sigma) = \varepsilon_n\,\overline{\sigma}$ $\quad$ where: $\varepsilon_n = (-1)^{\frac{n(n+1)}{2}}$

and $\overline{\sigma}|_{[v_0,\to v_n]} = \sigma|_{[v_n,\dots,v_0]}$ ⟵ reverse order of vertices: is product of $\underbrace{n+(n-1)+\dots+1}_{n(n+1)/2}$ transpositions

(idea: each transposition is a reflection in hyperplane, so reverses orientation of simplex, so insert ε_n to compensate)

one checks:
- r chain map
- $\underbrace{r^*\varphi\cup r^*\psi}_{\varepsilon_k\varepsilon_\ell} = \underbrace{r^*(\psi\cup\varphi)}_{\varepsilon_{k+\ell}}$
  differ by $(-1)^{k\ell}$
- $r\simeq id$ so can drop $r^* = id$ on cohomology

$\Big($ $r - id = P\partial + \partial P$ with $\quad$ v_i, w_i like for prism operator

$P\sigma = \sum(-1)^i\,\varepsilon_{n-i}\,(\sigma\circ\pi)|_{[v_0,\dots,v_i,\underline{\underline{w_n}},\dots,w_i]}$ $\Big)$ $\quad\square$

projection $\Delta^n\times I\xrightarrow{\pi}\Delta^n$

Naturality of cup product

Lemma $f: X\longrightarrow Y \implies f^*: H^*Y\longrightarrow H^*X$ hom of unital rings

Pf

$$\begin{aligned} f^*(\varphi\cup\psi)(\sigma) &= (\varphi\cup\psi)(f_*\sigma) \\ &= \varphi(f_*\sigma|_{[e_0,\dots,e_k]})\cdot\psi(f_*\sigma|_{[e_k,\dots,e_n]}) \\ &= ((\varphi\circ f_*)\cup(\psi\circ f_*))(\sigma) \\ &= (f^*\varphi\cup f^*\psi)(\sigma) \end{aligned}$$

unital: $f^*(1) = 1\circ f_* = 1$ $\quad\square$

UPSHOT

$H^*: \mathrm{Top}\longrightarrow$ {Graded-commutative unital rings with graded unital ring homs}

contravariant functor.

Warning An (iso)morphism $H^*(Y)\longrightarrow H^*(X)$ of groups will also preserve the ring structure if f^* is induced by a map of spaces $X\longrightarrow Y$ (by above Lemma).

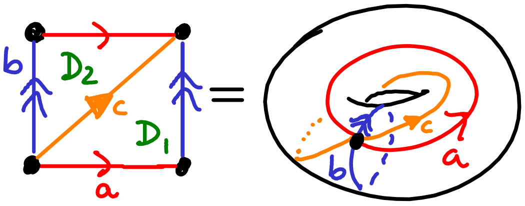
$\implies$ Cor The excision theorem iso on cohomology is an iso of rings.

However the connecting hom in M.V. or LES cannot possibly be a ring hom since it raises gradings by 1 ($\Rightarrow \delta(a\cup b)$ and $\delta(a)\cup\delta(b)$ have different grading!)

Example $H^1(T^2)\times H^1(T^2)\xrightarrow{\cup} H^2(T^2)$ is bilinear form $\mathbb{Z}^2\times\mathbb{Z}^2\to\mathbb{Z}$ with matrix $\begin{pmatrix}0 & +1\\ -1 & 0\end{pmatrix}$

Pf By the Useful Trick, it is enough to work with H^*_Δ instead of H^*.

Δ-complex structure for T^2:



$$C^\Delta_*:\ 0\to\mathbb{Z}^2\xrightarrow{\begin{pmatrix}1&1\\1&1\\-1&-1\end{pmatrix}}\mathbb{Z}^3\xrightarrow{0}\mathbb{Z}\to 0$$

gens: D_1, D_2 ; gens: a,b,c

dualise:

$$C^*_\Delta:\ 0\leftarrow\mathbb{Z}^2\xleftarrow{\begin{pmatrix}1&1&-1\\1&1&-1\end{pmatrix}}\mathbb{Z}^3\xleftarrow{0}\mathbb{Z}\to 0$$

gens: D_1^*, D_2^* ; gens: a^*,b^*,c^* ← dual basis for basis a,b,c (e.g. $a^*(a)=1$, $a^*(b)=0$, $a^*(c)=0$)

$\Longleftarrow$

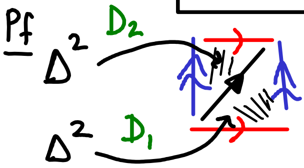
$*$		$H^\Delta_*(T^2)$	$H^*_\Delta(T^2)$
0	$\mathbb{Z}$	$\mathbb{Z}\cdot pt$	$\mathbb{Z}\cdot 1$
1	$\mathbb{Z}^2$	$\mathbb{Z}a\oplus\mathbb{Z}b$	$\mathbb{Z}\cdot(a^*+c^*)+\mathbb{Z}\cdot(b^*+c^*)$
2	$\mathbb{Z}$	$\mathbb{Z}\cdot(D_1-D_2)$	$\mathbb{Z}\cdot D_1^*$

← abbreviate $\begin{cases}A=a^*+c^*\\ B=b^*+c^*\end{cases}$

← (Remark $[D_1^*]=-[D_2^*]$ in $H^2_\Delta(T^2)$)

Claim $\boxed{A\cup B=D_1^*}$

Pf



$$(A\cup B)(D_1)=A(D_1|_{[e_0,e_1]})\cdot B(D_1|_{[e_1,e_2]})=A(a)B(b)=1\cdot 1=1$$

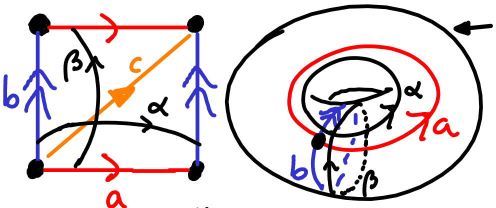
$$(A\cup B)(D_2)=A(b)B(a)=0\ .\ \square$$

(can also check these by hand)

Graded-comm. $\Rightarrow B\cup A=-D_1^*$, $A\cup A=(-1)^{1\cdot 1}A\cup A$ so $=0$, sim'ly $B\cup B=0$. $\square$

Rmk Recall that to specify a cochain in $C^k_\Delta(X)$ one needs to specify values on all generators of $C_k^\Delta(X)$ so not just on generators of $H_k^\Delta(X)$ (e.g. above A and a^* agree on gens a,b of $H_1^\Delta(T^2)$ but disagree on $c\in C_1^\Delta(T^2)$, note a^* is 1-cochain $\in C^1_\Delta(T^2)$ but is not a 1-cocycle)

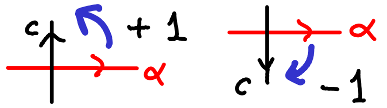
Some (but not all) k-cochains φ can be specified by drawing a "nice" $(n-k)$-dimensional subspace $\Sigma\subseteq X$ and defining $\boxed{\varphi(c)=\#(\text{times }\Sigma\text{ intersects }c)}$ for all $c\in C_k^\Delta(X)$ where one must explain with what sign ± an intersection point is counted and one has ensured that Σ intersects the generators of $C_*^\Delta(X)$ in a finite # points.



← We obtained the curves α,β by "pushing off" the curves a,b respectively away from themselves. Note the endpoints of α (and β) are the same so it is a loop (hence a 1-cycle in T^2).

$\Rightarrow$ get 1-cochains $\varphi_\alpha,\varphi_\beta\in C^1_\Delta(T^2)$:

$\varphi^*_\alpha(\underset{\nwarrow C_1^\Delta}{c})=\#\ \alpha$ intersects c counted with orientation signs:



↖Written $\alpha\cdot c$, called **intersection pairing**

Notice $\varphi_\alpha(a)=0$, $\varphi_\alpha(b)=1$, $\varphi_\alpha(c)=1$ so $\varphi_\alpha=B$. Similarly, $\varphi_\beta=-A$.

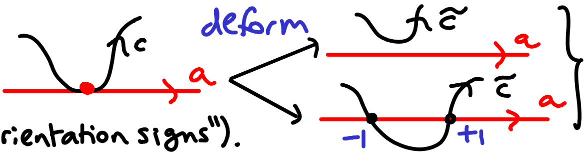
↙**Non-examinable** comments about intersection numbers

Fact Since T^2 is an orientable manifold, $\varphi_\alpha\cup\varphi_\beta=(\alpha\cdot\beta)\,\mathrm{vol}$ where vol is a generator of $H^2(T^2)$. Later in the course: vol is the "Poincaré dual" of the point class, and corresponds to the dual of the oriented sum of the top simplices. Above: $\mathrm{vol}=D_1^*$ and

$$\varphi_\alpha\cup\varphi_\beta=B\cup(-A)=A\cup B=(\alpha\cdot\beta)\,\mathrm{vol}=\mathrm{vol}=D_1^*.$$

Defining intersection numbers rigorously is tricky, even when using smooth chains. ↙("tangencies" are the key issue)

One can calculate $\varphi_\Sigma(c)$ on a cycle c by first deforming c to a smooth homologous cycle $\tilde{c}$ which is "transverse" to Σ, and then we count intersection points $\Sigma\cap\tilde{c}$ (with "orientation signs").

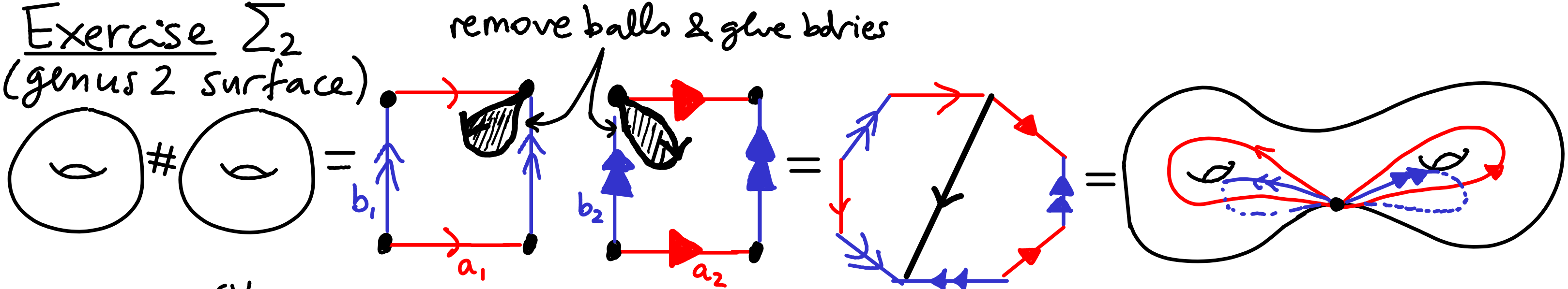


} in both cases $\varphi_a(c):=\varphi_a(\tilde{c})=0$

The fact that we consider the intersection number $a\cdot a=0$ is because we can push a off itself:

Exercise Σ_2

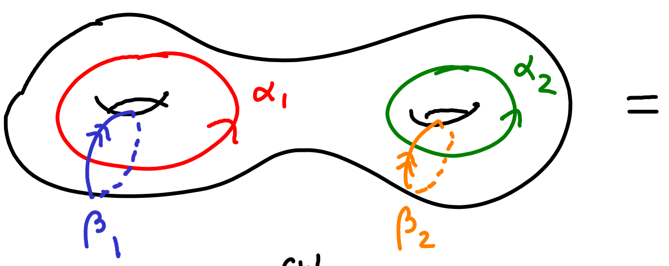
(genus 2 surface)

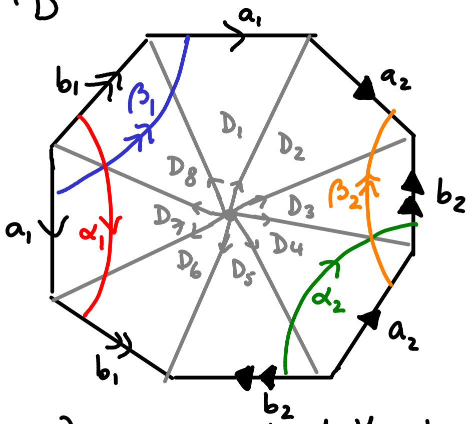


		$H_*^{CW}(\Sigma_2)$	$H^*_{cw}(\Sigma_2)$
0	$\mathbb{Z}$	$\mathbb{Z}\cdot pt$	$\mathbb{Z}\cdot 1$
1	$\mathbb{Z}^4$	$\mathbb{Z}a_1+\mathbb{Z}b_1+\mathbb{Z}a_2+\mathbb{Z}b_2$	$\mathbb{Z}\langle a_1^*, b_1^*, a_2^*, b_2^*\rangle$ ← dual basis in C^1_{cw}
2	$\mathbb{Z}$	$\mathbb{Z}\cdot D$	$\mathbb{Z}\cdot D^*$

(easy to define on $C^1_{cw}(\Sigma_2)$ but not so obvious on $C^1_\Delta(\Sigma_2)$)

Deform curves a_1, b_1, a_2, b_2 to get $\alpha_1, \beta_1, \alpha_2, \beta_2$:





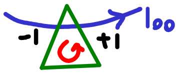
← Δ-cx structure on T^2

then notice for $c \in C_1^{cw}(\Sigma_2)$

$$a_i^*(c) = -\#(\beta_i \text{ intersects } c) \quad \text{(signed count)}$$
$$b_i^*(c) = \#(\alpha_i \quad '' \quad '')$$

so can extend this to a definition of $a_i^*, b_i^* \in C^1_\Delta(\Sigma_2)$ by allowing $c \in C_1^\Delta(\Sigma_2)$. Check that a_i^*, b_i^* are 1-cocycles in $C^1_\Delta(\Sigma_2)$.

(key idea: a loop has total signed intersection # = 0 with the boundary of a triangle



)

Exercise: $a_i^* \cup b_j^* = \delta_{ij}\cdot D^* = -\,b_j^* \cup a_i^*$

$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

$a_i^* \cup a_i^* = b_i^* \cup b_i^* = 0$

so same as geometric intersection numbers of corresponding curves.

Hint: represent D as a sum ± triangles in last picture.

(orientation signs: $D = -D_1 - D_2 + D_3 + D_4 - D_5 + D_6 + D_7 - D_8$ using + if outer edge is oriented anticlockwise)

Cultural Rmk on general theory (Intersection Theory/Differential Topology)

M^m oriented m-manifold

$N^n \subseteq M^m$ oriented compact n-dim submfd $\Rightarrow$ $H_n(N) \xrightarrow{incl_*} H_n(M)$ ← see later in course

$\exists$ generator $[N] \longmapsto [N]$

N, M also smooth (see Differential Geometry course) $\Rightarrow$ $\omega_N \in H^{m-n}(M)$ counts # intersections with N (with signs)

$N_1^{n_1}, N_2^{n_2} \subseteq M^m$ compact oriented smooth submfds and **transverse** (= at every $p \in N_1 \cap N_2$ the tangent spaces to N_1, N_2 at p span the tangent space to M at p) $\Rightarrow$

(can always "homotope" N_1 (or N_2) to achieve transversality, and class ω_{N_1} does not change if homotope) (homotope the inclusion map)

(tang. space means the "best" vector space approximation at p determined by the local smooth coordinates.)

> $N_1 \cap N_2$ is a compact orientable mfd of dim $= n_1+n_2-m$
>
> $\omega_{N_1} \cup \omega_{N_2} = \omega_{N_1\cap N_2} \in H^{2m-n_1-n_2}(M)$
>
> In particular if $n_1+n_2=m$, and M connected, then $H^m(M) \cong \mathbb{Z}$ s.t. $\omega_{N_1} \cup \omega_{N_2} \longmapsto \#(N_1 \cap N_2) \in \mathbb{Z}$.
>
> In non-orientable case, this all holds if work over $\mathbb{Z}/2$

($N_1 \cap N_2$ has an orientation induced by N_1, N_2. If $n_1+n_2=m$, get $N_1\cap N_2$ is sum $\Sigma \pm$ points. Sign compares oriented bases of tang. spaces: $T_pM = T_pN_1 \oplus T_pN_2$.)

Fact (Thom 1954)

Not all $a \in H^j(M)$ arise as ω_N for connected compact oriented codim=j smooth submfd N

But $\exists N \in \mathbb{N}$ s.t. $N\cdot a$ does arise. They do arise for $H^*(M;\mathbb{Q}), H^*(M,\mathbb{R}), H^*(M;\mathbb{Z}/2)$

11. KÜNNETH FORMULA AND PRODUCT SPACES

Algebra : tensor products

e.g. abelian groups = $\mathbb{Z}$-mods
vector spaces/$\mathbb{F}$ = $\mathbb{F}$-mods

R ring (comm. with 1)

Def A, B R-modules ⇒ **Tensor product** is R-module

$$\boxed{A \underset{R}{\otimes} B = \langle (a\,,\,b) : \ a\in A,\ b\in B\rangle / \text{relations of } \underline{\text{bilinearity}} \ \& \ \underline{\text{rescaling}}}$$

(or $A\otimes B$) R-mod generated write $a\otimes b$ for its class

bilinearity : $(a_1+a_2)\otimes b = a_1\otimes b + a_2\otimes b$

$a\otimes (b_1+b_2) = a\otimes b_1 + a\otimes b_2$

"can move $r\in R$ across the $\otimes$ symbol"

rescaling : $r.\,(a\otimes b) = (ra)\otimes b = a\otimes (rb)$ $\forall r\in R$

- So general element looks like $\sum a_k\otimes b_k$ (finite sum) ← NOT UNIQUELY!
- Don't confuse with $A\times B$: e.g. $0\otimes b = 0$ $\forall b$

Rmk Can define $A\underset{R}{\otimes}B$ also by a universal property : for all R-mods C,

$$\boxed{\mathrm{Hom}_R(A\underset{R}{\otimes}B\,;C) \xrightarrow[\text{natural}]{\cong} \{R\text{-bilinear maps } A\times B\to C)}$$

Using above description of $A\otimes B$: $\varphi \longmapsto (A\times B\to C, (a,b)\mapsto \varphi(a\otimes b))$

Example ($R=\mathbb{F}$) V, W v.s./$\mathbb{F}$ ⇒ $V\otimes W$ v.s./$\mathbb{F}$ basis $v_i\otimes w_j$

basis v_i basis w_j $\dim_{\mathbb{F}} V\otimes W = \dim V\cdot \dim W$

Exercise V, W finite dim/$\mathbb{F}$ ⇒ $V^*\otimes W \cong \mathrm{Hom}_{\mathbb{F}}(V,W)$

Hint $f : V\to\mathbb{F}$, $w\in W$, $f\otimes w \longmapsto (V\to W,\ v\longmapsto f(v)\cdot w)$

e.g. $(\mathbb{R}^n)^*\otimes\mathbb{R}^m \cong \mathrm{Mat}_{m\times n}(\mathbb{R}) \cong \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^m)$
$e_i^*\otimes e_j \longleftrightarrow$ matrix A with $A_{ji}=1$, 0 else.

Examples ($R=\mathbb{Z}$)

- $\mathbb{Z}^n \otimes \mathbb{Z}^m \cong \mathbb{Z}^{n\cdot m}$
- $\mathbb{Z}/n \otimes \mathbb{Z} \cong \mathbb{Z}/n \longleftarrow \sum a_i\otimes b_i = (\sum a_ib_i)\otimes 1$
- $\mathbb{Z}/2 \otimes \mathbb{Z}/3 = 0 \longleftarrow 1\otimes x = 3\otimes x = 1\otimes 3x = 0$
- $\mathbb{Z}/2 \otimes \mathbb{Z}/4 \cong \mathbb{Z}/2 \longleftarrow \begin{cases} 1\otimes 1 = 3\otimes 1 = 1\otimes 3 \\ 1\otimes 2 = 2\otimes 1 = 0 \end{cases}$

Examples

- $A\otimes B \cong B\otimes A$
- $(\bigoplus_i A_i)\otimes(\bigoplus_j B_j) \cong \bigoplus_i \bigoplus_j (A_i\otimes B_j)$
- $A\otimes R \cong A$ (so "$\cdot \underset{R}{\otimes} R$ does nothing")
- $A\otimes R/d \cong A/d\cdot A$

} hence now know $A\otimes B$ for any f.g. R-mods A, B.

for example $\mathbb{Z}/2\otimes\mathbb{Z}/4 \cong (\mathbb{Z}/4)/2\cdot\mathbb{Z}/4 \cong \mathbb{Z}/2$ ← (Rmk $(\mathbb{Z}/n)/m\cdot\mathbb{Z}/n \cong \mathbb{Z}/\gcd(m,n)$)

More generally: $\begin{cases} R/I \underset{R}{\otimes} R/J \cong R/(I+J) & \text{for ideals } I, J\subseteq R. \\ A\otimes_R R/J \cong A/J\cdot A \end{cases}$

Warning $\cdot\otimes A$ often not an exact functor, i.e. does not preserve exact sequences

indeed it can ruin injectivity: $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$ now take $\cdot \otimes \mathbb{Z}/2$ get $0 \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2$.

Fact $\boxed{\cdot \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \text{ and } \cdot \underset{\mathbb{Z}}{\otimes} \mathbb{R} \text{ are exact functors on } \mathbb{Z}\text{-mods}}$

More generally $\cdot \underset{R}{\otimes} \mathrm{Frac}(R)$ is exact on R-mods where Frac R is fraction field, and R is an integral domain

"localisation is an exact functor"

example A f.g. $\mathbb{Z}$-mod $\Rightarrow A \cong \mathbb{Z}^r \oplus \mathbb{Z}/d_1 \oplus \dots \oplus \mathbb{Z}/d_k$ some $d_i \neq 0$

$$\Rightarrow A\otimes\mathbb{Q} \cong \mathbb{Q}^r \Rightarrow \boxed{r = \operatorname{rank} A = \dim_{\mathbb{Q}}(A\otimes\mathbb{Q})}$$

Corollary Rank-nullity thm holds for $\mathbb{Z}$-modules if use rank instead of dim.

Pf $0 \to \underset{\substack{\shortparallel \\ \ker\varphi}}{A} \to B \xrightarrow{\varphi} \underset{\substack{\shortparallel \\ \operatorname{im}\varphi}}{C} \to 0$ exact $\Rightarrow 0 \to A\otimes\mathbb{Q} \to B\otimes\mathbb{Q} \to C\otimes\mathbb{Q} \to 0$ exact

$\Rightarrow \dim(C\otimes\mathbb{Q}) + \dim(A\otimes\mathbb{Q}) = \dim(B\otimes\mathbb{Q})$. □

rank-nullity for $\mathbb{Q}$-vector spaces.

Tensor product of chain cxes

$C_*, \tilde{C}_*$ chain cxes of R-mods $\Rightarrow (C_* \otimes \tilde{C}_*)_n = \bigoplus_{i+j=n} C_i \otimes \tilde{C}_j$

$$\boxed{\partial(x\otimes y) = (\partial x)\otimes y + (-1)^{\deg x}\, x \otimes \tilde{\partial} y}$$

"Leibniz rule"

Think of ∂ as an operator of deg $= -1$ acting from left

since ∂ "jumps over x" get $(-1)^{\deg\partial\cdot\deg x}$

Exercise $\partial\circ\partial = 0$ ← would fail without sign

recall $Z_* = \ker\partial$ = cycles, $B_* = \operatorname{im}\partial$ = boundaries

$$Z_i \otimes \tilde{Z}_j \subseteq Z_{i+j}(C_*\otimes\tilde{C}_*) \text{ and } \left.\begin{matrix} Z_i \otimes \tilde{B}_j \\ B_i \otimes \tilde{Z}_j \end{matrix}\right\} \subseteq B_{i+j}(C_*\otimes\tilde{C}_*)$$

Cor $\exists$ natural maps

$$\boxed{\begin{array}{ccc} H_i(C_*)\otimes H_j(\tilde{C}_*) & \longrightarrow & H_{i+j}(C_*\otimes\tilde{C}_*) \\ \sum [c_k]\otimes[\tilde{c}_k] & \longmapsto & \sum [c_k\otimes\tilde{c}_k] \end{array}}$$

FACT:

Algebraic Künneth Thm

$C_*, H_*(C_*)$ f.g. free R-mods (no assumption on $\tilde{C}_*$)

PID (principal ideal domain)

$$\Rightarrow \boxed{\bigoplus_{i+j=n} H_i(C_*)\otimes H_j(\tilde{C}_*) \cong H_n(C_*\otimes\tilde{C}_*)} \text{ via}$$

Algebra: Euler characteristic

C finitely generated graded abelian gp (so $\mathbb{Z}$-mod)

more generally: R-mod for PID R

Def **Euler characteristic** $\chi(C) = \sum (-1)^i \operatorname{rank} C_i$

Example/Motivation X finite CW-cx then take $C = C_*^{CW}(X)$ to get

$$\boxed{\chi(X) = \#(0\text{-cells}) - \#(1\text{-cells}) + \#(2\text{-cells}) - \dots}$$

Lemma If C_* f.g. chain cx $\Rightarrow \boxed{\chi(C_*) = \chi(H_*(C_*))}$ $\left(= \sum (-1)^i \operatorname{rank} H_i(C)\right)$

Pf Abbreviate $|C_i| = \text{rank } C_i$ $\left(= \dim_{\mathbb{Q}}(C_i \otimes_{\mathbb{Z}} \mathbb{Q})\right)$ ← for R-mods, do $\dim_{\mathbb{F}}(C_i \otimes_R \mathbb{F})$ with $\mathbb{F} = \text{Frac}(R)$ (R integral domain) [Corollary still holds, same proof]

By previous Corollary about rank-nullity:

$$0 \to Z_i \to C_i \xrightarrow{\partial} B_{i-1} \to 0 \Rightarrow |C_i| = |Z_i| + |B_{i-1}| \quad \Rightarrow |C_i| - |H_i| = |B_{i-1}| + |B_i|$$
$$0 \to B_i \to Z_i \to H_i \to 0 \Rightarrow |H_i| = |Z_i| - |B_i|$$

$$\Rightarrow \chi(C_*) - \chi(H_*) = \sum (-1)^i |B_{i-1}| + \sum (-1)^i |B_i| = \sum (-1)^i (-|B_i| + |B_i|) = 0. \ \square$$

Cor X space ⇒
$$\boxed{\begin{aligned}\chi(X) &:= \sum (-1)^i \text{ rank } H_i(X) \\ &= \sum (-1)^i \text{ rank } C_i(X)\end{aligned}}$$
← if finite rank $H_*(X)$
← if finite rank $C_*(X)$

So $\chi(X)$ is invariant up to hpy equivalence! Example $\chi(\text{platonic solid}) = \chi(S^2) = 2$

Product spaces

X, Y CW-cxes ⇒ $X \times Y$ CW-cx with cells $e_\alpha \times e_\beta$ attaching maps

e_α ↗ (X), e_β ↖ (Y)

$$\partial(\mathbb{D}^i_\alpha \times \mathbb{D}^j_\beta) = $$
$$\begin{tikzcd}
(\partial \mathbb{D}^i_\alpha) \times \mathbb{D}^j_\beta \arrow[d, "\varphi_\alpha \times id"'] & \cup & \mathbb{D}^i_\alpha \times (\partial \mathbb{D}^j_\beta) \arrow[d, "id \times \varphi_\beta"] \\
X^{i-1} \times Y^j \arrow[dr] & & X^i \times Y^{j-1} \arrow[dl] \\
& (X \times Y)^{i+j-1} &
\end{tikzcd}$$

Cor $\boxed{\chi(X \times Y) = \chi(X) \cdot \chi(Y)}$ ∀ finite CW-cxes X,Y

Pf $\sum (-1)^k \text{ rank } H_k^{CW}(X \times Y)$
$$= \sum (-1)^k \text{ rank } C_k^{CW}(X \times Y) = \sum_k \sum_{i+j=k} (-1)^{i+j} \text{ rank } C_i^{CW}(X) \cdot \text{ rank } C_j^{CW}(Y)$$
$\square$

Lemma $d(e^i_\alpha \times e^j_\beta) = (d e^i_\alpha) \times e^j_\beta + (-1)^i e^i_\alpha \times (d e^j_\beta)$

(proof later) hence $\boxed{C_*^{CW}(X \times Y) \cong C_*^{CW}(X) \otimes C_*^{CW}(Y)}$

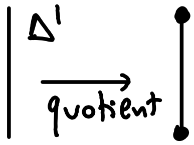
Hence if $H_*(Y)$ free then by Künneth $H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$.

Example

$*$	$H_*(S^1)$
0	$A \cong \mathbb{Z}$
1	$B \cong \mathbb{Z}$
2	0

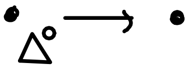
$*$	$H_*(S^1) \otimes H_*(S^1) \cong H_*(S^1 \times S^1)$ ← torus	
0	$A \otimes A$	$\cong \mathbb{Z}$
1	$(A \otimes B) \oplus (B \otimes A)$	$\cong \mathbb{Z}^2$
2	$B \otimes B$	$\cong \mathbb{Z}$
3	0	

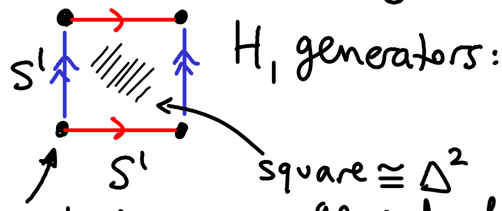
B generated by



$S^1 = \Delta^1/\text{endpts}$

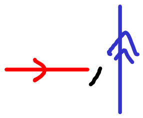
A generated by

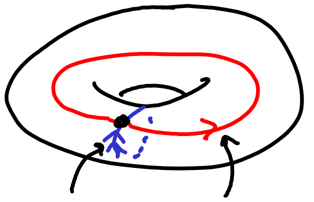




Ho generator is •

square ≅ Δ^2 generator of H_2

H_1 generators:




Pf $(\partial \mathbb{D}^i_\alpha)\times\mathbb{D}^j_\gamma \xrightarrow[\varphi_\alpha\times id]{} X^{i-1}\times Y^j \longrightarrow X^{i-1}\times Y^j \Big/ \underbrace{(X\times Y)^{i+j-2}\cap(X^{i-1}\times Y^j)}_{\| \leftarrow \text{easy check}}$

This proof is Non-examinable

$\circledast :=$

$X^{i-2}\times Y^j \cup X^{i-1}\times Y^{j-1}$

$X^{i-1} = X^{i-2}\cup\left(\mathbb{D}^{i-1}_\beta \cup \dots\right)/\sim$

$Y^j = Y^{j-1}\cup\left(\mathbb{D}^j_\gamma \cup \dots\right)/\sim$

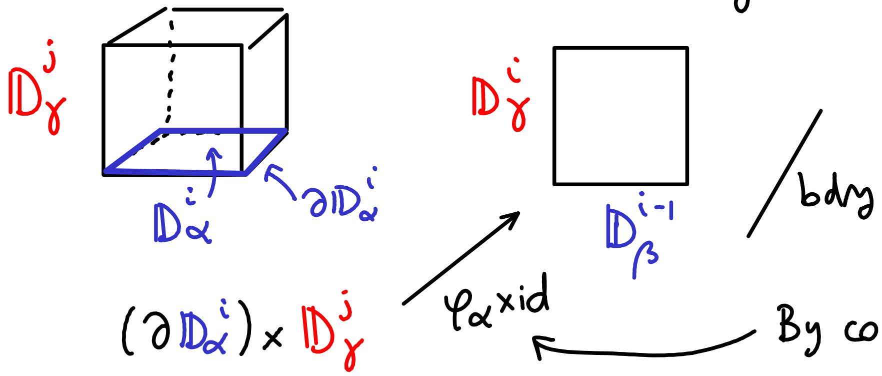
} get $\sim$ from attaching maps

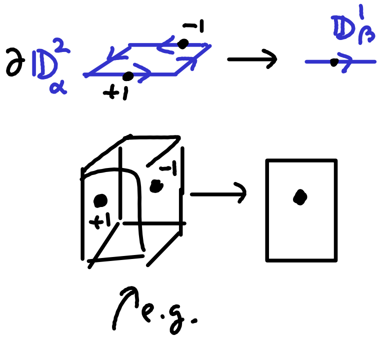
$X^{i-1}\times Y^j = X^{i-2}\times Y^j \cup X^{i-1}\times Y^{j-1} \cup \left(\mathbb{D}^{i-1}_\beta\times\mathbb{D}^j_\gamma \cup \dots\right)\Big/\sim$

$\Longrightarrow \circledast = \left(\mathbb{D}^{i-1}_\beta\times\mathbb{D}^j_\gamma \cup \dots\right)\Big/$ boundaries

$= \mathbb{D}^{i-1}_\beta\times\mathbb{D}^j_\gamma \Big/ \partial\left(\mathbb{D}^{i-1}_\beta\times\mathbb{D}^j_\gamma\right) \quad \vee \quad \dots$

$(\partial \mathbb{D}^i_\alpha)\times\mathbb{D}^j_\gamma \xrightarrow[\varphi_\alpha\times id]{} \mathbb{D}^{i-1}_\beta\times\mathbb{D}^j_\gamma \Big/ \text{bdry} \quad \vee \dots$





By considering local degrees now we see we get degree $= d_{\alpha\beta}$ for this.

$\Rightarrow$ get contribution $(de^i_\alpha)\times e^j_\beta$ ✓

similarly

$\mathbb{D}^i_\alpha\times\partial\mathbb{D}^j_\gamma \xrightarrow[id\times\varphi_\gamma]{} \mathbb{D}^i_\alpha\times\mathbb{D}^{j-1}_\delta \Big/ \text{bdry}$

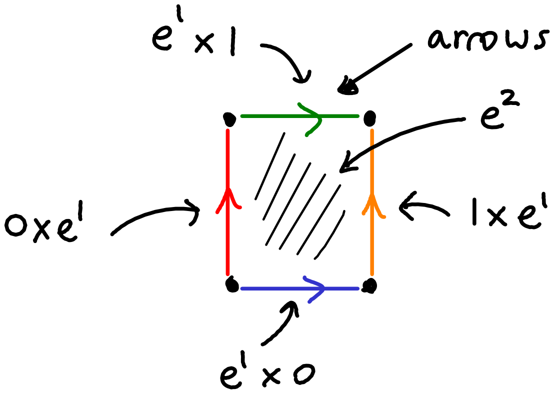
$\Longrightarrow$ degree $(-1)^i d_{\gamma\delta}$
so get $(-1)^i e^i_\alpha \times de^j_\beta$

$(-1)^i$ caused by orientations:

Could reorder factors: $\mathbb{D}^i_\alpha\times\mathbb{D}^j_\gamma \cong \mathbb{D}^j_\gamma\times\mathbb{D}^i_\alpha$ by $\begin{pmatrix}0 & Id_j\\ Id_i & 0\end{pmatrix}$ whose det $=(-1)^{ij}$. Then $\partial\mathbb{D}^j_\gamma\times\mathbb{D}^i_\alpha \to \mathbb{D}^{j-1}_\delta\times\mathbb{D}^i_\alpha/\text{bdry}$ gives degree $d_{\gamma\delta}$.

Swap factors $\mathbb{D}^{j-1}_\delta\times\mathbb{D}^i_\alpha/\text{bdry}$ by $\begin{pmatrix}0 & Id_i\\ Id_{j-1} & 0\end{pmatrix}$, det $=(-1)^{i(j-1)}$. Total sign $=(-1)^i$.

Example Recall after definition of H_*^{cw} we had example $I\times I$:



arrows here tell us how we map $[-1,1]\to$ edge (so orientation)

$$\partial e^2 = +e^1\times 0 + 1\times e^1 - e^1\times 1 - 0\times e^1$$

$$= (\partial e^1)\times e^1 \underset{\uparrow\, (-1)^{\dim e^1}}{-} e^1\times(\partial e^1) \qquad \checkmark$$

A further comment on orientation sign $(-1)^i$

$$\mathbb{D}^i\times\mathbb{D}^j \cong \underset{[v_0,\dots,v_i]}{\overset{\Delta^i}{\|}} \times \underset{[w_0,\dots,w_j]}{\overset{\Delta^j}{\|}}$$

$\leftarrow$ viewed in $\mathbb{R}^i, \mathbb{R}^j$

project $\mathbb{R}^{i+1}\to\mathbb{R}^i$

$(t_0,\dots,t_i)\mapsto\underline{\underline{(t_1,\dots,t_i)}}$

$$\partial(\mathbb{D}^i\times\mathbb{D}^j) \cong \underbrace{\partial\Delta^i}_{\|}\times\Delta^j \cup \Delta^i\times\underbrace{\partial\Delta^j}_{\|}$$

$$\sum_k (-1)^k[v_0,\dots,\hat{v}_k,\dots,v_i] \qquad\qquad \sum \underline{\underline{(-1)^k}}[w_0,\dots,\hat{w}_k,\dots,w_j]$$

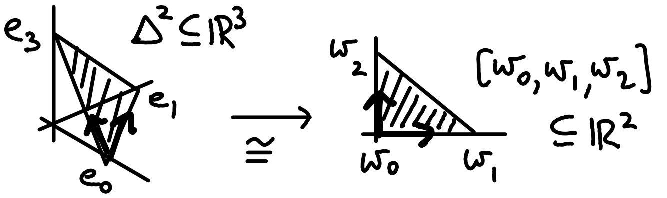
would be correct orientation sign for basis $w_1-w_0,\dots,\widehat{w_k-w_0},\dots,w_j-w_0$ but actually we have $[v_0,\dots,v_i]\times[w_0,\dots,\hat{w}_k,\dots,w_j]\subseteq\mathbb{R}^i\times\mathbb{R}^j$ and $(-1)^{i+k}$ is the orientation sign for the basis

$$v_1-v_0,\dots,v_i-v_0, w_1-w_0,\dots,\widehat{w_k-w_0},\dots,w_j-w_0$$

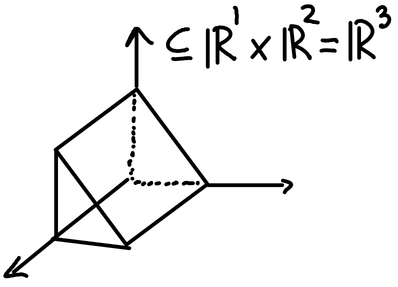
for the hyperplane in $\mathbb{R}^{i+j+1}$ containing

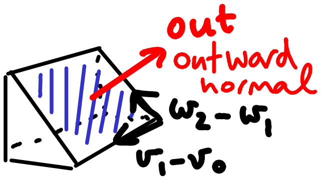
$\Rightarrow$ need $(-1)^i$ to fix orientation sign.

Example $\Delta^1\times\Delta^2$



$\Delta^1\times\Delta^2 \cong$





$$[v_0,v_1]\times\underbrace{[\hat{w}_0,w_1,w_2]}$$

out, w_2-w_1 is positive $\mathbb{R}^2$-basis

out, v_1-v_0, w_2-w_1 is negative $\mathbb{R}^3$-basis

differ due to $(-1)^i$, $i=1$.

Projections
$$\begin{tikzcd} & X \\ X \times Y \arrow[ur, "p_X"] \arrow[dr, "p_Y"'] & \\ & Y \end{tikzcd}$$

FACT:

e.g. $Y \simeq$ finite CW cx

no conditions on X

automatic if use field coefficients

Künneth Theorem If $H_n(Y)$ finitely generated, free $\forall n$

$$H_n(X \times Y) \cong \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y)$$

$$H^n(X \times Y) \cong \bigoplus_{i+j=n} H^i(X) \otimes H^j(Y)$$

$$p_X^* a \cup p_Y^* b \longleftarrow a \otimes b \qquad \text{(★)}$$

and extend linearly

Recall for cellular homology this on generators is: (chain level)

$$e^i_\alpha \times e^j_\beta \longleftarrow e^i_\alpha \otimes e^j_\beta$$

This is hom of rings if use following product

$$(a \otimes b) \cdot (\tilde{a} \otimes \tilde{b}) = (-1)^{|b| \cdot |\tilde{a}|} (a \cup \tilde{a}) \otimes (b \cup \tilde{b})$$

gradings

think of it as "exchanging order of $b, \tilde{a}$"

Rmk

An indirect proof the Thm is to write down two generalised cohomology theories $F(X,A) = H^*(X,A) \otimes H^*(Y)$ and $G(X,A) = H^*(X \times Y, A \times Y)$, and consider the natural transformation $\alpha : F \to G$ given by (★), notice for $X = pt$, $A = \emptyset$ both F, G give $H^*(Y)$.

Example $X = S^n$, $Y = S^m$ $\quad n \neq m$

$$H_*(S^n \times S^m) \cong \begin{cases} \mathbb{Z} & * = 0, n, m, n+m \\ 0 & \text{else} \end{cases} \cong H^*(S^n \times S^m)$$

where $a_i = \text{dual}(e^i)$

$$a_n \cup a_m = a_{n+m}$$

$$H_*(S^n \times S^n) \cong \begin{cases} \mathbb{Z} & * = 0, 2n \\ \mathbb{Z}^2 & * = n \leftarrow \text{gens: } \underset{a_n \otimes 1}{\overset{\shortparallel}{a_n^{(1)}}}, \underset{1 \otimes a_n}{\overset{\shortparallel}{a_n^{(2)}}} \\ 0 & \text{else} \end{cases} \cong H^*(S^n \times S^n)$$

$$a_n^{(1)} \cup a_n^{(2)} = a_{2n}$$

(but $a_n^{(i)} \cup a_n^{(i)} = 0$)

n-torus $S^1 \times \dots \times S^1$

exterior algebra

Cor $\quad H^k(T^n) \cong \Lambda^k[x_1, \dots, x_n]$ = free abelian gp. on gens. $\{x_{i_1} \wedge \dots \wedge x_{i_k} : i_1 < \dots < i_k\}$

so rank $= \binom{n}{k}$

where $x_i = p_i^*(\text{gen. of } H^1(S^1))$ $\quad$ deg $x_i = 1$

$p_i : T^n \to S^1$ projections to factors.

product is "$\wedge$" using the rule $x_i \wedge x_j = -x_j \wedge x_i$

(compare graded-commutativity of cup product)

Pf Künneth & induction $(T^n = T^{n-1} \times S^1)$ □

FACT cup product equals composition

$$\cup : H^i(X) \otimes H^j(X) \longrightarrow H^{i+j}(X \times X) \xrightarrow{\Delta^*} H^{i+j}(X)$$

$$(\Delta^i \xrightarrow[\sigma_1]{} X) \otimes (\Delta^j \xrightarrow[\sigma_2]{} X) \longmapsto (\underset{\Delta^{i+j}}{\overset{}{\Delta^i \times \Delta^j}} \xrightarrow[\sigma_1 \times \sigma_2]{} X \times X)$$

$\Delta^i \times \Delta^j \cong \Delta^{i+j}$

exterior product

Δ = diagonal map

$$X \longrightarrow X \times X$$
$$x \longmapsto (x, x)$$

12. UNIVERSAL COEFFICIENTS THEOREM

MOTIVATION: what is difference between $H^*(\mathrm{Hom}(C_*,\mathbb{Z}))$ and $\mathrm{Hom}(H_*(C_*),\mathbb{Z})$
Similarly: $H_*(C_*\otimes G)$ vs. $H_*(C_*)\otimes G$.

Proof is non-examinable. For (C_*,∂_*) chain cx:

$$\begin{tikzcd} \Longrightarrow 0 \arrow[r] & Z_* = \mathrm{Ker}\,\partial_* \arrow[loop below, "\partial=0"] \arrow[r, "incl"] & C_* \arrow[r, "\partial_*"] & B_{*-1} = \mathrm{Im}\,\partial_{*-1} \arrow[loop below, "\partial=0"] \arrow[r] & 0 \end{tikzcd}$$ is SES

FACT: Submodules of a free $\mathbb{Z}$-module are free

Rmk The same holds for R-mods if R is PID

($\mathbb{Z}$-module $\equiv$ abelian gp
free means: $\bigoplus_{\text{indexing set}} \mathbb{Z}$)

(PID = principal ideal domain
= integral domain R s.t.
every ideal $= R\cdot a$ some a)

Assume C_* free $\mathbb{Z}$-mod

$\overset{\text{FACT}}{\Rightarrow}$ Z_*, B_* free (as $\mathrm{Ker}\,\partial^*, \mathrm{Im}\,\partial^*$ are submods of C_*)

$\Rightarrow$ SES splits, choose splitting $\begin{tikzcd} C_* \arrow[r, bend left, "\partial_*"] & B_{*-1} \arrow[l, bend left, "s"] \end{tikzcd}$ so $\partial_* \circ s = id$

recall just pick preimages under ∂_* of a basis for B_*

dual SES $\Rightarrow$

$$\begin{tikzcd} 0 & Z^* \arrow[l] & C^* \arrow[l, "incl^*"'] & B^{*-1} \arrow[l, "\partial^*"'] & 0 \arrow[l] \end{tikzcd}$$

note: $incl^* =$ restrict to Z_*
since $incl^* \circ \phi : Z_* \xrightarrow{incl} B_* \xrightarrow{\phi} \mathbb{Z}$

$$\begin{tikzcd} 0 & Z^n \arrow[l] & C^n \arrow[l] & B^{n-1} \arrow[l, "\partial^*"'] & 0 \arrow[l] \\ 0 & Z^{n-1} \arrow[l] \arrow[u, "\partial=0"] & C^{n-1} \arrow[l] \arrow[u, "\partial"] & B^{n-2} \arrow[l, "\partial^*"'] \arrow[u, "\partial=0"] & 0 \arrow[l] \end{tikzcd}$$

(Rmk Although $\partial^* = 0 : B^* \to B^{*+1}$ the map $\partial^* : B^{n-1} \to C^n$ need not $=0$:
$\varphi : B_{n-1} \to \mathbb{Z}$
$\Rightarrow \partial^*\varphi = \varphi \circ \partial : C_n \xrightarrow{\partial} B_{n-1} \xrightarrow{\varphi} \mathbb{Z}$)

Connecting map
$\delta : Z^{n-1} \to B^{n-1}$
of LES:

$$\begin{tikzcd} 0 & \partial^*\varphi \arrow[l] & \varphi|_{B_*} = \phi|_{B_*} \arrow[l, "\partial^*"'] \\ \varphi|_{Z_*} = \phi \arrow[u, mapsto] & \exists\, \varphi \arrow[l, mapsto] \arrow[u, mapsto] & \end{tikzcd}$$

$B_* \subseteq Z_*$

$\Longrightarrow \boxed{\delta(\phi) = \phi|_{B_*}}$

LES $\Longrightarrow$
$(H^*B = B^*, H^*Z = Z^*$ since $\partial^* = 0)$

$$\begin{tikzcd} \cdots & Z^n \arrow[l, "\delta^n"'] & H^nC \arrow[l] & B^{n-1} \arrow[l, "\partial^*"'] & Z^{n-1} \arrow[l, "\delta^{n-1}"'] \\ & \varphi & {[\varphi]} \arrow[l, mapsto] & \phi|_{B_{n-1}} & \phi \arrow[l, mapsto] \end{tikzcd}$$

$$\Longrightarrow \quad 0 \leftarrow \mathrm{Ker}\,\delta^n \leftarrow H^nC \leftarrow B^{n-1}/\mathrm{Im}\,\delta^{n-1} \leftarrow 0$$

$\mathrm{Ker}\,\delta^n = \{\phi \in Z^n : \phi(B_n) = 0\}$ $\Longrightarrow$ so: $\begin{tikzcd} Z_n \arrow[rr, "\phi"] \arrow[dr] & & \mathbb{Z} \\ & Z_n/B_n = H_n(C_*) \arrow[ur] & \end{tikzcd}$

$= \mathrm{Hom}(H_n(C_*), \mathbb{Z})$

✪ **Universal Coefficients Thm:**

(evaluation of a cohomology class on cycles)

$$0 \to B^{n-1}/\mathrm{Im}\,\delta^{n-1} \to H^n(C) \to \mathrm{Hom}(H_n(C_*), \mathbb{Z}) \to 0$$ is SES and natural

$B^{n-1}/\mathrm{Im}\,\delta^{n-1} \overset{\shortparallel}{} \ \mathbf{Ext^1(H_{n-1}(C);\mathbb{Z})}$ (see next Lemma)

$[\varphi] \longmapsto (\varphi : H_n(C_*) \to \mathbb{Z})$

and SES splits (but not naturally): $\begin{tikzcd} B^{n-1}/\mathrm{Im}\,\delta^{n-1} \arrow[r, bend left, "\partial^*"] & H^n(C) \arrow[l, bend left, "s^*"] \end{tikzcd}$ $\quad s^* \circ \partial^* = id$
(since $\partial \circ s = id$
$\Rightarrow id = (\partial \circ s)^* = s^* \circ \partial^*$)

$\Rightarrow H^n(C) \cong \mathrm{Hom}(H_n(C), \mathbb{Z}) \oplus \mathrm{Ext}^1(H_{n-1}(C); \mathbb{Z})$

Lemma $\mathrm{Ext}^1(H_{n-1}(C);\mathbb{Z}) \cong B^{n-1}/\mathrm{Im}\,\delta^{n-1}$ canonically

Algebra background on Extension groups $\mathrm{Ext}^i(M;R)$ $\left(=\mathrm{Ext}^i_R(M,R)\right)$

general case

M R-module, R ring (comm. with 1)

$\Rightarrow \exists$ **free resolution**:

$$\cdots \to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} M \to 0$$ exact, P_i free R-mods

(pick gens x_α for M $\Rightarrow P_0 = \bigoplus_\alpha R \xrightarrow[\varphi_0]{} M$, $e_\alpha \mapsto x_\alpha$

" " y_β for $\mathrm{Ker}\,\varphi_0 \Rightarrow P_1 = \bigoplus_\beta R \xrightarrow[\varphi_1]{} \mathrm{Ker}\,\varphi_0$, $e_\beta \mapsto y_\beta$

continue inductively)

our case

$H_{n-1}(C_*)$ $\mathbb{Z}$-mod $(R=\mathbb{Z})$

$$\begin{tikzcd}
0 \arrow[r] & B_{n-1} \arrow[r, hook] \arrow[d, equal] & Z_{n-1} \arrow[r] \arrow[d, equal] & H_{n-1}(C) \arrow[r] \arrow[d, equal] & 0 \\
& P_1 & P_0 & M &
\end{tikzcd}$$

Take $\mathrm{Hom}(\cdot\,;R)$ and drop $\mathrm{Hom}(M;R)$

$$0 \to \mathrm{Hom}(P_0;R) \xrightarrow{\varphi_1^*} \mathrm{Hom}(P_1;R) \xrightarrow{\varphi_2^*} \cdots$$

Is cochain complex but not exact

$\Rightarrow$ take cohomology groups:

$$0 \to Z^{n-1} \to B^{n-1} \to 0$$

Def $\mathrm{Ext}^0(M;R) = \mathrm{Ker}\,\varphi_1^*$

$\mathrm{Ext}^1(M;R) = \mathrm{Ker}\,\varphi_2^* / \mathrm{Im}\,\varphi_1^*$

$\cdots$

Fact independent of choices P_i, φ_i

Example 1 $\mathrm{Ext}^0(M;R) \cong \mathrm{Hom}(M,R)$

$$\begin{tikzcd}
P_1 \arrow[r, "\varphi_1"] \arrow[dr, "0"'] & P_0 \arrow[r, "\varphi_0", two heads] \arrow[d, "\phi"] & M \arrow[dl, dashed] \\
& R &
\end{tikzcd}$$

descends: $m \longmapsto \phi(\varphi_0^{-1}m)$

well defined since $\phi(\underset{\mathrm{Im}\,\varphi_1}{\underset{\parallel}{\mathrm{Ker}\,\varphi_0}})=0$

Example 2 $\mathrm{Ext}^1(M;R) =$

$$\left\{\phi : \begin{tikzcd}
P_2 \arrow[r, "\varphi_2"] \arrow[dr, "0"'] & P_1 \arrow[r] \arrow[d, "\phi"] & P_0 \\
& R &
\end{tikzcd}\right\} \Big/ \left\{\phi = \psi\circ\varphi_1 : \begin{tikzcd}
P_1 \arrow[r, "\varphi_1"] \arrow[dr, "\phi"'] & P_0 \arrow[d, "\psi"] \\
& R
\end{tikzcd}\right\}$$

Proof of Lemma

By **Example 2**,

$\mathrm{Ext}^1(H_{n-1}(C_*);\mathbb{Z}) =$

$$= \left\{\begin{tikzcd}
0 \arrow[r] & B_{n-1} \arrow[r] \arrow[d, "\phi"] & Z_{n-1} \\
& \mathbb{Z} &
\end{tikzcd}\right\} \text{modulo}$$

those arising from restriction

$$\left\{\begin{tikzcd}
0 \arrow[r] & B_{n-1} \arrow[r] \arrow[dr, "\phi|_{B_{n-1}}"'] & Z_{n-1} \arrow[d, "\phi"] \\
& & \mathbb{Z}
\end{tikzcd}\right\}$$

Thus $B^{n-1}/\mathrm{Im}\,\delta^{n-1}$. $\square$

Rmk If R PID, then $\mathrm{Ker}(P_0 \to M)$ is free (since submod of free mod P_0)

$\Rightarrow$ can pick $P_1 = \mathrm{Ker}(P_0 \to M)$, $P_k = 0$ for $k \geq 2$ $\Rightarrow \mathrm{Ext}^k(M;R) = 0$ $k \geq 2$

(Co)homology with coefficients in a ring/field/module

Motivation

So far we had (C_*, ∂_*) ← group hom, chain cx of abelian groups } in graded sense

$\Rightarrow H_*(C_*) = \operatorname{Ker}\partial_* / \operatorname{Im}\partial_*$ abelian group (since $\operatorname{Ker}\partial$, $\operatorname{Im}\partial$ are)

We cannot use a chain cx of (non-abelian) groups, because $\operatorname{Im}\partial_*$ need not be a normal subgroup of $\operatorname{Ker}\partial_*$.

However, abelian groups can be thought of as $\mathbb{Z}$-modules, then given any **abelian group G**, define **homology with coeffs in G**

$$\boxed{H_*(C_*;G) = H_*(C_* \otimes_{\mathbb{Z}} G)} \leftarrow \text{with differential } \partial_* \otimes \mathrm{id}$$

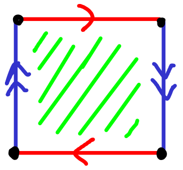
Def X space $\Longrightarrow$ $\boxed{H_*(X;G) = H_*(C_*(X) \otimes_{\mathbb{Z}} G)}$

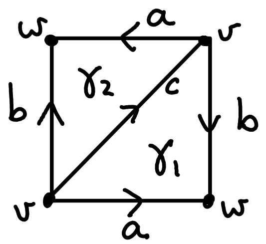
Explanation:

$C_k(X)$ free $\mathbb{Z}$-mod $\cong \bigoplus_{I_k} \mathbb{Z} \Rightarrow C_k(X) \otimes_{\mathbb{Z}} G \cong \bigoplus_{I_k} G$: just replace $\mathbb{Z}$ by G (as $\mathbb{Z} \otimes_{\mathbb{Z}} \cdot \cong \cdot$)

Why care? We hope to get more/new invariants of spaces

Example $X = \mathbb{RP}^2 =$





$*$	$C^{\Delta}_*(\mathbb{RP}^2;G)$
0	$Gv \oplus Gw$
1	$Ga \oplus Gb \oplus Gc$
2	$G\gamma_1 \oplus G\gamma_2$

for $G = \mathbb{Z}/2$: $0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow[\begin{pmatrix}1&1\\1&1\\1&1\end{pmatrix}]{\partial_2} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow[\begin{pmatrix}1&1&0\\1&1&0\end{pmatrix}]{\partial_1} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 0$

$$\Rightarrow H_*(\mathbb{RP}^2;\mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & *=0 \\ \mathbb{Z}/2 & *=1 \\ \mathbb{Z}/2 & *=2 \\ 0 & \text{else} \end{cases} \qquad \text{compare: } H_*(\mathbb{RP}^2) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}/2 & *=1 \\ 0 & \text{else} \end{cases}$$

($G = \mathbb{Z}$ case)

Form cochain complex using $\operatorname{Hom}_{\mathbb{Z}}(\cdot, G)$ (= group homs to G) in place of $\operatorname{Hom}(\cdot, \mathbb{Z})$

$$\boxed{\begin{aligned} H^*(C_*;G) &= H_*(\operatorname{Hom}_{\mathbb{Z}}(C_*, G)) \\ H^*(X;G) &= H_*(\operatorname{Hom}_{\mathbb{Z}}(C_*(X);G)) \end{aligned}}$$

X space → $H^*(X;G)$

with differential ∂^*: $\partial^*\phi = \phi \circ \partial_*$

so: $H^*(C_*(X);G)$

Universal coefficients thm (same proof using $\operatorname{Hom}_{\mathbb{Z}}(\cdot,G)$)

$$\boxed{\begin{aligned} 0 \to \operatorname{Ext}^1_{\mathbb{Z}}(H_{n-1}(C_*);G) \to H^n(C_*;G) &\longrightarrow \operatorname{Hom}_{\mathbb{Z}}(H_n(C_*), G) \to 0 \\ [\varphi] &\longmapsto (\varphi : H_n(C_*) \to G) \end{aligned}}$$

Example $X = \mathbb{RP}^2$, $G = \mathbb{Z}/2$, apply $\mathrm{Hom}_{\mathbb{Z}}(\cdot, G)$ to the complex in previous example

$$0 \leftarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xleftarrow[\begin{pmatrix}1&1&1\\1&1&1\end{pmatrix}]{\partial^1} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xleftarrow[\begin{pmatrix}1&1\\1&1\\0&0\end{pmatrix}]{\partial^0} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \leftarrow 0$$

$$\Rightarrow H^*(\mathbb{RP}^2; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$$

compare: $H^*(\mathbb{RP}^2) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$ ($G = \mathbb{Z}$ case)

compare $\mathrm{Hom}(H_*(\mathbb{RP}^2), \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ 0 & * \geq 2 \end{cases}$

caused by $\mathrm{Ext}^1_{\mathbb{Z}}(\underbrace{\mathbb{Z}/2}_{H_1(\mathbb{RP}^2)}; \underbrace{\mathbb{Z}/2}_{G = \mathbb{Z}/2}) \cong \mathbb{Z}/2$

Can generalise further:

C_* = chain cx of ...	coefficients in:	
abelian gps ($\mathbb{Z}$-mods)	abelian gp G ($\mathbb{Z}$-mod)	$H_*(C_*; G) = H_*(C_* \otimes_{\mathbb{Z}} G)$
R-modules ↖ ring (comm. with 1)	R-module M	$H_*(C_*; M) = H_*(C_* \otimes_R M)$

Rmk $H_*(C; M)$ will be an R-module since $\ker \partial, \mathrm{Im}\, \partial$ are (∂_* is R-linear hom by assumption)

X space $\Rightarrow C_k(X; R) = C_k(X) \otimes_{\mathbb{Z}} R \cong \bigoplus_{I_k} R$: just replace $\mathbb{Z}$ by R (as $\mathbb{Z} \otimes_{\mathbb{Z}} \cdot \cong \cdot$)

$$\Rightarrow \boxed{H_*(X; M) = H_*(C_*(X; R); M) = H_*(C_*(X; R) \otimes_R M)}$$

so just replace each $\mathbb{Z}$ by M in $C_*(X)$

Form cochain complex using $\mathrm{Hom}_R(\cdot, M)$ (= R-linear homs to M) in place of $\mathrm{Hom}(\cdot, \mathbb{Z})$ with differential ∂^*: $\partial^* \phi = \phi \circ \partial_*$

$$\boxed{\begin{aligned} H^*(C_*; M) &= H_*(\mathrm{Hom}_R(C_*, M)) \\ H^*(X; M) &= H_*(\mathrm{Hom}_R(C_*(X; R); M)) \end{aligned}}$$

X space

so: $H^*(C_*(X; R); M)$

Rmk These are R-mods. If we use $M = R$, then they are also rings via cup product

> **Universal Coefficients Thm** For R any PID, C_* chain cx of R-mods,
>
> $$0 \to \mathrm{Ext}^1_R(H_{n-1}(C_*); M) \to H^n(C_*; M) \to \mathrm{Hom}_R(H_n(C_*), M) \to 0$$ is SES and natural.
>
> $\mathrm{Ext}^1_R(H_{n-1}(C_*); M) \cong B^{n-1}/\mathrm{im}\,\delta^{n-1}$ working over R using homs to M
>
> $[\varphi] \longmapsto (\varphi : H_n(C_*) \to M)$
>
> and the SES splits but the splitting is not natural.
>
> Same proof using $\mathrm{Hom}_R(\cdot, M)$

Example $R = \mathbb{F}$ field $\Rightarrow C_*, H_*, H^*$ are vector spaces$/\mathbb{F}$.

Rmk all $\mathbb{F}$-mods (i.e. vector spaces$/\mathbb{F}$) are free $\mathbb{F}$-mods $\cong \bigoplus \mathbb{F} b_i$ (basis b_i)
up to iso they are determined by $\dim_{\mathbb{F}}$ = cardinality of basis.

Cor C_* = chain cx of $\mathbb{F}$-vector spaces $\Rightarrow \boxed{H^n(C_*;\mathbb{F}) \cong \left(H_n(C_*)\right)^*}$ ← dual v.s.: $\mathrm{Hom}_{\mathbb{F}}(H_n(C_*);\mathbb{F})$

Pf Pick any basis v_i for $\mathbb{F}$-v.s. B_{n-1}, extend it to a basis v_i, w_j of Z_{n-1} (also works in ∞ dim case).

$\Rightarrow$ can extend any $\mathbb{F}$-linear map $\psi : B_{n-1} \longrightarrow \mathbb{F}$ to $\phi : Z_{n-1} \longrightarrow \mathbb{F}$ just pick any values $\phi(w_j) \in \mathbb{F}$ e.g. $\phi(w_j) = 0$.

$\Rightarrow B^{n-1}/\mathrm{im}\,\delta^{n-1} = 0$ so $H^n(C_*;\mathbb{F}) \longrightarrow \mathrm{Hom}(H_n(C_*);\mathbb{F})$ iso □

Cor $\boxed{H^n(X;\mathbb{F}) \cong H_n(X;\mathbb{F})^*}$ ← dual v.s. for any field $\mathbb{F}$.

Cor $\boxed{H^n(X;M) \underset{\substack{\uparrow \\ \text{if } X \text{ is CW-cx}}}{\cong} H^n_{CW}(X;M) \underset{\substack{\uparrow \\ \text{if } X \text{ is } \Delta\text{-cx}}}{\cong} H^n_{\Delta}(X;M)}$

Pf Cor holds for homology and the isos are natural. ← i.e. functorial w.r.t. maps

The universal coeff. thm SES is natural. So result holds by 5-Lemma. □

Algebra : Structure thm for f.g. abelian groups and R-mods

Fact 1 A f.g. abelian gp $\Longrightarrow A \cong \underbrace{\mathbb{Z}^r}_{\text{free part } F} \oplus \underbrace{\mathbb{Z}/p_1^{n_1} \oplus \cdots \oplus \mathbb{Z}/p_a^{n_a}}_{\text{torsion part } T}$

where $p_i \in \mathbb{Z}$ prime (need not be distinct)

Also r, a, p_i, n_i are unique (up to reordering)

Example $\mathbb{Z}/4 = \mathbb{Z}/2^2 \not\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$

$\mathbb{Z}/6 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3$ (Chinese Remainder Thm)

Fact 2 $T \cong \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_k$ with $d_1 | d_2 | \cdots | d_k$ ($d_i \overset{\neq 0}{\in} \mathbb{N}$ unique)

Example $\mathbb{Z}/2 \oplus \mathbb{Z}/2^2 \oplus \mathbb{Z}/3 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3 \cdot 2^2$ $\quad d_1 = 2$, $d_2 = 12$

Fact 3 M f.g. R-mod, R PID, then:

$$\boxed{\begin{aligned} M &\cong F \oplus T \\ F &\cong R^r \\ T &\cong R/p_1^{n_1} \oplus \cdots \oplus R/p_a^{n_a} \\ &\cong R/d_1 \oplus R/d_2 \oplus \cdots \oplus R/d_k \end{aligned}}$$

$r \in \mathbb{N}$ unique, called **rank** of M

$p_i \in R$ primes, $p_i^{n_i}$ unique up to ordering & multⁿ by

$d_1 | \cdots | d_k$ non-zero, not invertible

d_i called **invariant factors**

unique up to multⁿ by invertible elements ← e.g. ± 1 if $R = \mathbb{Z}$

Rmk $T = \{m \in M : rm = 0 \text{ some } r \neq 0 \in R\}$ = **torsion elements**

$F \cong M/T$

Torsion shift

Easy Exercise $\boxed{\mathrm{Ext}^*_R\left(\bigoplus_i M_i \,;\, \prod_j N_j\right) \cong \prod_i \prod_j \mathrm{Ext}^*_R(M_i\,;N_j)}$ ← any R-mods M_i, N_j

↓

Upshot To compute $\mathrm{Ext}^1_R(M;R)$ for $M = R^r \oplus R/d_1 \oplus \cdots$ just need:

$$\boxed{\begin{aligned} \mathrm{Ext}^1_R(R\,;R) &= 0 \\ \mathrm{Ext}^1_R(R/d\,;R) &\cong R/d \end{aligned}}$$

← since $0 \to R \xrightarrow{id} R \to 0$, $\quad 0 = P_1$, $R = P_0$

← since $0 \to R \xrightarrow{d} R \xrightarrow{1} R/d \to 0$, $\quad R \xrightarrow{\phi} R$ so choice of $\phi(1) \in R$ modulo ϕ coming from

$$\begin{tikzcd} R \arrow[r, "d"] \arrow[d, "\phi"'] & R \arrow[dl, "\varphi"] \\ R & \end{tikzcd}$$

so $\phi(1) = d\cdot\varphi(1) \in d\cdot R$

$$\Longrightarrow \boxed{\mathrm{Ext}^1_R(M\,;R) \cong \mathrm{Torsion}(M)}$$

Exercises

- $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\,;\mathbb{Z}/m) \cong \mathbb{Z}/\gcd(m,n)$
- G abelian gp $\Rightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z};G) = 0$ and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/d\,;G) \cong G/d\cdot G$ ← $d \neq 0$
- R any ring (comm. with 1) $x \in R$ not zero divisor $\Rightarrow \mathrm{Ext}^*_R(R/(x)\,;\underset{R\text{-mod}}{N}) \cong \begin{cases} \{n \in N : x\cdot n = 0\} & * = 0 \\ N/x\cdot N & * = 1 \\ 0 & \text{else} \end{cases}$

Cor If $H_n(X;R)$ f.g. R-mod $\forall n$, R PID,

$\Longrightarrow H_n(X;R) = R^{r_n} \oplus T_n$ (free & torsion parts)

$$\Longrightarrow \boxed{H^n(X;R) \underset{\substack{\uparrow \\ \text{not natural}}}{\cong} R^{r_n} \oplus T_{n-1}}$$ ← torsion moves up!

Pf $0 \to T_{n-1} \to H^n(X;R) \longrightarrow \mathrm{Hom}(R^{r_n} \oplus T_{n-1}\,; R) \to 0$

$$\mathrm{Hom}(R^{r_n} \oplus T_{n-1}\,; R) \cong \underbrace{(\mathrm{Hom}(R\,;R))^{r_n}}_{\substack{R \to R \\ 1 \mapsto x}} \oplus \underbrace{\mathrm{Hom}(T_{n-1};R)}_{0}$$

$R \to R$, $1 \mapsto x$ $\longrightarrow$ $\cong R^{r_n}$

x determines the hom

0 since $T_{n-1} \to R$, $1 \longmapsto 0$ (R is integral domain, so no torsion elts $\neq 0$)

$$\Longrightarrow 0 \to T_{n-1} \to H^n(X;R) \to \underbrace{R^{r_n}}_{} \to 0$$

free, so can split the SES (pick lifts of basis). □ ← so not canonical

Example

*	$H_*(\mathbb{R}P^3)$	$H^*(\mathbb{R}P^3)$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}/2$	0
2	0	$\mathbb{Z}/2$
3	$\mathbb{Z}$	$\mathbb{Z}$

torsion moves up

Universal coefficients Theorem in homology

(recall $H_*(C_*\otimes_R M)=H_*(C_*;M)$)

FACT Theorem C_* chain cx of free R-mods (↖PID), M R-module

⇒ natural SES $0\to H_*(C_*)\otimes_R M\to H_*(C_*\otimes_R M)\to \mathrm{Tor}_1^R(H_{*-1}(C_*);M)\to 0$

$[c]\otimes m\mapsto[c\otimes m]$ ↖ defined below.

The SES splits, but the splitting is not natural.

Torsion groups

: A, B R-mods (R comm. ring with 1)

pick $\cdots\to P_2\xrightarrow{\varphi_2}P_1\xrightarrow{\varphi_1}P_0\xrightarrow{\varphi_0}A\to 0$ free resolution (exact sequence, P_i free R-mods)

take $\cdot\otimes B$, omit $A\otimes B$

$\Longrightarrow\ \cdots\to P_2\otimes B\xrightarrow[\varphi_2\otimes id]{}P_1\otimes B\xrightarrow[\varphi_1\otimes id]{}P_0\otimes B\to 0$ not exact but is chain cx

$\mathrm{Tor}_k^R(A,B)=H_k(\text{this complex})$ ← fact independent of choices of P_i,φ_i

Rmk R PID ⇒ $\ker\varphi_0$ free ⇒ pick $\begin{cases}P_1=\ker\varphi_0\\P_k=0 \text{ for } k\geq 2\end{cases}$ ⇒ only Tor_0^R, Tor_1^R can be non-zero

Example $\mathrm{Tor}_*^{\mathbb{Z}}(\mathbb{Z}/a,\mathbb{Z}/b)=?$

$0\to\mathbb{Z}\xrightarrow{\cdot a}\mathbb{Z}\xrightarrow[\text{quotient}]{\varphi_0}\mathbb{Z}/a\to 0$ free resolution

take $\cdot\otimes\mathbb{Z}/b$, drop $\mathbb{Z}/a\otimes\mathbb{Z}/b$ ↘

$\Longrightarrow 0\to\mathbb{Z}/b\xrightarrow{\cdot a}\mathbb{Z}/b\to 0$ (since $\mathbb{Z}\otimes_{\mathbb{Z}}G\cong G$ any G)

$\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/a,\mathbb{Z}/b)=(\mathbb{Z}/b)/a\cdot\mathbb{Z}/b\cong\mathbb{Z}/\langle a,b\rangle\cong\mathbb{Z}/\gcd(a,b)\cong\mathbb{Z}/a\otimes\mathbb{Z}/b$

$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/a;\mathbb{Z}/b)=\{x\in\mathbb{Z}/b : a\cdot x=0\}\cong\mathbb{Z}/\gcd(a,b)$ ↖ via: $\frac{b}{\gcd(a,b)}\leftarrow\!\shortmid 1$

Facts $\mathrm{Tor}_0^R(A,B)=P_0\otimes B/\mathrm{Im}(\varphi_1\otimes id)\cong A\otimes B$

$\mathrm{Tor}_*^R(A,B)\cong\mathrm{Tor}_*^R(B,A)$

Exercise $\mathrm{Tor}_*^R(\oplus A_i,\oplus B_j)\cong\bigoplus_i\bigoplus_j\mathrm{Tor}_*^R(A_i,B_j)$

$\mathrm{Tor}_*^R(A,B)=0$ for $*\geq 1$ if A or B is free (use $M\otimes_R R\cong M$)

$\Downarrow$ deduce $\mathrm{Tor}_1^R(A,M)$ for f.g. R-mods A (↖PID)

$\mathrm{Tor}_*^R(R/u;M)\cong\begin{cases}M/u\cdot M & *=0\\ u\text{-torsion}(M)=\{x\in M: u\cdot x=0\} & *=1\\ 0 \quad \text{else}\end{cases}$

↗ $u\in R$ not zero divisor, R any ring (comm. with 1)

Example $H_*(\mathbb{RP}^2;\mathbb{Z}/2)\cong\begin{cases}\mathbb{Z}/2 & *=0\\\mathbb{Z}/2 & *=1\\\mathbb{Z}/2 & *=2\end{cases}$ $\quad H_*(\mathbb{RP}^2)\otimes\mathbb{Z}/2\cong\begin{cases}\mathbb{Z}\otimes\mathbb{Z}/2\\\mathbb{Z}/2\otimes\mathbb{Z}/2\\0\end{cases}\cong\begin{cases}\mathbb{Z}/2\\\mathbb{Z}/2\\0\end{cases}$

↖ caused by $\mathrm{Tor}_1^{\mathbb{Z}}(H_1(\mathbb{RP}^2);\mathbb{Z}/2)=\mathrm{Tor}_1(\mathbb{Z}/2,\mathbb{Z}/2)\cong\mathbb{Z}/2$

Künneth Thm

R PID ⟹ natural SES: (C_* free ch. cx. R-mods, D_* any ch. cx. R-mods) →

$$\boxed{0\to\bigoplus_{i+j=n}H_i(C_*)\otimes H_j(D_*)\to H_n(C_*\otimes D_*)\to\bigoplus_{i+j=n-1}\mathrm{Tor}_1^R(H_i(C),H_j(D))\to 0}$$

and the SES splits but the splitting is not natural.

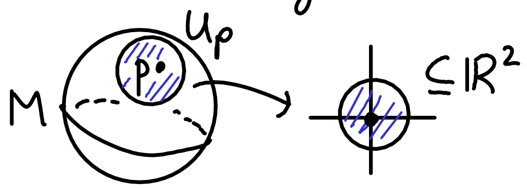
Example $R=$ field ⇒ that $\mathrm{Tor}_1=0$

Example (take $C_*:=C^{-*}(X)$, $D_*:=C^{-*}(Y)$): $0\to H^*(X)\otimes H^*(Y)\to H^*(X\times Y)\to\mathrm{Tor}_1(H^*(X),H^*(Y))\to 0$ exact (↙degree increases by 1)

(if $C_*(X)$ has ∞ rank then $C^{-*}(X)$ may not be free but it will be "flat" and Thm holds if C_* is flat R-mod)

13. MANIFOLDS : POINCARÉ–LEFSCHETZ DUALITY

- M **n-mfd** is Hausdorff topological space s.t. $\forall p \in M$ $\exists$ open neighbourhood $U_p \subseteq M$ homeomorphic to $\mathbb{R}^n$



(equivalently: to an open ball, or any open set in $\mathbb{R}^n$)

One also requires M **second countable** i.e. $\exists$ countable basis of open sets $\iff$ M is covered by countably many such U_p.

exercise

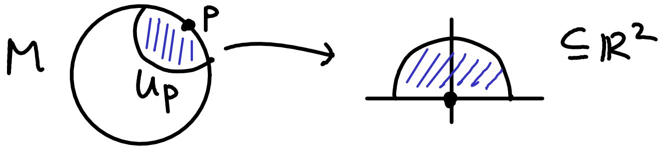
A **submanifold** $N \subseteq M$ is a mfd s.t. inclusion $N \to M$ is an embedding (i.e. a homeomorphism onto its image)

- M **n-mfd with boundary** if also allow $U_p \cong$ **upper half space $\mathbb{H}^n$**, $p \mapsto 0$

$$\{x \in \mathbb{R}^n : x_n \geq 0\} = \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0} = \mathbb{H}^n$$

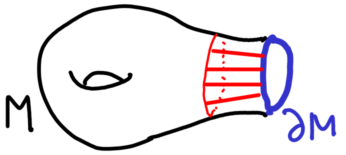
equivalently: any open nbhd of $0 \in \mathbb{H}^n$

Such p are called **boundary points**, they form the **boundary ∂M** which is an $(n-1)$-mfd without boundary.



FACT (collar nbhd thm) $\partial M \subseteq M$ has an open neighbourhood $\cong \partial M \times (0,1]$
$\partial M \longrightarrow \partial M \times 1$

M is **closed** if compact without boundary.

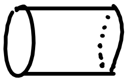
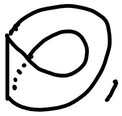
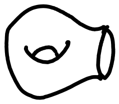


Rmk For manifolds, connected components = path components.
(since locally $\cong$ disc, so locally path-connected, so conn. $\iff$ path-conn.)

Examples

closed mfds : S^n, $\mathbb{RP}^n$, $T^n = S^1 \times \cdots \times S^1$ (**n-torus**), $\mathbb{CP}^n$, $O(n)$, $SU(n)$

non-compact mfds: $\mathbb{R}^n$, $\mathrm{Mat}_{m\times n} \cong \mathbb{R}^{mn}$, $GL(n,\mathbb{R})$

mfds with bdry: $\mathbb{D}^n$, $\mathbb{D}^1 \times S^1 =$ , Möbius band = , $T^2 \setminus$ open disc = 

FACT (Milnor 1959) Any mfd is homotopy equivalent to a CW-cx

fact If M is a compact manifold then $H_*(M)$ are finitely generated

Rmk M **triangulable** if $M \cong$ simplicial cx.
Not all mfds are triangulable, but most of those we encounter are.

Compact manifolds have f.g. homology ← Non-examinable proof

① X space is a **Euclidean neighbourhood retract** if $\exists$ **embedding** $j: X \longrightarrow \mathbb{R}^N$ some N, s.t. $i(X)$ is a retract of a nbhd $V \subseteq \mathbb{R}^n$.
↖(homeo onto image)

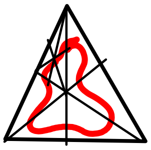
② X is **weakly locally contractible** if $\forall$ nbhd $x \in U \subseteq X$, $\exists$ nbhd $x \in V \subseteq U$ s.t. V is contractible inside U.

FACT compact $X \subseteq \mathbb{R}^n$ is ① $\Longleftrightarrow$ X is ②

Rmk If we find nbhd V as in ① with retraction $V \xrightarrow{f} X$ then any smaller nbhd V' also retracts using $f|_{V'}: V' \longrightarrow X$. Similarly in ② $V' \subseteq V$ is contractible: restrict the hpy.

Lemma A X compact & ① $\Longrightarrow$ X is the retract of a finite simplicial cx

pf $i(X) \subseteq \mathbb{R}^n$ compact $\Rightarrow$ lies inside some large n-simplex $\Delta^n \to \mathbb{R}^n$
Apply barycentric subdivision until simplices have diameter $<$ dist$(X, \partial V)$.
Simpl.cx. $= \bigcup \{$subsimplices which intersect $X\}$ using the restriction of retraction $V \to X$. □



Rmk Also deduce X has f.g. homology since retractions are surjective on H_*.
($\bigoplus \mathbb{Z} \twoheadrightarrow H_*(\text{finite simpl. cx}) \xrightarrow{\text{retract}} H_*(X)$ so get surjection from free $\mathbb{Z}$-mod, so f.g.)

Lemma B M compact mfd $\Longrightarrow$ M embeds into $\mathbb{R}^N$, some N.

Pf "Just do it proof":
$\forall p \in M$, $\exists$ homeo $\mathbb{D}^n \xrightarrow{\varphi_p} \text{nbhd}(p \in M)$
Pick finite subcover of φ_{p_i} of $M = \bigcup_{p \in M} \varphi_p(\mathbb{D}^n)$. Say $i = 1, \dots, k$
$\psi_{p_i}: M \xrightarrow{\varphi_{p_i}^{-1}} \mathbb{D}^n/\partial\mathbb{D}^n \cong S^n \subseteq \mathbb{R}^{n+1}$ define embedding $(\psi_{p_1}, \dots, \psi_{p_k}): M \to \mathbb{R}^{k\cdot(n+1)}$
↖(send $M \setminus \mathrm{Im}(\varphi_{p_i})$ to the point corresponding to $\partial\mathbb{D}^n \in \mathbb{D}^n/\partial\mathbb{D}^n$).
Finally use: a continuous bijection from a compact space to a Hausdorff space is $\cong$ □

Rmk Same works if M has boundary, just consider its **double** $\underset{\text{identify along } \partial M}{M \cup M}$ and apply the Lemma to the double.

Cor M compact mfd (possibly with bdry) $\Longrightarrow$ M has f.g. homology

Pf Mfds satisfy ② since locally ball $\simeq$ pt. M embeds in $\mathbb{R}^N$ by Lemma B.
① holds by FACT. Done by Lemma A. □

Local orientations and orientability

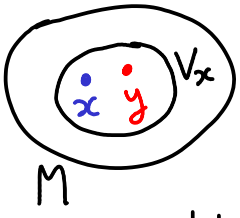
Def A **local orientation** of M at $x \in M$ is a choice of generator

excise complement of nbhd $V_x \cong \mathbb{D}^n$ ← choice of homeo is not canonical!

$$\mu_x \in H_n(M, M\setminus x) \cong H_n(\mathbb{D}^n, \underbrace{\mathbb{D}^n \setminus 0}_{\simeq\, \partial\mathbb{D}^n = S^{n-1}}) \cong \tilde{H}_n(S^n) \cong \mathbb{Z}$$

(see section 5 of these notes)

Def An **orientation** of M is a **locally consistent** choice $x \mapsto \mu_x$ meaning:



$V_x \cong \mathbb{D}^n \simeq pt$

$$\begin{tikzcd} & H_n(M, M\setminus V_x) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ H_n(M, M\setminus x) & & H_n(M, M\setminus y) \end{tikzcd}$$

← quotient maps

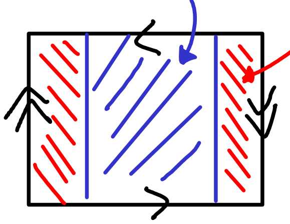
$$\begin{tikzcd} & \exists\, a \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ \mu_x & & \mu_y \end{tikzcd}$$

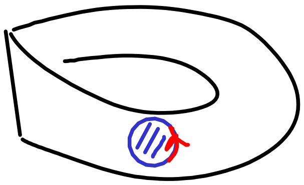
Def M **orientable** if $\exists$ orientation on M

oriented if we chose an orientation

Examples S^n, $\mathbb{R}^n$, $\mathbb{C}P^n$, orientable surfaces Σ_g, $\mathbb{R}P^n$ ← odd n

Non-example $\mathbb{R}P^2$ = Möbius band ∪ D^2

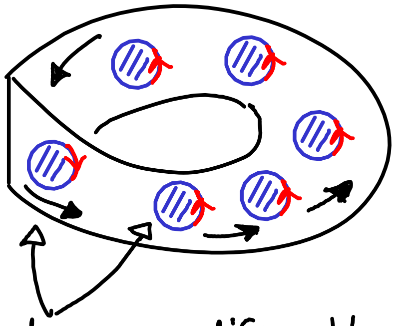




Choice of μ_x is choice of orientation of boundary circle of small disc containing x

$\Rightarrow \mathbb{R}P^2$ not orientable

by local consistency can move disc → continuously and preserves orientation



discs are differently oriented

$\Rightarrow$ contradicts local consistency.

The fundamental class [M]

FACT

Theorem For M closed n-mfd:

natural map from the LES

1) M orientable connected $\Longrightarrow$ $$H_n(M)\cong H_n(M,M\setminus x)=\mathbb{Z}\cdot\mu_x$$

once we choose an orientation

$$\Longrightarrow \quad \exists\,[M]\longleftrightarrow\mu_x$$

once we choose an orientation $(\mu_x)_{x\in M}$

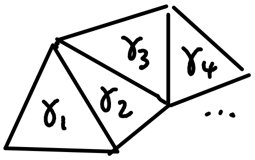
called **fundamental class**

(if swap orientation: for $-\mu_x$ get $-[M]$)

2) M not orientable connected $\Longrightarrow$ $$H_n(M)=0$$ $$H_n(M;\mathbb{F}_2)=\mathbb{F}_2\cdot[M]\cong\mathbb{F}_2$$

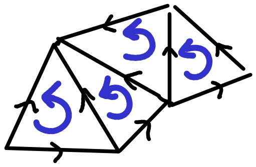
(or any field of characteristic 2)

Construction of [M] if M has Δ-complex structure



M compact $\Longrightarrow$ finite # n-simplices $\gamma_1,\dots,\gamma_N$

M oriented $\Longrightarrow$ pick orientations of $\gamma_1,\dots,\gamma_N$ to agree with given orientation of M: for $x\in\mathrm{Int}(\gamma_i)$



$$\mathbb{Z}\cdot\mu_x=H_n(M,M\setminus x)\xrightarrow[\text{exc.}]{\cong}H_n(\gamma_i,\gamma_i\setminus x)=\mathbb{Z}\cdot\gamma_i$$
$$\mu_x\longmapsto\gamma_i$$

$\Longrightarrow$ $\boxed{[M]:=\sum\gamma_i}$ satisfies $\partial[M]=0$ ✓ (each facet arises twice with opposite signs)

$$\begin{tikzcd} H_n(M) \arrow[r] & H_n(M,M\setminus x) \arrow[r, "\cong"] & H_n(\gamma_i,\gamma_i\setminus x) \\ {[M]} \arrow[rr, bend right, mapsto] & \mu_x \arrow[r, mapsto] & \gamma_i \end{tikzcd}$$

More generally: $[M]:=\sum\pm\gamma_i$ where signs come from
$$H_n(M,M\setminus x)\xrightarrow{\cong}H_n(\gamma_i,\gamma_i\setminus x)$$
$$\mu_x\longmapsto\pm\gamma_i$$
(so compare orientation μ_x with orientation of γ_i)

Not difficult to see that $H_n^{\Delta}(M)=\mathbb{Z}\cdot[M]$, so $\Rightarrow H_n(M)\cong H_n(M,M\setminus x)$
$$[M]\longmapsto\mu_x$$

Also $\mathbb{Z}\xrightarrow{\cong}\mathbb{Z}\cdot[M]$ since $C_{n+1}(M)=0$ ($\nexists$ $(n+1)$-simplices since $\dim M=n$)

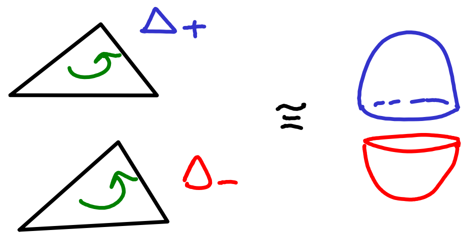
M non-orientble $\Longrightarrow$ each facet of γ_i appears twice in $\partial\sum\gamma_i$

$\Longrightarrow$ $\partial\sum\gamma_i=0$ over $\mathbb{F}_2$ independently of choices of orientations of γ_i. ✓

Examples

1) $S^n = \dfrac{\overset{\Delta_+}{\Delta^n} \cup \overset{\Delta_-}{\Delta^n}}{\text{glue bdries}}$

S^2:

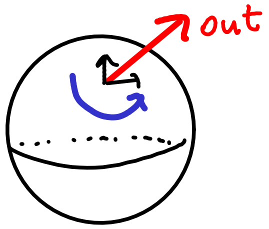


$[S^n] = \Delta_+ - \Delta_-$ if use canonical orientation we discussed

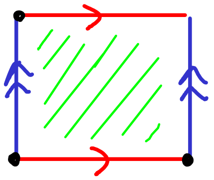
hence
$\partial[S^n] = \partial\Delta - \partial\Delta = 0$

$\mathbb{D}^n \subseteq \mathbb{R}^n$ canonical orientation

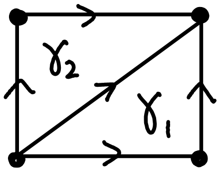
$\Rightarrow S^{n-1} = \partial \mathbb{D}^n$ " using outward normal first rule



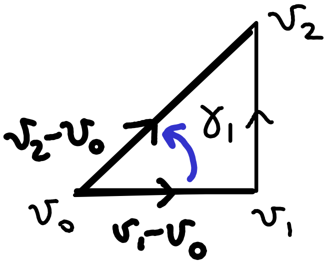
2) $T^2 =$



Δ-complex structure (compatibly with side identifications!)



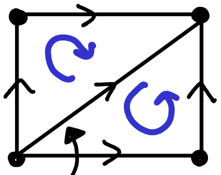
Want orientation induced by square $\subseteq \mathbb{R}^2$



$v_1 - v_0$, $v_2 - v_0$ positive $\mathbb{R}^2$-basis

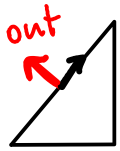
$\Rightarrow \gamma_1$ agrees with orientation

$\Longrightarrow$



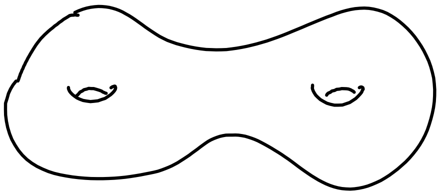
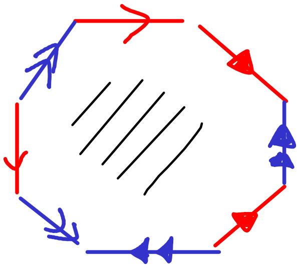
$[T^2] = +\gamma_1 \underset{\uparrow}{-} \gamma_2$

γ_2 orientation disagrees

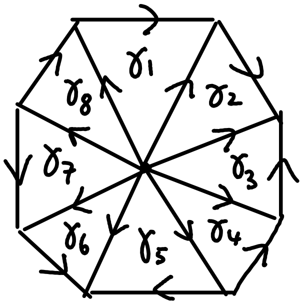


Rmk general feature: if two simplices are identified along a facet then the 2 respective outward normals are related by reflection.

So consistency $\Rightarrow$
- either simplices are compatibly oriented and the two induced orientations on facet are opposite
- or not compatibly oriented but facet orient^n is same, then need sign like in example when build $[T^2]$

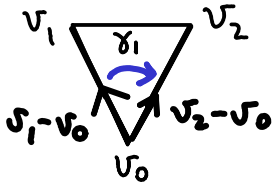
3) Recall Σ_2 =  = 

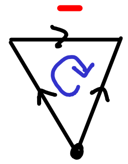
Δ-cx structure (compatible with side identifications!):



Use the orientation induced by polygon $\subseteq \mathbb{R}^2$

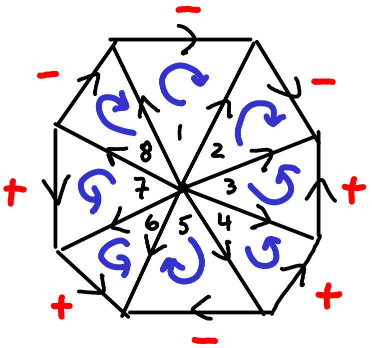
$$\Rightarrow [\Sigma_2] = -\gamma_1 - \gamma_2 + \gamma_3 + \gamma_4 - \gamma_5 + \gamma_6 + \gamma_7 - \gamma_8$$

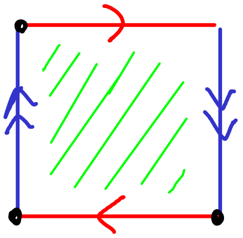
Explanation: 

$v_1 - v_0, v_2 - v_0$ is negative $\mathbb{R}^2$-basis $\Rightarrow$ 

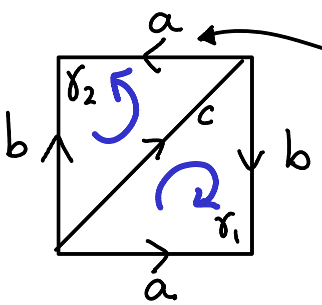
All simplices γ_i have v_0 = centre of polygon

$\Rightarrow$ sign $<\begin{matrix} - \text{ if outer edge clockwise} \\ + \text{ ——anti—— } \end{matrix}$



3) $\mathbb{R}P^2$ = 

(non-orientable) example



won't get Δ-cx structure if you try  since get issue here

Use the orientation induced by square $\subseteq \mathbb{R}^2$

$$\Rightarrow [\mathbb{R}P^2] = -\gamma_1 + \gamma_2$$

$$\partial [\mathbb{R}P^2] = -(b - a + c) + (a - b + c)$$

$$= -2b + 2a$$

$$\neq 0 \quad \text{so } \underline{\text{not}} \text{ cycle in } C_*^{CW}(\mathbb{R}P^2)$$

However, working modulo 2:

$$\partial [\mathbb{R}P^2] = 0 \in C_*^{CW}(\mathbb{R}P^2; \mathbb{F}_2) \text{ since } 2 = 0 \text{ in } \mathbb{F}_2$$

$$\Rightarrow [\mathbb{R}P^2] \in H^2(\mathbb{R}P^2; \mathbb{F}_2)$$

Degree

Def M, N oriented closed connected n-mfds, $f: M \to N$

$$f_*: H_n(M) \longrightarrow H_n(N)$$
$$[M] \longmapsto \underbrace{\deg(f)}_{\in \mathbb{Z}} \cdot [N]$$

Lemma If $f^{-1}(y)$ finite,

then $\deg(f) = \sum_{x \in f^{-1}(y)} \deg (f_y)_*$

local degree

local map like in chapter 7

pf

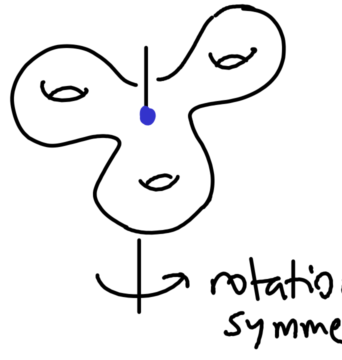
$$\begin{tikzcd}
{[M]} \arrow[r, phantom, "\in"] \arrow[dd, mapsto] & H_n(M) \arrow[r, "f_*"] \arrow[d] & H_n(N) \arrow[d, phantom, "\cong"] \arrow[r, phantom, "\ni"] & {[N]} \\
 & \bigoplus_{x \in f^{-1}(y)} H_n(M, M \setminus x) \arrow[r, "(f_x)_*"] & H_n(N, N \setminus y) \arrow[r, phantom, "\ni"] \arrow[d, phantom, "\ni"] & \mu^N_y \arrow[u, mapsto] \\
\bigoplus \mu^M_x \arrow[ur, phantom, "\in"] \arrow[rr, mapsto] & & \left(\sum \deg(f_x)_*\right) \cdot \mu^N_y &
\end{tikzcd}$$

$\square$

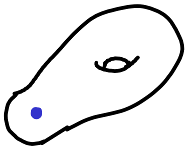
Examples

1) $S^1 \longrightarrow S^1$, $z \longmapsto z^n$, $[S^1] \longmapsto n \cdot [S^1]$ so $\deg = n$

2) $\Sigma_3 =$



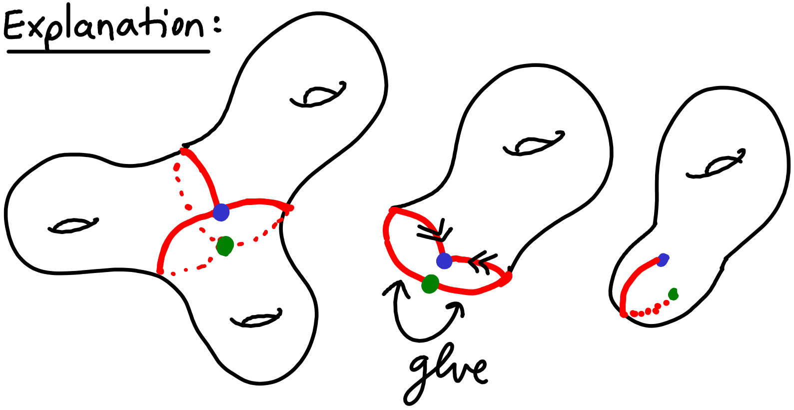
$\xrightarrow[q]{\text{quotient}} \Sigma_3 \big/ \left(\mathbb{Z}/3\text{-rotation action}\right) =$



$= \Sigma_1$

torus

Explanation:



Easy check: $\deg(q) = 3$
(e.g. use local degrees)

Cultural Rmk

For M, N, f smooth, the $\deg f = \#$(preimages of a generic point of N)

Idea: $\deg f$ tells you how many times you cover N. (almost all points work)

Poincaré duality

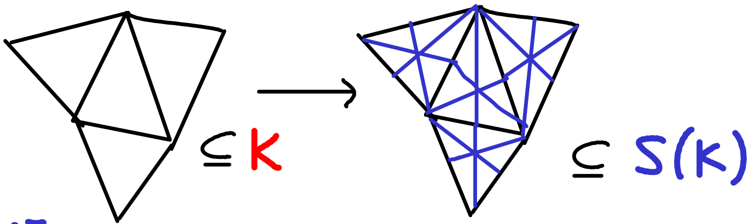
FACT Theorem For M closed n-mfd

M oriented ⟹ $\boxed{H^k(M) \cong H_{n-k}(M)}$ s.t. $\begin{matrix}1 & \leftrightarrow & [M]\\ \cap & & \cap \\ H^0(M) & \cong & H_n(M)\end{matrix}$

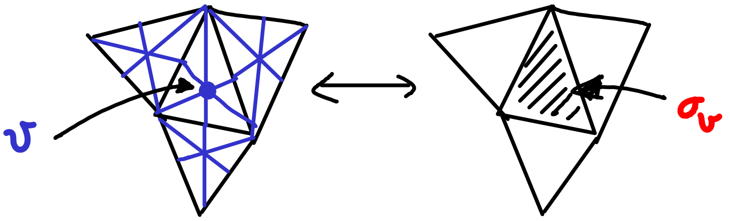
M non-oriented ⟹ same holds with $\mathbb{F}_2$ coefficients

Sketch proof when M is a simplicial complex K (Non-examinable)

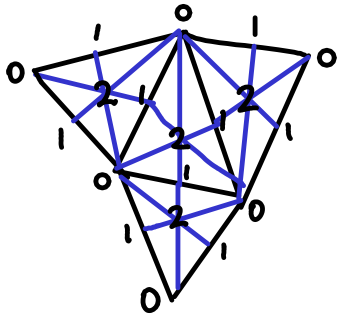
S(K) = barycentric subdivision



1) simplex $\sigma = \sigma_v$ of K with barycentre v $\longleftrightarrow$ $v = v_\sigma$ vertex of S(K)

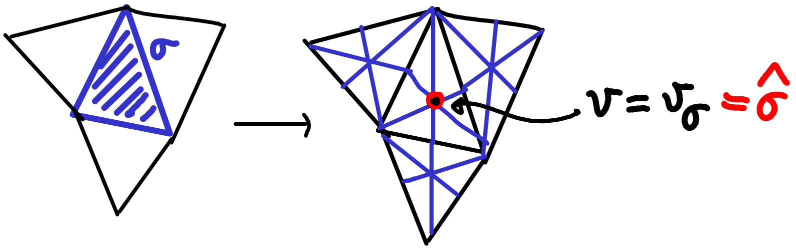


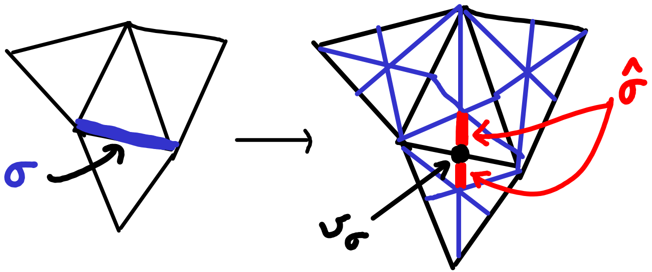
2) $ht(v) = (\underline{\text{height}} \text{ of } v) = \dim \sigma_v$

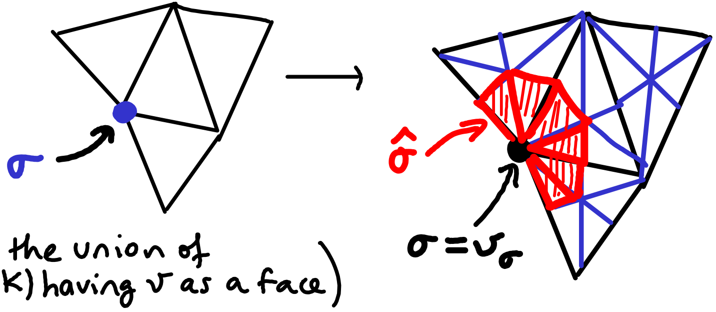


3) σ k-simplex of K
$\downarrow$
dual simplex

$$\hat\sigma = \bigcup_{\substack{\tau \in S(K),\ v_\sigma \in \tau,\\ ht(v_\sigma) \text{ is min}\\ \text{of heights of}\\ \text{vertices of } \tau}} \tau$$







Rmk: $\bigcup \tau$ with $ht(v_\sigma)$ max will give back σ.
Thus $\hat\sigma, \sigma$ intersect transversely at v_σ.
One can also describe $\hat\sigma$ as

$\hat\sigma = \bigcap_{\substack{\text{vertices}\\ v \in \sigma}} \overline{\mathrm{Star}_{S(K)}}(v)$ ⟵ (closed star is the union of simplices of S(K) having v as a face)

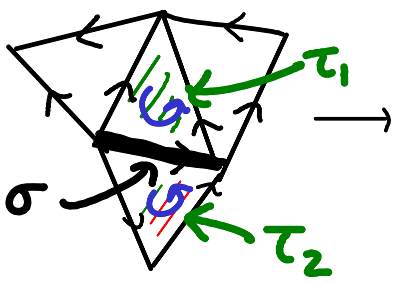
FACTS

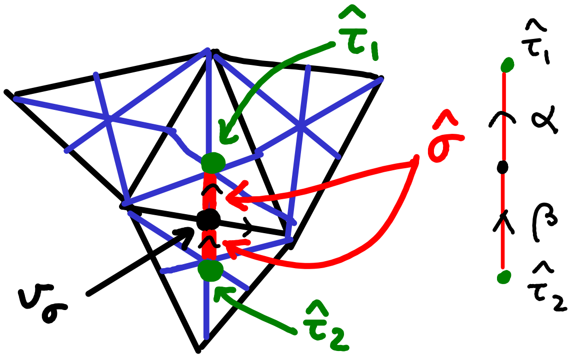
- $\dim \hat{\sigma} = n - \dim \sigma$
- dual cells $\hat{\sigma}$ give a cell decomposition of M

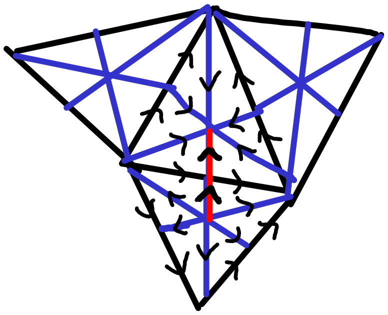
("polygonal" complex rather than Δ-cx)

- ⊛ $\partial \hat{\sigma} = \sum_{\substack{\text{facet} \\ \sigma \subsetneq \tau \\ \tau \in K}} \pm \hat{\tau}$

need compare orientations of σ, τ
(+ if σ as a facet of τ has boundary orientation)

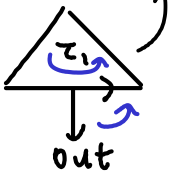


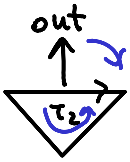




$\hat{\sigma} = \alpha + \beta$

$\partial \hat{\sigma} = + \hat{\tau}_1 - \hat{\tau}_2$





4) **dual chain complex**

D_{n-k} = free abelian group on dual chains $\hat{\sigma}$

$H_*(M) \cong H_*(D_*, \partial_*)$ (since $\hat{\sigma}$ give a cell decomp. of M)

5) $\varphi : D_{n-k} \longrightarrow C^k(M)$

$\hat{\sigma} \longmapsto \sigma^*$ where $\sigma^*(\alpha) = \begin{cases} 0 & \text{for } k\text{-cells } \alpha \neq \sigma \\ 1 & \text{for } \alpha = \sigma \end{cases}$

Rmk notice that
$\sigma^*(\alpha) = \#\,\alpha$ intersects $\hat{\sigma}$
counted with orientation signs.

- φ linear bijection ✓
- chain map:

see ⊛

$\varphi(\partial \hat{\sigma}) = \varphi \sum \pm \hat{\tau} = \sum \pm \tau^*$

$\partial^* \varphi(\hat{\sigma}) = \partial^* \sigma^* = \left(\sigma^* \circ \partial : \tau \xmapsto{\partial} \sum_{\substack{\text{facets } \sigma_i \\ \text{of } \tau}} \pm \sigma_i \xmapsto{\sigma^*} \begin{cases} \pm 1 & \text{if one of } \sigma_i = \sigma \\ 0 & \text{else} \end{cases} \right)$

$= \sum \pm \tau^* = \varphi(\partial \hat{\sigma})$ ✓

UPSHOT φ is chain iso so get iso:

$$H_*(M) \underset{④}{\cong} H_*(D_*, \partial_*) \xrightarrow[\varphi]{\cong} H^{n-*}(M)$$

Cor $\chi(\text{odd dimensional closed orientable mfd}) = 0$

Pf **Betti numbers** $b_i = \text{rank } H_i(M) \underset{\uparrow}{=} \text{rank } H^i(M) \underset{\uparrow}{=} \text{rank } H_{n-i}(M)$

universal coeff. thm. Poincaré duality

$$\chi(M) = b_0 - b_1 + \dots + b_{\dim M-1} - b_{\dim M}$$

equal. □

(Poincaré–)Lefschetz duality

Theorem

M compact oriented n-mfd with boundary ⇒

$$\boxed{H^k(M) \cong H_{n-k}(M,\partial M)}$$

$1 \in H^0(M) \longmapsfrom [M,\partial M] \in H_n(M,\partial M)$

relative fundamental class

$$\boxed{H_k(M) \cong H^{n-k}(M,\partial M)}$$

Non-oriented ⇒ same holds with $\mathbb{F}_2$ coefficients.

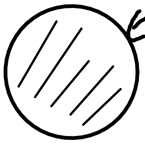
either by universal Coefficient thm since $H_0(M,\partial M)=0$ or by hand since given $p\in M$, $\varphi\in C^0(M,\partial M)$ consider $(\partial\varphi)(\gamma)$ for γ path from p to any $q\in\partial M$.

Pf basically same as Poincaré duality. □

Cor M compact, connected, $\partial M \neq \emptyset \implies \begin{cases} H^n(M) \cong H_0(M,\partial M) = 0 \\ H_n(M) \cong H^0(M,\partial M) = 0 \end{cases}$

Examples

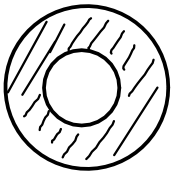
1) $\mathbb{D}^n$



$\partial\mathbb{D}^n = S^{n-1}$

$\mathbb{Z} \cong H^0 \mathbb{D}^n \cong H_n(\mathbb{D}^n, S^{n-1})$

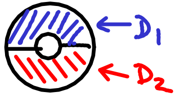
2)



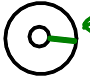
$A = \text{annulus} \subseteq \mathbb{R}^2$

$\simeq S^1$

$\mathbb{Z} \cong H^0 A \cong H_2(A,\partial A) \leftarrow$ generator $D_1 - D_2$



$\mathbb{Z} \cong H^1 A \cong H_1(A,\partial A) \leftarrow$ generator

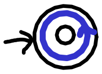


$\leftarrow$ copy of $\mathbb{D}^1$

$0 \cong H^2 A \cong H_0(A,\partial A)$

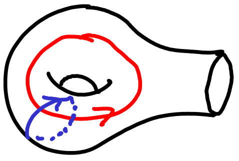
(notice $\partial\mathbb{D}^1 \to \partial A$)

Rmk notice gen. of $H_1(A)$ is →



which intersects gen. of $H_1(A,\partial A)$ once transversely.

3) $M = T^2 \setminus$ open ball =



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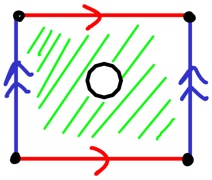
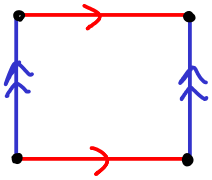
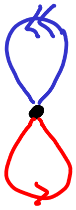


$\simeq$ def. retract



$\simeq$



$\simeq S^1 \vee S^1$

$$\Rightarrow H_*(M,\partial M) \cong H^{2-*}(S^1 \vee S^1) = \begin{cases} \mathbb{Z}^2 & * = 1 \\ \mathbb{Z} & * = 2 \\ 0 & \text{else} \end{cases}$$

$\leftarrow$ gen. by 2 loops

$\leftarrow$ gen. by $[M,\partial M]$

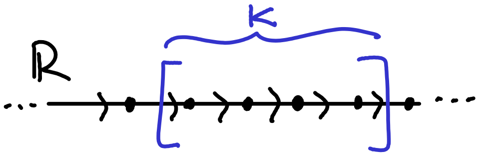
What happens in the non-compact case?

Locally finite homology (Borel-Moore)

$C_*^{lf}(X)$ allow infinite sums $\sum \overset{\in\mathbb{Z}}{n_i}\, \sigma_i$ ← generators of $C_*(X)$
s.t. given any compact subset $K \subseteq X$,
$\#\{n_i \neq 0 : K \cap \mathrm{Im}\, \sigma_i \neq \emptyset\} < \infty$.

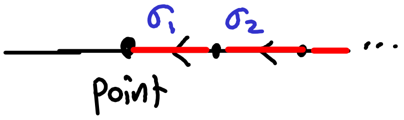
Examples

- $C_1^{lf}(\mathbb{R}) \ni \sum_{m\in\mathbb{Z}} \sigma_m$, $\sigma_m : I \cong [m,m+1] \subseteq \mathbb{R}$



$\Rightarrow$ get cycle $[\mathbb{R}] \in H_1^{lf}(\mathbb{R})$

- $C_0^{lf}(\mathbb{R}) \ni [\text{point}] \ni \partial(\sigma_1 + \sigma_2 + \cdots)$ is a boundary:



exercise $H_*^{lf}(\mathbb{R}) = \begin{cases} \mathbb{Z} & *=1 \\ 0 & \text{else} \end{cases}$ $(\cong H^{1-*}(\mathbb{R}))$

FACT Theorem M orientable n-mfd (possibly not compact) $\Rightarrow$ $\boxed{H^*(M) \cong H_{n-*}^{lf}(M)}$

cohomology with compact supports $H_c^*(X)$

$C_c^*(X)$: only allow cochains $\phi : C_*X \to \mathbb{Z}$ s.t. $\exists$ compact $K \subseteq X$ with (depends on ϕ)
$\phi(C_*(X\setminus K)) = 0$ (vanish on chains in $X\setminus K$)

Example $\underset{\text{cycle}}{c \in C_*(X)}$ $\Rightarrow$ $\phi(\alpha) =$ signed # intersections of c with α
(geometric intersection #)
$\Rightarrow \phi \in C_c^*(X)$ since $\phi(\alpha) = 0$ if $\alpha \subseteq X \setminus \mathrm{Im}(c)$

Thm M orientable n-mfd (possibly not compact) $\Rightarrow$ $\boxed{H_*(M) \cong H_c^{n-*}(M)}$

Warning H_*^{lf}, H_c^* are not homotopy invariant (indeed non-trivial for $\mathbb{R}^n$)
Caused because they are not functorial. They are however functorial for **proper maps** (preimages of compact sets are compact)
Mayer-Vietoris holds for H_c^* but not for H_*^{lf}.

Fact $H_c^*(X) \cong \varinjlim H^*(X, X\setminus K)$ where compacts $K_1 \subseteq K_2$ give $H^*(M, M\setminus K_1) \to H^*(M, M\setminus K_2)$

Direct limit $\varinjlim G_i$ via maps $G_i \to G_j$ means $\sqcup G_i /$ identify $g \in G_i$ with its images under those maps
(The indices are partially ordered & directed: $\forall i,j, \exists k \geq i,j$ so can compare G_i, G_j inside G_k (via $G_i \to G_k$, $G_j \to G_k$)

Fact $\varinjlim$ is an exact functor.

Cap product and Poincaré duality revisited

X space, $k \geq \ell$

(sometimes write $\phi \cap \sigma$)

$$\cap : C_k(X) \times C^\ell(X) \longrightarrow C_{k-\ell}(X)$$

cap product

$$(\sigma : \Delta^k \to X) \cap (\phi : C_\ell(X) \to \mathbb{Z}) = \underbrace{\phi(\underbrace{\sigma|_{[e_0,\dots,e_\ell]}}_{\text{"bottom face"}})}_{\in \mathbb{Z}} \cdot \underbrace{\underbrace{\sigma|_{[e_\ell,\dots,e_k]}}_{\text{"top face"} \cong \Delta^{k-\ell}}}_{\in C_{k-\ell}(X)}$$

(easy) Properties

- $\cap$ bilinear
- $\partial(\sigma \cap \phi) = (-1)^\ell (\partial\sigma \cap \phi - \sigma \cap \partial^*\phi)$
- cycle $\cap$ cocycle is cycle
- boundary $\cap$ cocycle / cycle $\cap$ coboundary are boundaries

$$\Longrightarrow \boxed{\cap : H_k(X) \times H^\ell(X) \longrightarrow H_{k-\ell}(X)} \quad \text{bilinear}$$

Theorem (Poincaré duality) The map $\boxed{\phi \longmapsto [M] \cap \phi}$ gives following isos

① For M closed oriented n-mfd

$$[M] \cap \cdot \; : H^*(M) \xrightarrow{\cong} H_{n-*}(M)$$

② For M non-compact oriented n-mfd,

$$[M] \cap \cdot \; : H^*_c(M) \xrightarrow{\cong} H_{n-*}(M) \qquad ✪$$

$$[M] \cap \cdot \; : H^*(M) \xrightarrow{\cong} H^{\ell f}_{n-*}(M)$$

Sketch Pf of ① for smooth mfds (Non-examinable)

If M smooth $\Longrightarrow$ $\exists$ "good cover" $\mathcal{U}_i$ of M meaning open cover s.t.

FACT from Riemannian geometry ("convex neighbourhoods")

$\mathcal{U}_i \cong \mathbb{R}^n$

$\mathcal{U}_{i_1} \cap \dots \cap \mathcal{U}_{i_k} \cong \mathbb{R}^n$ or $\emptyset$

Then compute $H^*_c(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{else} \end{cases}$ and ✪ holds for $\mathbb{R}^n$.

$\Rightarrow$ ★ holds $\forall\, \mathcal{U}_i$

$\Rightarrow$ by naturality of ✪ and of Mayer-Vietoris get ✪ for $\bigcup_{\text{finite}} \mathcal{U}_i$

use 5-lemma

$\Rightarrow$ ✪ for M, which is ①. $\square$

General Pf of Poincaré duality ← Non-examinable

Step 1 : holds for $\mathbb{R}^n$

Pf $H^k_c(\mathbb{R}^n) \cong H^k(D^n, S^{n-1}) \cong \begin{cases} 0 & k \neq n \\ \mathbb{Z} & k = n \end{cases} \cong H_{n-k}(\mathbb{R}^n)$

(recall Fact: $H^*_c(X) \cong \varinjlim H^*(X, X \setminus K)$ ← can make K larger by picking K = large ball then $H^k(\mathbb{R}^n, \mathbb{R}^n \setminus K) \cong H^k(K, \partial K)$)

Pick Δ-cx structure for $\mathbb{R}^n$. So $[\mathbb{R}^n] = \sum \pm \sigma_i$ ← sum over n-simplices.

Say ∃ simplex $\sigma_0 : \Delta^n \to \mathbb{R}^n$. Define $\phi : C^{CW}_c(\mathbb{R}^n) \to \mathbb{Z}$, $\phi(\sigma_0) = \pm 1$ ⊛

$\phi\left(\begin{smallmatrix}\text{other}\\ \text{simplices}\end{smallmatrix}\right) = 0$

$\Rightarrow \delta\phi = 0$ for dim reasons

$[\mathbb{R}^n] \cap \phi = \sum \phi(\pm\sigma_i) = \phi(\pm\sigma_0) = 1$ (pick sign in ⊛)

Step 2 holds for $A, B, A \cap B \Longrightarrow$ holds for $A \cup B$

Pf Mayer-Vietoris for H^*_c, naturality, 5-Lemma ✓

Step 3 holds for A_i, and $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \Rightarrow$ holds for $\bigcup A_i$

Pf By applying $\varinjlim$: both sides of P.D. iso commute with limits ✓

Step 4 holds for open subsets in $\mathbb{R}^n$

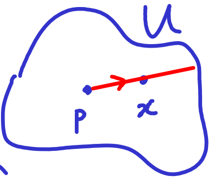
Pf Every such set is a union of convex open sets (e.g. balls)

By Step 3 enough to consider case of finite union.

By induction on # convex open sets:

1 convex set $U \cong \mathbb{R}^n$ via a proper homeomorphism,

now use Step 1 ✓



fix $p \in U$

$U \longrightarrow \mathbb{R}^n$

$x \longmapsto \dfrac{x - p}{\text{dist } p \text{ to } \partial U \text{ along ray to } x}$

2 convex sets : KEY TRICK convex set ∩ convex set is convex in $\mathbb{R}^n$!

$\Rightarrow$ use Step 2 & previous case

k+1 convex sets : $A = \bigcup\{\text{first } k \text{ convex sets}\}$, $B =$ last convex set

$\Rightarrow A \cap B \subseteq B$ is a union of k convex sets

$\Big\} \Rightarrow$ use Step 2 & Inductive hypothesis ✓

Step 5 holds for mfd M

Consider open sets in M for which it holds.

By a Zorn's Lemma argument we get a maximal open subset U where holds.

If $U \neq M$ pick $p \in M \setminus U$ and nbhd $V \cong \mathbb{R}^n$ of p. Then holds for $U, V, U \cap V$ (note $U \cap V \subseteq V \cong \mathbb{R}^n$ open, so Step 4 applies) so by Step 2 holds for $U \cup V$

Contradicts maximality. ✓ □

Recall there is a well-defined evaluation of H^*-classes on H_*:

$$\begin{array}{rcl} \langle\cdot,\cdot\rangle : H_k(M;R)\otimes H^k(M;R) & \longrightarrow & R \\ c\otimes\alpha & \longmapsto & \langle c,\alpha\rangle = \overset{\text{any representative cocycle } \varphi \text{ for } \alpha}{\varphi(c)} \end{array}$$

__Easy exercise__ $\boxed{\langle c, \alpha\cup\beta\rangle = \langle c\cap\alpha, \beta\rangle}$ any $\alpha,\beta\in H^*$, $c\in H_*$

__Corollary of Poincaré duality__

M compact oriented n-mfd, $\mathbb{F}$ field.

$$\Rightarrow \boxed{\begin{array}{l} H^k(M;\mathbb{F})\otimes H^{n-k}(M;\mathbb{F}) \xrightarrow{(\star)} \mathbb{F} \\ \qquad \alpha\otimes\beta \longmapsto \langle [M],\alpha\cup\beta\rangle \\ \text{is a } \underline{\text{non-singular}} \text{ bilinear form.} \end{array}}$$

hence: $H^*(M;\mathbb{F}) \cong (H^{n-*}(M;\mathbb{F}))^{\overset{\text{dual}}{*}}$

__Pf.__ By exercise, $\langle [M], \alpha\cup\beta\rangle = \langle [M]\cap\alpha, \beta\rangle \underset{\text{definition of Poincaré duality PD}}{=} \langle PD(\alpha),\beta\rangle$

So the following diagram commutes:

PD is iso over $\mathbb{Z}$, hence iso over $\mathbb{F}$ by universal coefficients.

$$\begin{tikzcd}
H^k(M;\mathbb{F})\otimes H^{n-k}(M;\mathbb{F}) \arrow[dr, "\text{pairing } (\star)"] \arrow[dd, "\cong"', "PD\otimes id"] & \\
& \mathbb{F} \\
H_{n-k}(M;\mathbb{F})\otimes H^{n-k}(M;\mathbb{F}) \arrow[ur, "\langle\cdot,\cdot\rangle \leftarrow (\star\star)"'] &
\end{tikzcd}$$

By universal coefficients, $H^*(M;\mathbb{F})\cong \mathrm{Hom}(H_*(M;\mathbb{F}),\mathbb{F})$ via $\beta\mapsto\langle\beta,\cdot\rangle$

Hence $(\star\star)$ is a non-degenerate bilinear pairing ← (using that for any $\mathbb{F}$-vector space V $V\otimes \underbrace{V^*}_{\mathrm{Hom}(V,\mathbb{F})}\longrightarrow\mathbb{F}$, $v\otimes\varphi\mapsto\varphi(v)$ is non-deg. pairing.

Hence so is the pairing $(\star)$ in the diagram. □

__Remark__ For M non-orientable, the same holds for $\mathbb{F}$ of characteristic 2, e.g. $\mathbb{Z}/2$.

For $\mathbb{Z}$ coefficients it can fail if $H^*(M)\not\cong \underbrace{\mathrm{Hom}(H_*M,\mathbb{Z})}_{\text{has no torsion}}$. So we define:

__Betti group__ $B^k(M) = H^k(M)/\text{torsion}(H^k(M))$

$B_k(M) = H_k(M)/\text{torsion}(H_k(M))$

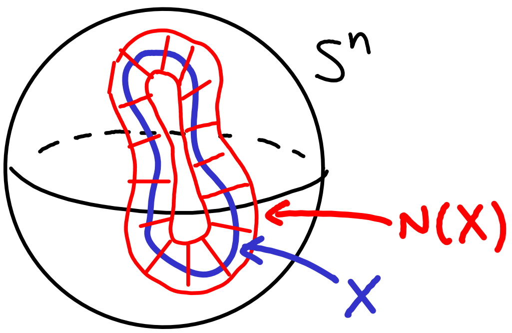
By what we proved in the section on universal coefficients, $B^q(M)\cong\mathrm{Hom}(B_q(M),\mathbb{Z})$ whenever $H_{q-1}(M)$ is finitely generated (which we know holds for compact mfds). The iso is given by $\langle\cdot,\cdot\rangle$ again: this descends to quotients since $\langle c,\alpha\rangle = 0\in\mathbb{Z}$ if c or α has finite order (i.e. torsion). The same proof as above yields:

M compact oriented n-mfd $\Rightarrow \boxed{\begin{array}{l} B^k(M)\otimes B^{n-k}(M)\longrightarrow\mathbb{Z},\ \alpha\otimes\beta\mapsto\langle[M],\alpha\cup\beta) \\ \text{is non-degenerate bilinear form.}\end{array}}$

Also the __Remark__ holds.

__Example__ Use this to prove ex. 4(c) sheet 3. (__Hint__: $H^{2k}(\mathbb{C}P^n)\cup H^{2n-2k}(\mathbb{C}P^n) = H^{2n}(\mathbb{C}P^n)$)

Alexander duality



(in fact, enough to assume X is locally contractible)

$\emptyset \neq X \subsetneq S^n$ compact subset s.t.

∃ open neighbourhood $N(X)$ which deformation retracts to X such that $\overline{N(X)} \subseteq S^n$ is an n-mfd with boundary.

Theorem $\boxed{\tilde{H}_*(X) \cong \tilde{H}^{n-*-1}(S^n \setminus X)}$

Pf later

Example $X \subseteq S^3$ **knot** (i.e. $X = \text{image}(S^1 \xrightarrow[\text{onto the image}]{\text{homeomorphism}} S^3)$)

↑ **embedding**

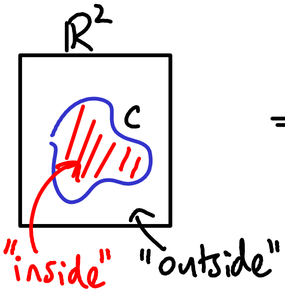
$\Longrightarrow$ $N(X) \cong$ solid torus $\simeq S^1$

$\Longrightarrow$

$$\tilde{H}_0(X) = 0 = \tilde{H}^2(S^3 \setminus X)$$
$$\tilde{H}_1(X) = \mathbb{Z} = \tilde{H}^1(S^3 \setminus X)$$
$$\tilde{H}_2(X) = 0 = \tilde{H}^0(S^3 \setminus X)$$

so the homology of a **knot complement** does not tell knots apart (always same)

Theorem (**Jordan curve theorem**)



e.g. by stereographic projection $S^2 \cong \mathbb{C} \cup \infty$

$C \cong S^1$ closed curve in $\mathbb{R}^2 \subseteq S^2$

$\Longrightarrow$ $\mathbb{R}^2 \setminus C$ has 2 path-components (= connected components)

Similarly for $S^n \cong C \subseteq \mathbb{R}^{n+1} \subseteq S^{n+1}$.

Pf $S^n \cong C \subseteq \mathbb{R}^{n+1} \subseteq S^{n+1}$ $\Rightarrow$ $\mathbb{Z} \cong \tilde{H}_n(S^n) \overset{\text{Alexander duality}}{\cong} \tilde{H}^0(S^{n+1} \setminus C)$

$\Rightarrow$ $H^0(S^{n+1} \setminus C) \cong \mathbb{Z}^2$

$\Rightarrow$ $S^{n+1} \setminus C$ has 2 path components. ∎

Proof Alexander duality

Abbreviate $N = N(X)$ (nbhd of X which is $\simeq X$)

$$Y := S^n \setminus N \simeq S^n \setminus X$$

for $* < n-1$

$$\tilde{H}^{n-*-1}(Y) = H^{n-*-1}(Y) \underset{\text{Lefschetz}}{\cong} H_{*+1}(Y, \partial Y) \underset{\text{exc.}}{\cong} H_{*+1}(S^n, \bar{N}) \underset{\substack{\text{LES}\\ \text{using } *<n-1}}{\cong} \tilde{H}_*(\underset{\underset{X}{\wr}}{\bar{N}})$$

for $* = n-1$

$$\tilde{H}^0(Y) \oplus \mathbb{Z} \cong H^0(Y) \underset{\text{Lef.}}{\cong} H_n(Y, \partial Y) \underset{\text{exc.}}{\cong} H_n(S^n, \bar{N}) \cong \tilde{H}_{n-1}(\underset{\underset{X}{\wr}}{\bar{N}}) \oplus \mathbb{Z}$$

Explanation of ↗:

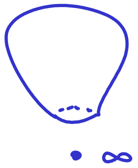
$$\begin{tikzcd}
\text{LES:} & 0 \arrow[r] & \tilde{H}_n(S^n) \arrow[r] \arrow[dr, "\cong"'] & H_n(S^n, \bar{N}) \arrow[r] \arrow[d, "\text{quotient}"] & \tilde{H}_{n-1}(\bar{N}) \arrow[r] & \overset{\tilde{H}_{n-1}(S^n)\,=}{0} \\
 & & & H_n(S^n, S^n \setminus \infty) \cong \mathbb{Z} & &
\end{tikzcd}$$

is SES

⊛ $\tilde{H}_n(\bar{N}) = H_n(\bar{N}) = 0$ ↗ 0

(see Cor. to Poincaré-Lefschetz, using: each (path-) connected component of the manifold $\bar{N}$ has non-empty boundary)

$S^n = \mathbb{R}^n \cup \infty$ ↖ $S^n \setminus \infty$



$\Rightarrow$ Hence that quotient map gives a splitting of the SES.

for $* = n$ $\quad H^{n-*-1}(Y) = H^{-1}(Y) = 0$

$$H_n(X) \cong H_n(N) \underset{\underset{\text{Lefschetz duality}}{\uparrow}}{\cong} H^0(N, \partial N) \underset{\nwarrow \text{ see ⊛}}{=} 0. \ \square$$