# C3.1 Algebraic Topology

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Please be aware there are likely typos in these notes: comments/corrections are welcome!

## Course Book

- **Hatcher**, *Algebraic Topology* — Chp. 2 & 3
  This is also freely available from the author's website.

## Expectations

- You are expected to read chapters 2 & 3 of Hatcher
- You should read the technical remarks about orientation signs in these notes: we will likely not have time for those in lectures.
- This course will not discuss intersection numbers rigorously. The notes often mention these in order to develop your intuition. The books by Bott & Tu and Guillemin & Pollack discuss these ideas rigorously.

## Other references

- Ulrike Tillmann's C3.1 notes — see course page
- The discussion of orientations in my B3.2 Geometry of Surfaces notes may be helpful.

## Other books

**Massey**, A basic course in Algebraic Topology
(Not "Algebraic Topology: An Introduction" ← does not treat homology)

MORE BASIC but full of ideas:

**Fulton**, Algebraic Topology: a first course.

MORE ADVANCED:

**May**, A concise course in Algebraic Topology
**Davis & Kirk**, Lecture Notes in Algebraic Topology
**Bredon**, Topology and Geometry
Classics by **Spanier**, **Dold**, also see references in May's book
**Bott & Tu**, Differential forms in Algebraic Topology
**Guillemin & Pollack**, Differential Topology

# CONTENTS

## 6. MAYER-VIETORIS SEQUENCE

MV LES, $H_*(S^n)$

wedge sum $X \vee Y$, cone $CX$, suspension $\Sigma X$, connected sum $X \# Y$

## 7. DEGREE OF MAPS OF SPHERES

degree, application to tangent vector fields on sphere, hairy ball theorem

local degree, proof of fundamental thm of algebra

## 8. CELLULAR HOMOLOGY

CW complexes, cellular complex, rank $H_n^{CW} \leq$ # n-cells

$H_*^{CW}(\mathbb{D}^1 \times \mathbb{D}^1)$, $H_*^{CW}(\mathbb{RP}^n)$, $H_*^{CW}(S^n)$, $H_*^{CW}(\Sigma_g)$

$\Delta$-cx $\Rightarrow$ CW cx, $H_*^{CW}(X) \cong H_*^{\Delta}(X) \cong H_*(X)$, Axioms for homology

## 9. COHOMOLOGY

cochains, cohomology, $H^*(X)$, $H^*_{CW}(X)$, $H^*_{\Delta}(X)$, $H^*(\mathbb{RP}^3)$

functoriality, homotopy invariance, cochain homotopy, dual of a SES

excision, LES, Mayer-Vietoris for $H^*$, axioms for cohomology

## 10. CUP PRODUCT

Cup product, $H^*(X)$ unital graded-commutative ring, pull-back is ring hom,

examples: $H^*(T^2)$, $H^*(\Sigma_2)$, remarks about intersection theory

## 11. KÜNNETH FORMULA AND PRODUCT SPACES

Tensor products of R-mods, tensor product of chain cxes,

algebraic Künneth thm, product spaces $X \times Y$, Euler characteristic $\chi$

CW-cx for product space, Künneth thm, $H^*(S^n \times S^m)$, $H^*(T^n)$

## 12. UNIVERSAL COEFFICIENTS THEOREM

Universal coeff. thm, Background on Ext groups and free resolutions

(Co)homology with coefficients in a ring/field/module, $H^*(\mathbb{RP}^2; \mathbb{Z}/2)$

Univ. coeff. thm for PID R, Duality $H^*(X;\mathbb{F}) \simeq H_*(X;\mathbb{F})$ over fields

Structure thm for f.g. mods M over PID R, $\mathrm{Ext}^1_R(M;R)$, torsion shift $H_*$ to $H^{*+1}$

## 13. MANIFOLDS: POINCARÉ-LEFSCHETZ DUALITY

Local orientation, orientation, fundamental class, degree, P. duality, L. duality,

Locally finite homology $H_*^{lf}$, cohomology with compact supports $H_c^*$, Cap product and P.D.,

Alexander duality, Knot complements, Jordan curve thm

# 0. OVERVIEW OF THE COURSE

## Motivation

Space $X$ $\xrightarrow{\text{associate}}$ Algebraic object $A(X)$ like numbers, groups, rings, ...

Isomorphism of spaces $X \cong Y$ $\Longrightarrow$ Isomorphism $A(X) \cong A(Y)$

PAY-OFF Can tell spaces apart: compute $A(X), A(Y)$ $\rightsquigarrow$ if $A(X) \not\cong A(Y)$ then $X \not\cong Y$

## Examples

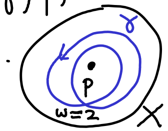
1) Set $X$ $\longrightarrow$ $A(X) = \# X \in \mathbb{N} \cup \{\infty\}$
   (bijection $X \to Y$) $\Longrightarrow$ same size

2) Vector space $X$ $\longrightarrow$ $A(X) = \dim X \in \mathbb{N} \cup \{\infty\}$
   (linear iso $X \to Y$) $\Longrightarrow$ same dim

3) Topological Space $X$ $\longrightarrow$
   - $\# \pi_0(X) = $ **# path components** $\in \mathbb{N} \cup \{\infty\}$
   - **# connected components** $\in \mathbb{N} \cup \{\infty\}$
   - $\chi(X) = $ **Euler characteristic** $\in \mathbb{Z}$
   - (for $X \subseteq \mathbb{R}^2$) Function $X \times \mathcal{L}X \longrightarrow \mathbb{Z} \cup \{\infty\}$, where loops $= C^0(S^1, X)$

     $(p, \gamma) \longmapsto w(\gamma; p)$

     **Winding number** of $\gamma$ around $p$.




(Homeomorphism $X \to Y$) $\longrightarrow$ $A(X) = A(Y)$

CONVENTION for this course:

Space = topological space

Map of spaces = continuous map = cts map

For spaces, "$\cong$" means homeomorphism

"id" = identity map

All diagrams commute unless we say otherwise, e.g. $\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \gamma \downarrow & & \downarrow \beta \\ C & \xrightarrow{\delta} & D \end{array}$ means $\beta \circ \alpha = \delta \circ \gamma$

## Category theory is the best language to phrase all this

Algebraic topology builds <u>functors</u> such as

$$F : \left\{\begin{array}{l}\text{Topological spaces}\\ \text{and continuous maps}\end{array}\right\} \begin{array}{l}\nearrow \left\{\begin{array}{l}\text{Groups}\\ \text{and group homs}\end{array}\right\}\\ \longrightarrow \left\{\begin{array}{l}\text{Abelian groups}\\ \text{and group homs}\end{array}\right\}\\ \searrow \left\{\text{Rings and ring homs}\right\}\end{array}$$

We will not use much <u>category theory</u>, just basic terminology:

<u>Def</u> A **category** $\mathcal{C}$ consists of the data:

$Ob(\mathcal{C})$ = a collection of **objects**

$Hom(A,B)$ = a set of **morphisms** between any $A, B \in Ob\,\mathcal{C}$ ("**arrows**")

- with **composition** rule $Hom(B,C) \times Hom(A,B) \xrightarrow{\circ} Hom(A,C)$ which is associative. $A \xrightarrow{f} B \xrightarrow{g} C$, $g \circ f$

- with **identity** morphs $id_A \in Hom(A,A)$ s.t. $f \circ id_A = id_B \circ f = f$ $\forall (f : A \to B) \in Hom(A,B)$

<u>Example</u>
Sets = { sets with all maps between sets }
Top = { topological spaces with continuous maps }
Gps = { groups with group homs }

<u>Def</u> A (covariant) **functor** $F : \mathcal{C}_1 \to \mathcal{C}_2$ is the data:
- an assignment $(A \in Ob\,\mathcal{C}_1) \longmapsto (F(A) \in Ob\,\mathcal{C}_2)$
- an assignment $(A \xrightarrow{f} B) \longmapsto (F(A) \xrightarrow{F(f)} F(B))$, with $f \in Hom_{\mathcal{C}_1}(A,B)$ and $F(f) \in Hom_{\mathcal{C}_2}(F(A),F(B))$

Compatible with identities and compositions.

$$F(id_A) = id_{F(A)} \qquad F(g \circ f) = F(g) \circ F(f)$$

A **contravariant functor** is defined similarly except it reverses the direction of arrows: $(F(A) \xleftarrow{F(f)} F(B)) \in Hom(F(B), F(A))$
(so $F(g \circ f) = F(f) \circ F(g)$ reverses order of compositions)

<u>Examples</u>

1) $F$ : Top $\longrightarrow$ Sets, $A \mapsto A$, $f \mapsto f$ "forget the topology and continuity"

2) $F$ : Sets $\longrightarrow$ Gps, $A \longmapsto$ free abelian group generated by $A$

$$\mathbb{Z}\langle A\rangle = \left\{ \sum_{\text{finite}} n_i \cdot a_i \;:\; \begin{array}{l} a_i \in A \\ n_i \in \mathbb{Z}\end{array}\right\}$$

$$(A \xrightarrow{f} B) \longmapsto \left( F(A) : \begin{array}{l}\mathbb{Z}\langle A\rangle \longrightarrow \mathbb{Z}\langle B\rangle \\ \sum n_i \cdot a_i \longmapsto \sum n_i \cdot f(a_i)\end{array}\right)$$

When we say a construction is **natural** we mean functorial:

$$\begin{array}{ccc} X & \xrightarrow{A} & A(X) \\ f\downarrow & & \downarrow A(f) \\ Y & \xrightarrow{A} & A(Y) \\ g\downarrow & & \downarrow A(g) \\ Z & \xrightarrow{A} & A(Z) \end{array} \qquad g\circ f, \quad A(g \circ f) = A(g) \circ A(f)$$

$A$ : (a category of spaces) $\to$ (a cat. of algebraic objects)
The algebraic objects we assigned are assigned compatibly with maps of spaces, <u>and</u> the compatibility maps $A(f)$ are also compatible w.r.t. composition.
So we made <u>compatible choices</u> in constructing $A$.

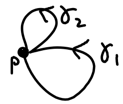
Not to be confused with <u>natural transformations</u> of functors (later) which is about relating two such <u>constructions</u> $A_1, A_2$ in a compatible way

## Example of a functor in algebraic topology
(see B3.5 Topology and Groups course)

$\pi_1(X, p)$ = **Fundamental group** $= \left\{ \begin{array}{l}\gamma \in \mathcal{L}X \\ \gamma(1) = p\end{array}\right\} \Big/$ continuous deformations of loops based at $p$

($X$ ↑ topological space, $p \in X$)

Group multiplication: concatenate loops $\gamma_1 * \gamma_2$ (each travelling twice as fast)



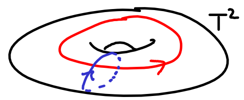

Examples

$\pi_1(\mathbb{R}^n) = 0$ ← (for basepoint $= 0 \in \mathbb{R}^n$: deform: $h : S^1 \times [0,1] \to \mathbb{R}^n$, $h(t,s) = (1-s)\gamma(t)$)


$\pi_1(S^1) \cong \mathbb{Z}$ ← total # times wind around circle

$\pi_1(S^n) \cong 0$ $n \geq 2$ (not obvious) ($S^n$ = n-sphere)

$\pi_1(\text{torus}) \cong \mathbb{Z}^2$ ←



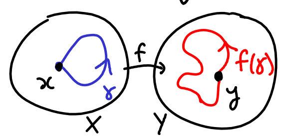

those loops generate $\pi_1$

FUNCTOR

$$\text{BasedTop} = \left\{\begin{array}{l}\text{Topological spaces with choice of basepoint,}\\ \text{and continuous basepoint-preserving maps}\end{array}\right\} \xrightarrow{\pi_1} \text{Gps}$$

$$(X,p) \longmapsto \pi_1(X,p)$$

$$((X,x) \xrightarrow[f]{\text{cts}} (Y,y)) \longmapsto \left( \begin{array}{l}\pi_1(X,x) \xrightarrow{\text{gp. hom.}} \pi_1(Y,y) \\ \gamma \longmapsto f \circ \gamma \end{array}\right)$$



**Lemma** Functors map isomorphisms to isomorphisms (iso. means ∃ inverse w.r.t. composition)

**Pf** $A \xrightarrow{f} B \xrightarrow{g} A$ (composite $= id$) $\Rightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FA$ (composite $= F(id) = id$), similarly for $B \xrightarrow{g} A \xrightarrow{f} B$ (composite $= id$). □

**Def** **Natural transformation** $\alpha : F \longrightarrow G$ between functors $C_1 \overset{F}{\underset{G}{\rightrightarrows}} C_2$ is an association $(A \in \mathrm{Ob}\, C_1) \longmapsto (\alpha_A : F(A) \longrightarrow G(A))$ $\in \mathrm{Hom}_{C_2}(F(A), G(A))$

such that $(A \xrightarrow{f} B) \in \mathrm{Hom}_{C_1}(A,B) \Rightarrow$

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\alpha_B} & G(B) \end{array}$$

(commutes)

It is called a **natural isomomorphism** if each $\alpha_A$ is an isomorphism in $C_2$

**Example of a natural transformation in algebraic topology**

Let $H_1(X,p) =$ abelianisation of $\pi_1(X,p)$ (want to identify $ab=ba$ so quotient by $\langle aba^{-1}b^{-1}\rangle$ ← **commutators**)

$\Rightarrow$ natural trans. $(\text{Based Top} \xrightarrow{\pi_1} \text{Gps}) \xrightarrow{\alpha} (\text{Based Top} \xrightarrow{H_1} \text{Gps})$

which associates $\underset{\in \text{BasedTop}}{(X,p)} \longmapsto \left(\alpha_{(X,p)} : \pi_1(X,p) \xrightarrow{\text{quotient}} H_1(X,p)\right)$

**Cultural Rmk** **higher homotopy groups** $\pi_n(X,p) = \left\{ \begin{array}{l} S^n \xrightarrow{\text{cts}} X \\ \text{basept} \mapsto p \end{array} \right\} \Big/ \text{cts "deform"}$

FACT abelian for $n \geq 2$, but hard: e.g. $\pi_k(S^n)$ not all known.

We will not study these in this course.

We will study simpler invariants called HOMOLOGY groups $H_n(X)$

FACT (Hurewicz) ∃ natural transformation $\pi_n \longrightarrow H_n$ which will make sense at the end of course:

$f : S^n \xrightarrow{\text{cts}} X$ gives rise to a class $f_*[S^n] \in H_n(X)$.

**Exercise** to practice these notions from category theory:

Summarise your undergraduate linear algebra as follows:

1) ∃ functor $F : \underbrace{\left\{ \begin{array}{l} \mathbb{R}^n \text{ for } n \in \mathbb{N} \\ \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) = \{m \times n \text{ matrices}\} \end{array} \right\}}_{\text{Mat}} \to \underbrace{\left\{ \begin{array}{c} \text{finite dimensional real vector spaces} \\ \text{and linear maps} \end{array} \right\}}_{\text{Vect}}$

2) A choice of basis for each vector space $V$ determines a functor $G : \text{Vect} \to \text{Mat}$

3) Construct natural isomorphisms $G \circ F \xrightarrow{\alpha} \mathrm{Id}_{\text{Mat}}$, $F \circ G \xrightarrow{\beta} \mathrm{Id}_{\text{Vect}}$

When functors satisfying such natural isos exist, the categories are called **equivalent** (not isomorphic). So Mat, Vect are equivalent categories.

**Aim of the course**: build a functor

HOMOLOGY $H_* : \text{Top} \longrightarrow$ **Graded abelian groups**

$X \longmapsto H_*(X)$ ← grading $* \in \mathbb{Z}$

$(X \to Y) \longmapsto \left( H_*(X) \longrightarrow H_*(Y) \right.$ grading preserving hom $\left.\right)$

and a contravariant functor

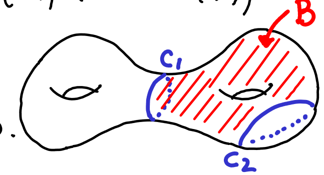
COHOMOLOGY $H^* : \text{Top} \longrightarrow$ **Graded rings**

$X \longmapsto H^*(X)$

$(X \to Y) \longmapsto (H^*(X) \longleftarrow H^*(Y))$

Rough idea:

$H_* X$ is generated by "nice" subspaces $C \subseteq X$ which have no boundary: $\partial C = \emptyset$, modulo identify $C_1, C_2$ if $C_1 \cup C_2$ arises as a boundary $\partial B$. Call such $C_1, C_2$ **homologous**.




Facts

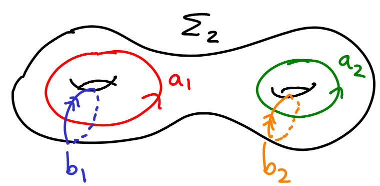
- $H_0(X) \cong \bigoplus_{\pi_0 X} \mathbb{Z}$ ← $\pi_0 X = \{\text{path-connected components}\}$ ← generated by a point in each path-comp.
- $X = \sqcup X_i$ path-components $\Rightarrow H_*(X) \cong \bigoplus H_*(X_i)$
- $\chi(X) = \sum_{d \geq 0} (-1)^d \operatorname{rank} H_d(X)$

$\chi$ ← **Euler characteristic**; rank ← max # $\mathbb{Z}$-linearly independent elements

Example: compact surfaces

$$H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}^{2g} & * = 1 \\ \mathbb{Z} & * = 2 \\ 0 & \text{else} \end{cases}$$

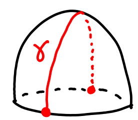
$\Sigma_g$ ← orientable surface genus $g$, $\chi = 2 - 2g$




We will show that those 4 loops generate $H_1(\Sigma_2)$

$$H_*(N_h) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/_2 \oplus \mathbb{Z}^{h-1} & * = 1 \\ 0 & \text{else} \end{cases}$$

$N_h$ ← non-orientable surface $S^2$ with $h$ Möbius bands attached, $\chi = 2 - h$




$N_1 = \mathbb{RP}^2 = S^2 / \pm \mathrm{Id}$

Notice $\gamma$ is a loop. It generates $H_1(N_1)$

## Examples of homology calculations

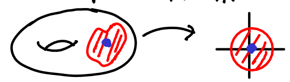
$H_*(\mathbb{R}^n) \cong H_*(\mathbb{D}^n) \cong \begin{cases} \mathbb{Z} & *=0 \\ 0 & \text{else} \end{cases}$

n-dimensional ball
$\mathbb{D}^n = \{x : \|x\| \leq 1\} \subseteq \mathbb{R}^n$

$H_*(S^n) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z} & *=n \\ 0 & \text{else} \end{cases}$

$S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ n-dim sphere

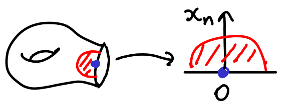
Hausdorff top. space s.t. each pt has an open neighbourhood homeo to an open ball in $\mathbb{R}^n$ → n-dimensional manifolds $X$



$H_*(X) = \begin{cases} 0 & \text{for } * < 0 \\ 0 & \text{for } * > n \\ \mathbb{Z} & \text{for } * = n \text{ for connected orientable compact manifold} \\ 0 & \text{for } * = n \text{ for non-orientable, non-compact, connected manifolds with boundary} \neq \emptyset \end{cases}$

for **n-dimensional manifolds** $X$

**boundary point** has an open nbhd homeo to open nbhd of $0 \in$ half-space: $\{x \in \mathbb{R}^n : x_n \geq 0\}$

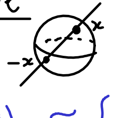



Rmk $M$ compact connected n-mfd

$\Rightarrow H_{n-1}(M) \cong \begin{cases} \mathbb{Z}^k & \text{some } k \geq 0 \text{ if orientable} \\ \mathbb{Z}^k \oplus \mathbb{Z}/2 & \text{if non-orientable} \end{cases}$

$H_*(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & \text{odd } * = 1, 3, 5, \ldots < n \\ \mathbb{Z} & * = n \text{ if } n \text{ odd} \\ 0 & \text{else} \end{cases}$

$\mathbb{RP}^n = S^n / \pm id$ real projective space




$\mathbb{RP}^n$ orientable $\Leftrightarrow$ n odd (e.g. $\mathbb{RP}^1 \cong S^1$)

$H_*(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & \text{even } * = 0, 2, 4, \ldots, 2n \\ 0 & \text{else} \end{cases}$

e.g. $\mathbb{CP}^1 \cong S^2$ stereographic projection


$\mathbb{CP}^n$: space of complex lines through $0 \in \mathbb{C}^{n+1}$ — complex projective space

$\cong (\mathbb{C}^{n+1} \setminus 0) / \mathbb{C}^*\text{-rescaling}$

$= \{[z_0, z_1, \ldots, z_n] : z_j \in \mathbb{C} \text{ not all } 0\} / [z] = [\lambda z] \text{ for } \lambda \in \mathbb{C}^*$

## Examples of cohomology calculations

$H^0(X) = \prod_{\pi_0 X} \mathbb{Z}$ ← if $\pi_0 X$ finite, then $\cong \bigoplus_{\pi_0 X} \mathbb{Z} \cong H_0 X$
but if infinite then not: here allow only finite sums

$H^*(X) \cong \prod H^*(X_i)$ ← $X_i$ path-components of $X$

FACT: If $H_n(X)$ finitely generated abelian gp, so
$H_n(X) \cong \mathbb{Z}^{r_n} \oplus T_n$ ← $T_n$ = torsion elements = elements of finite order
Then $H^n(X) \cong \mathbb{Z}^{r_n} \oplus T_{n-1}$ as abelian groups

$\Rightarrow H^*(\Sigma_g), H^*(\mathbb{R}^n), H^*(\mathbb{D}^n), H^*(S^n), H^*(\mathbb{CP}^n)$ same as for $H_*$, but:

$H^*(N_h) = \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}^{h-1} & *=1 \\ \mathbb{Z}/2 & *=2 \\ 0 & \text{else} \end{cases}$

$H^*(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & *=0 \\ 0 & *=1 \\ \mathbb{Z}/2 & *=2 \\ 0 & \text{else} \end{cases}$ $(h=1)$

$H_*(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}/2 & * \text{ even} = 2, 4, \ldots \leq n \\ \mathbb{Z} & * = n \text{ if } n \text{ odd} \\ 0 & \text{else} \end{cases}$

and $H^n(\text{non-orientable compact n-mfd}) \cong \mathbb{Z}/2$.

$\Rightarrow$ The interesting feature is the ring structure:

$H^*(\mathbb{CP}^n) \cong \mathbb{Z}[x]/x^{n+1}$ grading: $|x| = 2$ — $\mathbb{Z}[x]$ = polynomials in $x$ with $\mathbb{Z}$-coefficients

$H^*(S^n) \cong \mathbb{Z}[x]/x^2$ $|x| = n$

$H^*(T^n) \cong \Lambda[x_1, \ldots, x_n]$ $|x_i| = 1$
$T^n = S^1 \times \cdots \times S^1$ **n-torus**

e.g. $n=2$ (torus): $\begin{cases} \mathbb{Z} \cdot 1 & *=0 \\ \mathbb{Z}x_1 \oplus \mathbb{Z}x_2 & *=1 \\ \mathbb{Z}x_1 \wedge x_2 & *=2 \\ 0 & \text{else} \end{cases}$

**exterior algebra** generated/$\mathbb{Z}$ by symbols $x_{i_1} \wedge \ldots \wedge x_{i_k}$ with $i_1 < \ldots < i_k$
product given by $\wedge$ using relations $x_i \wedge x_j = -x_j \wedge x_i$ and $x_i \wedge x_i = 0$. (and $\wedge$ is bilinear)

$H^*(\mathbb{RP}^{2n}) \cong \mathbb{Z}[x]/(2x, x^{n+1})$ where $|x| = 2$

$H^*(\mathbb{RP}^{2n+1}) \cong \mathbb{Z}[x,y]/(2x, x^{n+1}, y^2, xy)$ $|y| = 2n+1$

$H^*(\Sigma_g) \cong \Lambda[a_1, \ldots, a_g, b_1, \ldots, b_g] / \langle a_i b_j \text{ for } i \neq j,\ a_i b_i - a_j b_j,\ a_i a_j,\ b_i b_j \rangle$

$|a_i| = |b_i| = 1$ ← exterior alg. instead of poly. alg since $a_i b_i = -b_i a_i$

**Why more information?**

$S^2 \times S^2$ and $\mathbb{CP}^2 \# \mathbb{CP}^2$ have same $H_* = \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z} \oplus \mathbb{Z} & *=2 \\ \mathbb{Z} & *=4 \end{cases}$ (connected sum: remove a ball in each, glue along $\partial$ball)

but the rings $H^*$ are not iso, hence $S^2 \times S^2 \not\cong \mathbb{CP}^2 \# \mathbb{CP}^2$.

## Example of why such functors are useful

Suppose $\exists\ F_*: \text{Top} \longrightarrow \text{Gps}$ functors s.t.

① $F_*(S^n) \neq 0 \Leftrightarrow * = n$ and ② $F_*(\mathbb{D}^n) = 0$ **all** $*$

Rmk We'll build such an $F_*$: *reduced homology* $\tilde{H}_*$

s.t. $\tilde{H}_* = H_*$ for $* \neq 0$, and $\tilde{H}_0 \cong \mathbb{Z}^{\#(\text{path-components})-1}$

Theorem *Invariance of dimension* (Brower ~1910)

| | | | |
|---|---|---|---|
| $S^n \cong S^m$ | $\Leftrightarrow$ | $n=m$ | by ① |
| $\mathbb{R}^n \cong \mathbb{R}^m$ | $\Leftrightarrow$ | $n=m$ | |

"homeomorphisms preserve dimension"
Non-trivial result because there are space-filling curves. e.g. Peano (1890) $\exists$ cts surjection $[0,1] \to [0,1]^2$ (interval → square). The theorem implies this is not injective. (cts. bij. compact → Hausdorff ⟹ homeo)

Pf Lemma $\Rightarrow F_n(S^n \xrightarrow{\cong} S^m)$ is iso $F_n(S^n) \xrightarrow{\cong} F_n(S^m)$ of gps. ($F_n(S^n) \neq 0$, $F_n(S^m) = 0$ if $n \neq m$ ✓)

If $\mathbb{R}^n \xrightarrow[\varphi]{\cong} \mathbb{R}^m$, then can extend $\varphi$ to the *one-point compactifications*: $S^n \cong \mathbb{R}^n \cup \{\infty\} \xrightarrow{\cong} \mathbb{R}^m \cup \{\infty\} \cong S^m$, $\infty \mapsto \infty$. □

("Alexandroff extension")

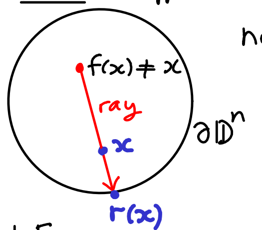
stereographic projection $(x_0, \dots, x_n) \mapsto \frac{(x_1, \dots, x_n)}{1 - x_0}$

Rmk new open neighbourhoods at $\infty$ are $\{\infty\} \cup (\mathbb{R}^n \setminus C)$ where $C$ is (closed &) compact. The extended map is cts since $\varphi^{-1}(C)$ is (closed &) compact since $\varphi^{-1}$ is homeo.

Theorem *Brower fixed point thm* by ① & ②

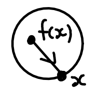
$f: \mathbb{D}^n \longrightarrow \mathbb{D}^n$ continuous $\Rightarrow f$ has a fixed point ($f(p) = p$ some $p$)

Proof Suppose not. Let $r(x) = (\text{ray from } f(x) \text{ to } x) \cap \partial\mathbb{D}^n$




notice: • $r: \mathbb{D}^n \longrightarrow \partial\mathbb{D}^n = S^{n-1}$ continuous

• $r|_{\partial\mathbb{D}^n} = id_{S^{n-1}}$

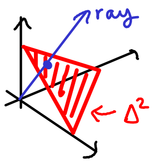



$S^{n-1} = \partial\mathbb{D}^n \xrightarrow{\text{inclusion } i} \mathbb{D}^n \xrightarrow{r} S^{n-1}$, $r \circ i = id$

apply $F_{n-1}$ $\Longrightarrow F_{n-1}(r) \circ F_{n-1}(i) = id \Longrightarrow F_{n-1}(i)$ injective $F_{n-1}(S^{n-1}) \to F_{n-1}(\mathbb{D}^n)$ ↯ (since $F_{n-1}(S^{n-1}) \neq 0$, $F_{n-1}(\mathbb{D}^n) = 0$) □

Example (Brower) $A = n \times n$ matrix, $A_{ij} > 0$ real $\Longrightarrow \exists$ real evalue $\lambda > 0$ with real evector $(v_1, \dots, v_n)$ with $v_i \geq 0$

Hint




$\mathbb{R}^3 \supseteq$ $X = \{\text{rays in "positive octant"}\}$ $x \in \mathbb{R}^n: x_i \geq 0\ \forall i$

notice $AX \subseteq X$

notice $X \cong \Delta^n = \{x \in \text{octant}: \sum x_i = 1\} \cong \mathbb{D}^n$

ray $\mapsto$ ray $\cap\ \Delta^n$

# 1. ALGEBRAIC PRELIMINARIES : CHAIN COMPLEXES

## Graded abelian groups

Def A ***$\mathbb{Z}$-graded abelian group*** $C$ is an abelian group together with a direct sum decomposition

$$C = \bigoplus_{n \in \mathbb{Z}} C_n \quad \leftarrow \text{abelian group}$$

Convention: always grade by $\mathbb{Z}$ unless say otherwise.

Example $C = \mathbb{Z}[x] = $ integer polynomials in $x$, $C_n = \mathbb{Z} \cdot x^n$ ← so grading by degree

A graded ab. gp. $A$ is a ***graded subgp*** of $C$ if • subgp • $A_n \subseteq C_n$.

A ***homomorphism*** $h: C \longrightarrow D$ of gr. ab. gps is hom of gps s.t.

$$h(C_n) \subseteq D_n$$

A ***hom of degree k*** is hom with

$$h(C_n) \subseteq D_{n+k}$$

***Shift by k***: $\mathbb{Z}$-gr. ab. gp. $C[k]$ with

$$C[k]_n = C_{k+n}$$

Notice: $C[k]_0 = C_k$ is now in degree zero, so shifted down by $k$

$\Rightarrow$ Can view gr. hom of deg $k$ as a gr. hom $h: C \longrightarrow D[k]$

## Abelian groups which are *finitely generated*

recall f.g. means $\exists$ surjection $\mathbb{Z}^m \to G$ for some $m$

FACT Finitely generated abelian groups are classified:

$$G \cong \mathbb{Z}^r \oplus \mathbb{Z}/_{p_1^{n_1}} \oplus \dots \oplus \mathbb{Z}/_{p_k^{n_k}}$$

$n_i \neq 0 \in \mathbb{N}$, $p_i$ primes (possibly not distinct)

***free part*** ($\mathbb{Z}^r$), $r$ called ***rank G***, ***torsion part*** (the $\mathbb{Z}/_{p_i^{n_i}}$ summands)

compare finite dimensional vectorspaces / field $\mathbb{F}$: $V \cong \mathbb{F}^r$, $r = \dim V$

## Chain complexes

**Def** A **chain complex** $(C_*, \partial_*)$ is a gr. ab. gp. C together with a hom $\partial$ of degree $-1$ such that $\partial \circ \partial = 0$.
($\partial_*$: **differential** or **boundary homomorph**)

Thus:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \qquad \partial_n \circ \partial_{n+1} = 0$$

hence $\operatorname{Im} \partial_{n+1} \subseteq \operatorname{Ker} \partial_n$, where $\operatorname{Im} \partial_{n+1} =: B_n$ (**n-boundaries**) and $\operatorname{Ker} \partial_n =: Z_n$ (**n-cycles**)

**n-chains** = elements of $C_n$

Now consider "cycles modulo boundaries":

**Def** The **homology** of $(C_*, \partial_*)$ is the gr. ab. gp.

$$H_n(C_*, \partial_*) = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}$$

often abbreviate by $H_*(C)$

Two cycles are called **homologous** if they differ by a boundary

**Def** A **chain map** $h : (C_*, \partial_*) \longrightarrow (\tilde{C}_*, \tilde{\partial}_*)$ is a graded hom such that $h \circ \partial_* = \tilde{\partial}_* \circ h$

**Example** A chain **subcomplex** $C_* \subseteq \tilde{C}_*$ is a graded subgp with $\partial_* =$ restriction of $\tilde{\partial}_*$ to $C_*$.
So the inclusion $\operatorname{incl} : (C_*, \partial_*) \longrightarrow (\tilde{C}_*, \tilde{\partial}_*)$ is a chain map.
Also get **quotient complex** $\tilde{C}_*/C_*$
with $\tilde{\partial}_*[\tilde{c}] = [\tilde{\partial}_* \tilde{c}]$ (well-defined: $\tilde{\partial}_* C_* = \partial_* C_* \subseteq C_*$)

**Lemma** A chain map induces a hom on homology

$$H_*(h) = h_* : H_*(C_*, \partial_*) \longrightarrow H_*(\tilde{C}_*, \tilde{\partial}_*)$$
$$[x] \longmapsto [h(x)]$$

**Proof** $\operatorname{Ker} \partial_n \xrightarrow{\text{hom}} \operatorname{Ker} \tilde{\partial}_n$, $x \longmapsto h(x)$ since $\tilde{\partial}(h(x)) = h(\partial x) = 0$ (as $\partial x = 0$)

Need $\operatorname{Im} \partial_n \longrightarrow \operatorname{Im} \tilde{\partial}_n$ to get well-defined hom

$$H_n(C) = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1} \longrightarrow \operatorname{Ker} \tilde{\partial}_n / \operatorname{Im} \tilde{\partial}_{n+1} = H_n(\tilde{C})$$

Proof: $h(b) = h(\partial c) = \tilde{\partial} h(c) \in \operatorname{Im}(\tilde{\partial})$. □ ($b = \partial c \in \operatorname{Im}(\partial)$)

The last step was a very simple example of a proof by **"diagram chasing"**

$$\begin{array}{ccccccccc} \cdots \to & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \to \cdots \\ & \downarrow h_{n+1} & & \downarrow h_n & & \downarrow h_{n-1} & \\ \cdots \to & \tilde{C}_{n+1} & \xrightarrow{\tilde{\partial}_{n+1}} & \tilde{C}_n & \xrightarrow{\tilde{\partial}_n} & \tilde{C}_{n-1} & \to \cdots \end{array}$$

(commutativity of this diagram is the definition of h being a chain map)

$$\begin{array}{ccc} c & \xrightarrow{\partial} & \partial c = b \\ h \downarrow & & \downarrow h \\ hc & \xrightarrow{\tilde{\partial}} & \tilde{\partial}(hc) = h\partial c = h(b) \end{array}$$

□

**Def** $(C_*, \partial_*)$ is **exact** (or **acyclic**) if $H_*(C) = 0$
so $\operatorname{Im} \partial_{n+1} = \operatorname{Ker} \partial_n$

In general **exact sequence** means "Im(previous map) = Ker (next map)"

A **short exact sequence (SES)** is an exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \longrightarrow 0$$

**Easy exercise**

$\left(0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0\right)$ exact $\iff \begin{cases} i & \text{injective} \\ \pi & \text{surjective} \\ B/i(A) \cong C & \text{via } [b] \mapsto \pi(b) \end{cases}$

**Examples**

$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2 \to 0$

$0 \longrightarrow \mathbb{Z} \xrightarrow{\text{inclusion}} \mathbb{Z} \oplus \mathbb{Z}/2 \xrightarrow{\text{project}} \mathbb{Z}/2 \to 0$

Note $A, C$ do not determine $B$.

**Snake Lemma** A SES of chain complexes and chain maps yields a **long exact sequence (LES) on homology**:

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{\pi_*} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*[-1]} \cdots$$

$\delta$: degree $-1$ map $H_*(C) \to H_*(A)[-1]$ called **connecting map**

(So **exact triangle**: $H_*(A) \longrightarrow H_*(B) \longrightarrow H_*(C) \xrightarrow{[-1]} H_*(A)$ )

Pf simplify notation by identifying $A$ with $i(A) \subseteq B$: $\underset{\in A}{a} \equiv \underset{\in B}{i(a)}$, $\partial a \equiv i \partial a = \partial i a$

$\implies$ now $A_* \subseteq B_*$ inclusion of subcomplex:

$$0 \longrightarrow (A_*, \partial_*) \xrightarrow{\text{incl}} (B_*, \partial_*) \xrightarrow{\pi} (C_*, \tilde{\partial}_*) \longrightarrow 0$$

$$\begin{array}{ccccccccc} 0 & \to & A_n & \to & B_n & \to & C_n & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_{n-1} & \to & B_{n-1} & \to & C_{n-1} & \to & 0 \end{array}$$

$$\begin{array}{ccccc} & & \exists\, b & \xrightarrow{\text{surj.}} & \text{cycle } c = \pi(b) \\ & & \downarrow & & \downarrow \\ \partial b & \longrightarrow & \partial b & \longrightarrow & \tilde{\partial} c = 0 \end{array}$$

($\partial b$ lifts to $A$ by exactness)

**Define** $\delta : H_*(C) \longrightarrow H_*(A)[-1]$, $c \longmapsto \partial b$ where $b \in \pi^{-1}(c)$ (typically $b$ is not in $A$, so $\partial b$ need not be a bdry in $A$)

**Well-defined?**
- $\pi^{-1}(c) = \{b + a : a \in A\}$ and $\partial(b+a) = \partial b + \underbrace{\partial a}_{\text{boundary in } A}$ ✓
- cycle $\to$ cycle: $\partial(\partial b) = 0$ ✓
- boundary $\to$ boundary: $\exists\, \beta \xrightarrow{\text{surj.}} x \in C_{n+1}$

$$\begin{array}{ccc} \exists\,\beta & \xrightarrow{\text{surj.}} & x \in C_{n+1} \\ \downarrow & & \downarrow \\ \partial\beta & \longrightarrow & \text{boundary } c = \tilde{\partial} x \\ & & \downarrow \\ & & 0 \end{array}$$

$\implies$ can pick $b = \partial \beta$

$\implies \partial b = \partial\partial\beta = 0$ ✓

**Exactness at $H_n(C)$** (exercise: check exactness at $H_*A$, $H_*B$):

Need $\operatorname{Im} \pi_* = \operatorname{Ker} \delta$:

$\subseteq$ : $\delta(\pi_* b) = \partial b = 0$ ✓ ($b$ cycle)

$\supseteq$ :

$$\begin{array}{ccccc} \exists\, a & & b & \longrightarrow & c = \pi_* b \\ \downarrow & & \downarrow & & \downarrow \\ \partial a = \delta c = \partial b & \longrightarrow & \partial b & \longrightarrow & 0 \end{array}$$

($b$ not necessarily cycle! $c \in \operatorname{Ker}\delta$; $\partial a = \delta c$ by assumption $\delta c = 0 \in H_*A$)

$\pi_*(b-a) = c$ (since $\pi_* A = 0$)

$\underbrace{\partial(b-a)}_{\text{thus cycle!}} = \partial b - \partial a = 0$

$\implies c = \pi_*(b-a) \in \operatorname{Im} \pi_*$ ∎

**Rmk** $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ SES $\Rightarrow$ the **connecting map of LES** is

$$\boxed{\begin{array}{l} \delta : H_*(C) \longrightarrow H_*(A)[-1] \\ \quad c \longmapsto i^{-1}(\partial b) \end{array}}$$

$\forall\, b \in B$ with $\pi(b) = c$.

**Lemma** The construction of $\delta$ is natural (i.e. functorial)

**Pf** (all chain maps)

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{i} & B & \xrightarrow{\pi} & C & \to & 0 \\ & & f \downarrow & & g \downarrow & & h \downarrow & & \\ 0 & \to & \tilde{A} & \xrightarrow{\tilde{i}} & \tilde{B} & \xrightarrow{\tilde{\pi}} & \tilde{C} & \to & 0 \end{array}$$

$$\begin{array}{ccccccc} \delta c = a & \to & \partial b & & b & \to & c \\ f\downarrow & & \downarrow g & & g \downarrow & & \downarrow h \\ fa & \to & g\partial b & & gb & \to & hc \end{array}$$

($\tilde{\partial} g b = \delta h c$)

$\Rightarrow \delta h c = \tilde{i}^{-1} \tilde{\partial} g b = \tilde{i}^{-1} g \partial b = f a = f \delta c$ ∎

**Exercise** Deduce the LES is natural, so

$$\begin{array}{ccccccccc} \cdots \to & H_* A & \xrightarrow{i_*} & H_* B & \xrightarrow{\pi_*} & H_* C & \xrightarrow{\delta} & H_{*-1}(A) & \to \cdots \\ & f_* \downarrow & & g_* \downarrow & & h_* \downarrow & & f_* \downarrow & \\ \cdots \to & H_* \tilde{A} & \longrightarrow & H_* \tilde{B} & \xrightarrow{\tilde{\pi}_*} & H_* \tilde{C} & \xrightarrow{\delta} & H_{*-1}(\tilde{C}) & \to \cdots \end{array}$$

## 5-Lemma

$$\begin{array}{ccccccccc} A & \to & B & \to & C & \to & D & \to & E \\ \cong\downarrow\alpha & & \cong\downarrow\beta & & \downarrow\gamma & & \cong\downarrow\delta & & \cong\downarrow\varepsilon \\ A' & \to & B' & \to & C' & \to & D' & \to & E' \end{array}$$

exact rows $\Rightarrow \gamma$ also iso.

Pf exercise (diagram chase) □

## Splitting Lemma

Cor $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ SES of abelian gps

If $B \xrightarrow{\beta} C$, $\exists \gamma$ s.t. $\beta \circ \gamma = id_C$ then the SES **splits**: $B \cong A \oplus C$

(converse is obvious)

Pf

$$\begin{array}{ccccccccc} 0 & \to & A & \to & A\oplus C & \to & C & \to & 0 \\ \| & & \| & & \downarrow \alpha+\gamma & & \| & & \| \\ 0 & \to & A & \to & B & \to & C & \to & 0 \end{array}$$

□

Exercise If $A \xrightarrow{\alpha} B$, $\exists \mu$ s.t. $\mu \circ \alpha = id_A$ then it splits: $B \xrightarrow[\mu \oplus \beta]{\cong} A \oplus C$

Exercise If $C$ is a free abelian group ($C \cong \bigoplus_{i \in I} \mathbb{Z}$) then the SES splits.

Rmk $A$ free $\not\Rightarrow$ splits, e.g. $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$

Cultural Rmk Splitting Lemma generalises the **rank-nullity theorem** from linear algebra: $V \xrightarrow{\beta} W$ linear map of vector spaces $\Rightarrow \mathrm{Im}\beta \oplus \mathrm{Ker}\beta \cong V$

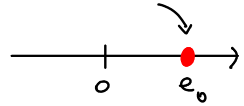
Pf $0 \to \mathrm{Ker}\,\beta \xrightarrow{incl} V \xrightarrow{\beta} \mathrm{Im}\,\beta \to 0$ is SES, and splits since $\mathrm{Im}\,\beta$ free.

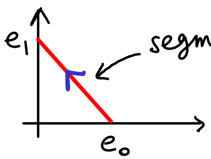
# 2. Δ-COMPLEXES AND SIMPLICIAL HOMOLOGY

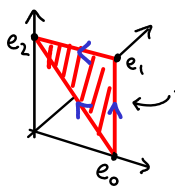
**Standard n-simplex** $\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} : \begin{array}{l} t_i \geq 0 \\ \sum t_i = 1 \end{array} \right\}$

$(t_0,\dots,t_n) = \sum t_i e_i$ ← standard basis of $\mathbb{R}^{n+1}$ $e_0, \dots, e_n$ $(e_0 = (1,0,\dots,0), \dots$

Examples




Def For $\{v_0, \dots, v_n\} \subseteq \mathbb{R}^{n+k}$ (any $k \geq 0$) s.t. $v_1 - v_0, \dots, v_n - v_0$ $\mathbb{R}$-linearly independent

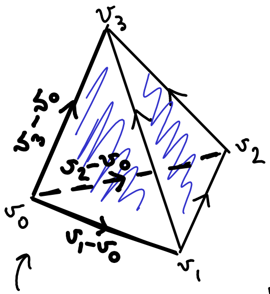
$[v_0, \dots, v_n]$ = **n-simplex** spanned by $v_0, \dots, v_n$

= convex hull of $v_0, \dots, v_n$

$= \{ \sum t_i v_i : t_i \geq 0 \text{ and } \sum t_i = 1 \}$

= Image of linear homeo $\sigma : \Delta^n \to \mathbb{R}^{n+k}$, $\sigma(e_i) = v_i$

↑ **canonical homeomorphism**




(Solid prism: includes inside)

Will often blur the distinction between map $\sigma$ and its image,

$$\sigma \equiv [\sigma e_0, \dots, \sigma e_n]$$

but the ordering of the $v_j$ will be important (so the map $\sigma$ is more precise)

We encode this extra data by orienting the edges $v_i \to v_j$ if $i<j$

Def **d-dimensional faces** $[v_{i_0}, \dots, v_{i_d}]$ for $i_0 < \dots < i_d$

Example 0-dim faces are the vertices $v_0, \dots, v_n$

**facets** = $(n-1)$-dimensional faces

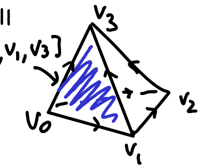
$= [v_0, \dots, \hat{v_\ell}, \dots, v_n]$ where we omit $v_\ell$

$= \{ \sum t_i v_i \in [v_0, \dots, v_n] : t_\ell = 0 \}$

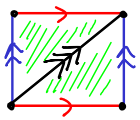
$= \mathrm{Image}\ \sigma|_{\Delta^{n-1}_\ell} : \Delta^{n-1}_\ell \to \mathbb{R}^{n+k}$

$\Delta^{n-1}_\ell = \{ t \in \Delta^n : t_\ell = 0 \}$

$[v_0, v_1, \hat{v_2}, v_3] = [v_0, v_1, v_3]$



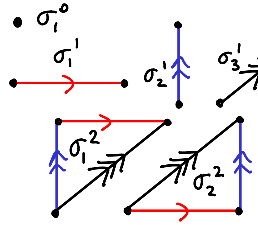
**Example** Can build a torus out of simplices:

$T^2 =$ 

1 0-simplex $\bullet\ \sigma_1^0$

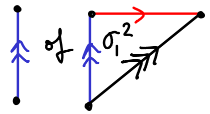
3 1-simplices

2 2-simplices




each facet is associated to another simplex, and we identify them linearly

$T^2 =$ quotient space $\bigsqcup \sigma_i^n$ / canonical homeos associated to the facets (don't confuse the abstract simplices with their images in $T^2$ = quotient space)

for example identify facet  of $\sigma_1^2$ with $\sigma_2^1$ via linear homeo (orientation-preserving)

**Def** **$\Delta$-complex** is determined by data
- indexing set $I_n$, for each $n \in \mathbb{N}$
- choice of $n$-simplex $\sigma_\alpha^n$ (not necessarily standard) for each $\alpha \in I_n$
- gluing data: for each $\alpha \in I_n$, $0 \le i \le n$, associate some $\beta(\alpha, i) \in I_{n-1}$
- consitency condition (see later)

The $\Delta$-complex is the quotient space

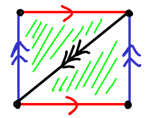
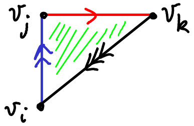
$$X = \bigsqcup_{\substack{\alpha \in I_n \\ n \in \mathbb{N}}} \sigma_\alpha^n \Big/ \begin{array}{l} i\text{-th facet of } \sigma_\alpha^n \text{ is identified with } \sigma_{\beta(\alpha,i)}^{n-1} \\ \text{via the order-preserving canonical linear homeo} \end{array}$$

(quotient topology: $U \subseteq X$ is open $\Leftrightarrow$ $U$ intersects $\sigma_\alpha^n$ in an open set, $\forall \alpha, n$)

A **$\Delta$-complex structure** on a top. space $Y$ is a homeo from a $\Delta$-cx $X \cong Y$.

Explicit description of the facet identification

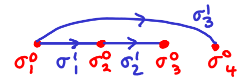
$$\begin{array}{ccccc} \{\sum s_j w_j\} = [w_0, \ldots, w_{n-1}] & \longrightarrow & [v_0, \ldots, v_n] & = & \{\sum t_j v_j\} \\ \uparrow \sigma_{\beta(\alpha,i)}^{n-1} & & \uparrow \sigma_\alpha^n & & \cup\!| \\ \Delta^{n-1} & \longrightarrow & \Delta_i^{n-1} \subseteq \Delta^n & \xrightarrow{\sigma_\alpha^n|_{\Delta_i^{n-1}}} & \{s_0 v_0 + \ldots + s_{i-1} v_{i-1} + s_i v_{i+1} + \ldots + s_{n-1} v_n\} \\ (s_0, \ldots, s_{n-1}) & \longmapsto & (s_0, \ldots, s_{i-1}, 0, s_i, \ldots, s_{n-1}) & & = [v_0, \ldots, \hat{v}_i, \ldots, v_n] \end{array}$$

**Non-example**  because:  vertices are not totally ordered: $i < j < k < i$ ↯

This decomposition for $T^2$ is not a $\Delta$-complex.

## Consistency condition

We want to additionally ensure that each point of $X$ lies in the interior of exactly one $\sigma_\alpha^n$, because we want to avoid unexpected identifications.

Example: $X =$  then glue $\sigma_1^2 =$ via 



notice how $\sigma_3^0, \sigma_4^0$ get identified in the quotient, but we only notice this after gluing $\sigma_1^2$
(If you try to run the definition of simplicial homology – defined later – you notice that the differential cannot satisfy $\partial_1 \circ \partial_2 = 0$)

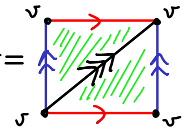
Equivalently: the facet gluing maps are compatible under double restriction: $\forall i < j$

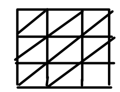
$$[v_0, \ldots, v_n] \overset{\text{facet}}{\rightsquigarrow} [v_0, \ldots, \hat{v}_i, \ldots, v_n] \overset{\text{identify}}{\cong} [w_0, \ldots, w_{n-1}] \overset{\text{facet}}{\rightsquigarrow} [w_0, \ldots, \hat{w}_{j-1}, \ldots, w_{n-1}] \overset{\text{identify}}{\cong} [x_0, \ldots, x_{n-2}]$$
$$[v_0, \ldots, v_n] \overset{\text{facet}}{\rightsquigarrow} [v_0, \ldots, \hat{v}_j, \ldots, v_n] \overset{\text{identify}}{\cong} [z_0, \ldots, z_{n-1}] \overset{\text{facet}}{\rightsquigarrow} [z_0, \ldots, \hat{z}_i, \ldots, z_{n-1}] \overset{\text{identify}}{\cong} [x_0, \ldots, x_{n-2}]$$

this ensures that $[v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_n]$ is identified with the same $[x_0, \ldots, x_{n-2}]$ whether we first restrict to $t_i = 0$ (omit $v_i$) or first restrict to $t_j = 0$ (omit $v_j$).

Another equivalent condition: can define the **k-th skeleton** of $\Delta$-cx $X$, $X^k$ = quotient space you get by gluing all simplices of dimensions $\le k$. Consistency is the condition that the boundary of each $\sigma_\alpha^n$ should map continuously into $X^{n-1}$
(in the above Example consider the vertex $\rightarrow \triangle = \partial \sigma_1^2$)

**Rmk** (see Part A Topology) A **simplicial complex** is a $\Delta$-complex in which each $d$-dim face is uniquely determined by $d$ distinct vertices. (more precisely, the "topological realisation" of a simpl. complex)
A homeo from such a complex to $X$ is a **triangulation** of $X$.

**Non-example** $T^2 =$  both 2-simplices have vertices $v, v, v$

whereas $T^2 =$  is a triangulation.

## Simplicial chain complex

**Def** For a $\Delta$-complex $X$, let $X_n$ = set of $n$-simplices of $X$

$$C_n^\Delta(X) = \text{free abelian group generated by the set } X_n = \Big\{ \sum_{\alpha \in I_n} c_\alpha \cdot \sigma_\alpha^n : c_\alpha \in \mathbb{Z} \text{ and only finitely many } c_\alpha \neq 0 \Big\}$$

differential: $\partial_n \sigma_\alpha = \sum (-1)^i \cdot \sigma_{\beta(\alpha,i)}$

so: $\partial_n [v_0,\dots,v_n] = \sum (-1)^i \cdot [v_0,\dots,\hat{v}_i,\dots,v_n]$ } and extend linearly

Will show $\partial\circ\partial=0$, so get **simplicial homology**: $H_*^\Delta(X) = H_*(C_*^\Delta, \partial_*)$

**Examples**

$\partial_1(\overrightarrow{v_0 \quad v_1}) = -v_0 + v_1$

$\partial_2([v_0,v_1,v_2]) = +[v_1,v_2] - [v_0,v_2] + [v_0,v_1]$

Later: The $(-1)^i$ signs keep track of whether the orientation agrees/disagrees with geometric boundary orientation, so ... versus $\partial$ ... = ...

$\partial_2\circ\partial_1$ (this) $= +(v_2-v_1) - (v_2-v_0) + (v_1-v_0) = 0$

$\partial\circ\partial=0$ fails for ... (not $\Delta$-complex), try!

**Lemma** $\partial\circ\partial = 0$

**Pf** $\partial_{n-1}(\partial_n [v_0,\dots,v_n]) = \sum(-1)^i \partial_{n-1}[v_0,\dots,\hat{v}_i,\dots,v_n]$

$= \sum_{j<i} (-1)^i(-1)^j [v_0,\dots,\hat{v}_j,\dots,\hat{v}_i,\dots,v_n]$ ⟲ antisymmetric if swap $i,j$

$+ \sum_{j>i} (-1)^i \underline{(-1)^{j-1}} [v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n]$

$= 0$ □

**Example** $S^1 = $ (circle with vertex $v$, edge $e$) $\Delta$-cx: $X_0$: 1 0-simplex • $e_1^0 = e_{\beta(1,0)} = e_{\beta(1,1)}$

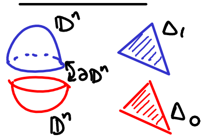
$X_1$: 1 1-simplex → $e_1^1$

$0 \to C_1^\Delta \to C_0^\Delta \to 0$, $C_1^\Delta = \mathbb{Z}e$, $C_0^\Delta = \mathbb{Z}v$

$e \longmapsto v - v = 0$

$\Rightarrow H_*^\Delta(S^1) = \begin{cases} \mathbb{Z} & *=0,1 \\ 0 & \text{else} \end{cases}$

**Example** $\Delta$-cx structure on $S^n$:

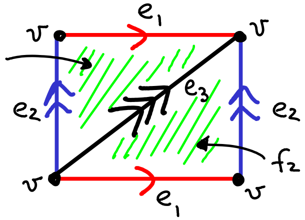



$S^n = \Delta^n \sqcup \Delta^n / $ glue along $\partial\Delta^n$ (call this $\Delta_1$, this $\Delta_0$)

One can deduce: ← but messy!

$H_* S^n = \begin{cases} \mathbb{Z}\langle \text{point}\rangle \cong \mathbb{Z} & *=0 \\ \mathbb{Z}\langle \Delta_1 - \Delta_0\rangle \cong \mathbb{Z} & *=n \\ 0 & \text{else} \end{cases}$ (pick any vertex)

**Example** $X = T^2 =$




$0 \to C_2^\Delta \longrightarrow C_1^\Delta \longrightarrow C_0^\Delta \longrightarrow 0$

$C_2^\Delta = \mathbb{Z}f_1 + \mathbb{Z}f_2$, $C_1^\Delta = \mathbb{Z}e_1+\mathbb{Z}e_2+\mathbb{Z}e_3$, $C_0^\Delta = \mathbb{Z}v$

$f_1 \mapsto e_1 - e_3 + e_2$

$f_2 \mapsto e_2 - e_3 + e_1$

$e_1, e_2, e_3 \longmapsto v - v = 0$

$H_*^\Delta(T^2) = \begin{cases} \mathbb{Z}v & *=0 \\ (\mathbb{Z}e_1+\mathbb{Z}e_2+\mathbb{Z}e_3)/\mathbb{Z}(e_1-e_3+e_2) & *=1 \quad \leftarrow \text{freely generated by } e_1, e_2 \\ \mathbb{Z}\cdot(f_1-f_2) & *=2 \\ 0 & \text{else} \end{cases}$

$\cong \begin{cases} \mathbb{Z} & *=0,2 \\ \mathbb{Z}^2 & *=1 \\ 0 & \text{else} \end{cases}$

Alternative useful method using a trick from algebra: Smith normal form of $\partial_2$: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix} \xrightarrow{\text{row op.}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \xrightarrow{\text{col. op.}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ so after $\mathbb{Z}$-isos of $C_2, C_1$ we get $\mathbb{Z}^2 \to \mathbb{Z}^3$, $(a,b) \mapsto (a,0,0)$

hence: $H_2 \cong \ker \partial_2 \cong \mathbb{Z}$, $H_1 \cong \operatorname{coker}\partial_2 \cong \mathbb{Z}^2$

**Remark about orientations** (see also my B3.2 Geometry of Surfaces notes)

For a vector space an **orientation** is a choice of basis modulo linear endomorphisms of $\det > 0$

**Example** $\mathbb{R}^2$: ($e_1$, $e_2$) right-hand orientation (positive) $\xrightarrow[\det<0]{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$ ($e_1$, $e_2$) left-hand orientation (negative)

**Fact** $GL(n,\mathbb{R})$ has 2 path-components $\begin{cases} A : \det A > 0 \\ A : \det A < 0 \end{cases}$ so can always continuously deform a basis to another within same orientation

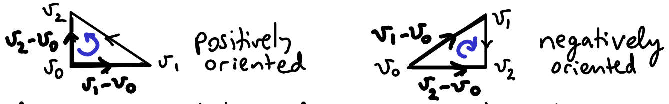
**Canonical orientation on $\mathbb{R}^n$**: $e_1,\dots,e_n$ standard basis ← "positive orientation"

**Example** $[v_0,\dots,v_n]$ simplex $\Rightarrow$ $v_1 - v_0, \dots, v_n - v_0$ is a basis of vector subspace

$V = \{\sum a_i v_i : \sum a_i = 0\} \subseteq \mathbb{R}^{n+k}$

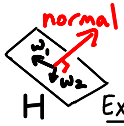
hence a choice of orientation of $V$, and each transposition of vertices $v_0,\dots,v_n$ switches the orientation class.

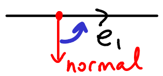
If $v_0,\dots,v_n \in \mathbb{R}^n$ then $V=\mathbb{R}^n$ so simplex's orientation can be compared with $\mathbb{R}^n$-orient$^n$.

<u>Example</u> In $\mathbb{R}^2$: 


- No canonical choice of orientation for an abstract vector space.
  Need choose basis $v_1,\dots,v_n$ then <u>declare</u> another basis <u>positively oriented</u> if the change of basis matrix has $\det > 0$.

- For hyperplane $H \subseteq \mathbb{R}^n$ <u>with choice of normal</u> can <u>declare</u> orientation of basis $w_1,\dots,w_{n-1}$ of $H$ <u>positive</u> if normal, $w_1,\dots,w_{n-1}$ is positive $\mathbb{R}^n$-basis
  <u>convention</u> "outward normal first"



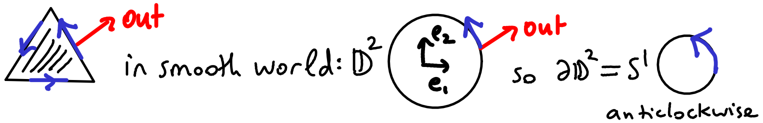

<u>Example</u> 

$H \subseteq \mathbb{R}^2$ $\Rightarrow$ $e_1$ positive basis for $H$

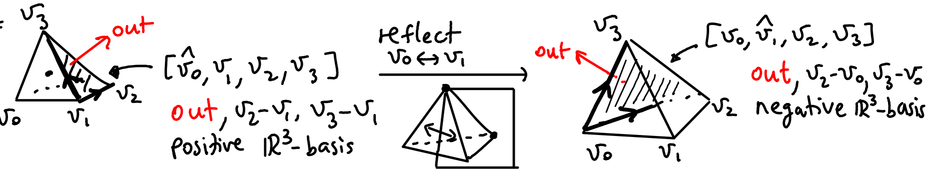
$(\text{normal}, e_1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\det = +1 > 0$

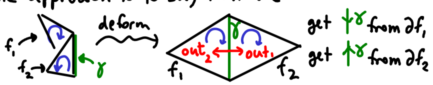
<u>Example</u> $\Delta^n \subseteq \mathbb{R}^{n+1}$ with normal $(1,1,\dots,1)$ is positively oriented.

<u>UPSHOT</u> For an $n$-simplex $[v_0,\dots,v_n]$ in $\mathbb{R}^n$, each facet lies in a hyperplane and have canonical choice of normal: <u>outward normal</u>. Hence <u>facets are canonically oriented</u>.

<u>Example</u> $\mathbb{R}^2 \supseteq$ 


Any <u>reflection of $\mathbb{R}^n$ will swap orientation</u>: after $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ get clockwise

<u>Example</u> 

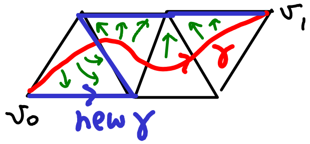

<u>UPSHOT</u> $(-1)^i$ in $(-1)^i[v_0,\dots,\hat{v}_i,\dots,v_k]$ in definition of simplicial $\partial$ secretly keeps track of whether the orientation of the simplex agrees or not with the orientation induced geometrically by the above conventions. From this point of view, the equation $\partial\circ\partial=0$ holds because a codimension 2 face $\gamma$ of a simplex $\sigma$ arises as the facet of exactly two facets $f_1, f_2$ of $\sigma$, and the geometric orientations of $f_1, f_2$ induce <u>opposite</u> geometric orientations on $\gamma$ (therefore if we keep track of orientation signs we count $+\gamma-\gamma=0$).
(checking that they are opposite requires some thought, one approach is to say that we can deform $f_1, f_2$ until they make a flat angle, and then their outward normals will be opposite.) Picture: 


<u>Lemma</u> $H_*^\Delta(X) \cong \bigoplus H_*^\Delta(X_i)$ where $X_i$ are the path-components of $X$.

<u>Pf</u> $\bigoplus C_*^\Delta(X_i) \longrightarrow C_*^\Delta(X)$, $\oplus c_i \longmapsto \sum c_i$ is chain isomorphism since any simplex $\sigma: \Delta^k \to X$ has path-connected image, so $\subseteq X_i$ some $i$. □ (since $\Delta^k$ path-conn.)

<u>Theorem</u> $X$ has $\Delta$-cx structure $\Rightarrow$ $H_0^\Delta(X) \cong \bigoplus_{\text{path-conn. components}} \mathbb{Z}$

<u>Pf</u> By Lemma, WLOG $X$ path-connected
- vertex $v$ $\Rightarrow$ $v \in C_0(X) \xrightarrow{\partial} C_{-1}(X) \equiv 0$ $\Rightarrow [v] \in H_0(X)$
- vertices $v_0, v_1 \in X$ $\Rightarrow$ $\exists$ path $\gamma$ from $v_0$ to $v_1$
  $\Rightarrow$ can homotope path so that go along edges (continuously deform)




$\Rightarrow$ $\gamma$ is sum of 1-chains s.t. $\partial\gamma = v_1 - v_0$
$\Rightarrow$ $[v] \in H_0(X)$ independent of choice of $v$
$\Rightarrow$ $H_0(X) = \langle [v] \rangle$

- $H_0(X) = \langle [v] \rangle \leftarrow \mathbb{Z}$ is injective?
  $nv \leftarrow\!\shortmid n$ Suppose $nv = \partial c$ some $c \in C_1(X)$
  consider the <u>augmentation hom</u>

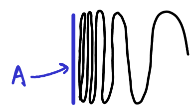
$$C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$$
$$\textstyle\sum n_i \sigma_i \longmapsto \sum n_i$$
(0-simplices)

notice composite is 0 since $\partial\left(\text{1-simplex } \sigma_0 \to \sigma_1\right) = \sigma_1 - \sigma_0 \xrightarrow{\varepsilon} 1-1=0$

$\Rightarrow$ $n = \varepsilon(nv) = \varepsilon\partial c = 0$. □

<u>Rmk</u> $X$ top. space $\Rightarrow$ **path conn. component $\subseteq$ connected component** since path-conn. $\Rightarrow$ connected. For $\Delta$-cx, these are same (since connected + locally path-conn. $\Rightarrow$ path-connected).
However for top. space they need not be the same, e.g. <u>Topologist's sine curve</u>

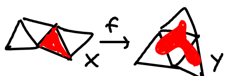
$\{(x, \sin\frac{1}{x}) : x \in (0,1]\} \cup \underbrace{0 \times [0,1]}_{A} \subseteq \mathbb{R}^2$
2 path-conn. components




- <u>connected</u>
- not path-connected
- not locally path-connected

# 3. SINGULAR HOMOLOGY

Motivation  Not obvious that $H_*^\Delta$ is functorial: $\Delta^k \xrightarrow{\sigma} X \xrightarrow{f} Y$ ← continuous map

then $f\circ\sigma$ typically not a simplex: $\triangle \xrightarrow{\sigma}$ 

Solution 1: only allow **simplicial maps** $f: X\to Y$ (so $f\circ\sigma$ simplex $\forall\sigma$)

Solution 2: show that any cts map $f: X\to Y$ can be approximated arbitrarily well by a simplicial map after having performed barycentric subdivision on $X,Y$ enough times. Also any two such approximations induce the same map $H_*^\Delta(X)\to H_*^\Delta(Y)$

Solution 3: develop new homology $H_*(X)$ which allows any cts map $\Delta^n \to X$ and prove $H_*^\Delta(X)\cong H_*(X)$ for $\Delta$-complexes $X$.

↑ WILL DO THIS.

**Def** **Singular n-simplex** is any continuous map $\sigma: \Delta^n \longrightarrow X$ (X is any top. space)

**Singular n-chains** $C_n(X)$ = free abelian group generated by ↑

$$= \left\{ \sum_{\substack{\text{singular} \\ n\text{-simplices } \sigma}} c_\sigma \cdot \sigma \;:\; c_\sigma \in \mathbb{Z} \text{ only finitely many } c_\sigma \neq 0 \right\}$$

$$\boxed{\partial_n \sigma = \sum (-1)^i \cdot \sigma|_{\Delta_i^{n-1}}}$$ (and extend linearly) ← i-th facet

Rmk Here $\Delta_i^{n-1} = [e_0,..,\hat{e}_i,..,e_n]$ is identified canonically with $\Delta^{n-1}$ (send $e_k\to e_k$ $k<i$, $e_k\to e_{k-1}$ for $k>i$)

Lemma $\partial\circ\partial = 0$

Proof $\partial_{n+1}(\partial_n\sigma) = \partial_{n+1}\left(\sum(-1)^i\sigma|_{\Delta_i^{n-1}}\right)$ ← $[e_0,..,\hat{e}_i,..,e_n]$

$$= \sum_{j<i}(-1)^i(-1)^j\sigma|_{[e_0,..,\hat{e}_j,..,\hat{e}_i,..,e_n]}$$

$$+\sum_{j>i}(-1)^i\underline{\underline{(-1)^{j-1}}}\sigma|_{[e_0,..,\hat{e}_i,..,\hat{e}_j,..,e_n]}$$ antisymmetric if swap i,j

$= 0$. □

$\Longrightarrow$ **singular homology**: $\boxed{H_*(X) = H_*(C_*,\partial_*)}$

For $\Delta$-complex $X$ have inclusion of subcomplex $C_*^\Delta \to C_*$

$\Rightarrow$ induces $H_*^\Delta(X) \longrightarrow H_*(X)$  Fact: isomorphism (proof later, see cellular $H_*^{CW}\cong H_*$)

Corollary $H_*^\Delta(X)$ is independent of choice of $\Delta$-cx structure on $X$

Example $X = \text{point} \Rightarrow C_n(X) = \mathbb{Z}\cdot(\sigma_n: \Delta^n\to X$ the constant map$)$

$\partial\sigma_n = \sum(-1)^i\sigma_n|_{\Delta_i^{n-1}} = \underbrace{\sum(-1)^i}_{\{1 \text{ if } n \text{ even},\ 0 \text{ if } n \text{ odd}\}}\sigma_{n-1}$ $\Rightarrow \cdots \xrightarrow{1}\mathbb{Z}\xrightarrow{0}\mathbb{Z}\xrightarrow{1}\mathbb{Z}\xrightarrow{0}\mathbb{Z}\to 0$ (with $\mathbb{Z}\cong C_1$, $\mathbb{Z}\cong C_0$)

$\Rightarrow H_*(pt) \cong \begin{cases}\mathbb{Z} & *=0\\ 0 & \text{else}\end{cases}$

Lemma $\boxed{H_*(X) \cong \bigoplus H_*(X_i)}$ where $X_i$ are path-components of $X$

Pf Image of cts map $\Delta^n \to X$ is path conn. so lies in some $X_i$. □

Cor $\boxed{H_0(X) = \bigoplus_{X_i}\mathbb{Z}}$

Pf By Lemma, WLOG $X$ path-connected. $\Delta^0 = pt \to X$ is cycle since $C_{-1}(X) = 0$ (← generators of $C_0(X)$)

Given 2 points $x,y\in X$, a path $\Delta^1 = [0,1]\xrightarrow{\gamma} X$, $\gamma(0)=x$, $\gamma(1)=y$ is also a 1-chain!

So $y - x = \partial\gamma$, so $x,y$ are homologous. Finally if $n\cdot[x] = 0 \in H_0(X)$ then $nx = \partial c$ some $c\in C_1(X)$ = generated by paths. Now run the augmentation hom. trick like we did for $H_0^\Delta$: $n = \varepsilon(nx) = \varepsilon\partial c = 0$ as $\varepsilon\circ\partial = 0$. □

## Naturality (i.e. functoriality)

Lemma $f: X\to Y$ continuous

$\Longrightarrow H_*(f) = f_*: H_*(X)\to H_*(Y)$ induced by chain map

**induced map** ↗ $f_*: C_*(X)\longrightarrow C_*(Y)$

$\boxed{f_*(\sigma) = f\circ\sigma}$ and extend linearly  ($\Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y$, $f_*\sigma$)

Pf $\partial_n(f_*\sigma) = \sum(-1)^n f\circ\sigma|_{\Delta_i^{n-1}} = f_*\left(\sum(-1)^n\sigma|_{\Delta_i^{n-1}}\right) = f_*(\partial_n\sigma)$ □

Properties 1) $X\xrightarrow{f}Y\xrightarrow{g}Z \Longrightarrow (g\circ f)_* = g_*\circ f_*$

2) $id_* = id$

Pf 1) $(g\circ f)_*\sigma = g\circ f\circ\sigma = g_*(f\circ\sigma) = g_*(f_*\sigma)$ ✓

2) $id_*\sigma = id\circ\sigma = \sigma$ ✓ □

Cor $H_*: \left\{\begin{matrix}\text{topological spaces}\\ \&\\ \text{cts maps}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{graded abelian groups}\\ \&\\ \text{graded homs}\end{matrix}\right\}$ is a functor

Cor $X\cong Y \Rightarrow H_*(X)\cong H_*(Y)$

# 4. CHAIN HOMOTOPIES AND HOMOTOPY INVARIANCE

## Algebra: chain homotopies

$f_*, g_* : (C_*, \partial_*) \longrightarrow (\tilde{C}_*, \tilde{\partial}_*)$ chain maps

**Def** $f_*, g_*$ are **chain homotopic** if $\exists$ (degree $+1$) hom $h : C_* \longrightarrow \tilde{C}_*[1]$ s.t.

$$\boxed{\tilde{\partial} \circ h + h \circ \partial = f_* - g_*}$$

$h$ is called a **chain homotopy**

**Consequence** $f_* = g_* : H_*(C_*, \partial_*) \longrightarrow H_*(\tilde{C}_*, \tilde{\partial}_*)$ on homology

**Pf**

$$\begin{array}{ccccc} C_{n+1} & \longrightarrow & C_n & \xrightarrow{\partial_n} & C_{n-1} \\ & \swarrow^{h_n} & f_n \downarrow\downarrow g_n & \swarrow^{h_{n-1}} & \\ \tilde{C}_{n+1} & \xrightarrow{\tilde{\partial}_{n+1}} & \tilde{C}_n & \longrightarrow & \tilde{C}_{n-1} \end{array}$$

$c$ cycle $\in C_n$

$\Rightarrow f_n(c) - g_n(c) = \underbrace{\tilde{\partial}_{n+1} \circ h_n(c)}_{\text{boundary}} + h_{n-1} \circ \underbrace{\partial_n(c)}_{=0}$

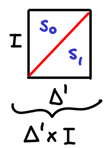
$\Rightarrow f_n(c) = g_n(c)$ in $H_*(\tilde{C})$ $\square$

**Theorem** $i_0 : X \longrightarrow X \times I$, $i_0(x) = (x, 0)$ where $I = [0,1]$

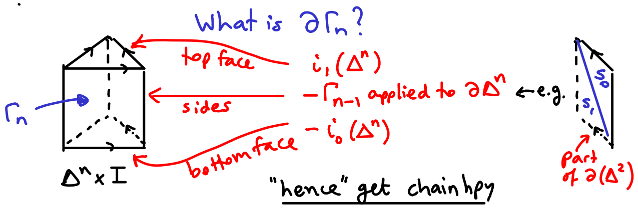
$i_1 : X \longrightarrow X \times I$, $i_1(x) = (x, 1)$

$\Longrightarrow i_{0*}, i_{1*} : C_*(X) \to C_*(X \times I)$ are chain hpic.

**Key idea** Need the "prism operator" which cuts $\Delta^n \times I$ into a sum $\Gamma_n$ of $(n+1)$-simplices in $\Delta^n \times I$:




$\Gamma_1 = s_0 - s_1$




**Pf** ← Non-examinable

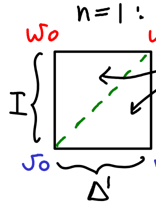
bottom facet $\Delta^n \times 0 = [v_0, \ldots, v_n]$ ← $v_i = e_i \times 0$

top facet $\Delta^n \times 1 = [w_0, \ldots, w_n]$ ← $w_i = e_i \times 1$

$\subseteq \Delta^n \times [0,1] \subseteq \mathbb{R}^{n+1}$

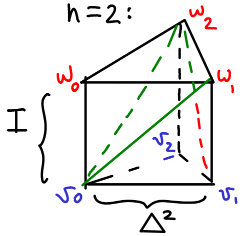
**Examples**

$n=1$:




two 2-simplices $[v_0, v_1, w_1]$, $[v_0, w_0, w_1]$

$n=2$:




three 3-simplices: $[v_0, v_1, v_2, w_2]$, $[v_0, v_1, w_1, w_2]$, $[v_0, w_0, w_1, w_2]$

Let $s_i = [v_0, \ldots, v_i, w_i, \ldots, w_n]$

**Claim** The $s_i$ cover $\Delta \times [0,1]$ and give $\Delta$-cx structure on $\Delta^n \times I$

**Pf** $\sum_{k \leq i} t_k v_k + \sum_{k \geq i} s_k w_k = (t_0, \ldots, t_{i-1}, \underline{t_i + s_i}, s_{i+1}, \ldots, s_n, s_i + \cdots + s_n)$

So given $(x_0, \ldots, x_n, a) \in \Delta^n \times I$, equate and solve:

$t_0 = x_0, \ldots, t_{i-1} = x_{i-1}$, $s_{i+1} = x_{i+1}, \ldots, s_n = x_n$, and $\begin{cases} s_i = a - x_{i+1} - \cdots - x_n \\ t_i = x_i - s_i \end{cases}$

Note $x_k \geq 0$, $\Sigma x_k = 1$, $a \in [0,1]$ hence $\Sigma t_k + \Sigma s_k = 1$ ✓ $\begin{cases} t_k \geq 0 \text{ for } k < i \ ✓ \\ s_k \geq 0 \text{ for } k > i \end{cases}$

but $\begin{matrix} s_i \geq 0 \\ t_i \geq 0 \end{matrix}\Bigg\} \Leftrightarrow \begin{cases} a \geq x_{i+1} + \cdots + x_n \\ x_i + x_{i+1} + \cdots + x_n \geq a \end{cases}$. Thus a solution exists if we pick $i = \min\{k : a \geq x_{i+1} + \cdots + x_n\}$

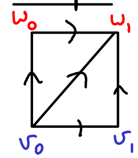
There are multiple solutions if $x_{i+1} = x_{i+2} = \cdots = x_j = 0$, but that is as expected: those points of $\Delta^n \times I$ belong to the faces of $s_i, s_{i+1}, \ldots, s_j$. $\square$

**Def** $\Gamma_n = \sum (-1)^i s_i \in C_{n+1}(\Delta^n \times [0,1])$ ← geometrically this "represents" $\Delta^n \times I$ as a simplicial chain

$\Rightarrow \partial\Gamma_n = \sum_i \sum_{j \leq i} (-1)^i (-1)^j [v_0, \ldots, \hat{v}_j, \ldots, v_i, w_i, \ldots, w_n]$
$+ \sum_i \sum_{j \geq i} (-1)^i (-1)^{j+1} [v_0, \ldots, v_i, w_i, \ldots, \hat{w}_j, \ldots, w_n]$

} geometrically this "represents" $\partial(\Delta^n \times I) = (\partial\Delta^n \times I) \sqcup (\Delta^n \times \partial I)$

**Example**




$\Gamma_1 = [v_0, w_0, w_1] - [v_0, v_1, w_1]$ "is the square"

$\partial\Gamma_1 = [w_0, w_1] - [v_0, w_1] + [v_0, w_0] - [v_1, w_1] + [v_0, w_1] - [v_0, v_1]$

"is $\partial$ of square" "inside facets" cancel

**Prism operator** $P : C_n(X) \longrightarrow C_{n+1}(X\times[0,1])$

$$\boxed{P(\sigma) = (\sigma\times id)_*(\Gamma_n)}$$

$\sigma : \Delta^n \to X$

$\sigma\times id : \Delta^n\times[0,1] \to X\times[0,1]$

$(\sigma\times id)(x,t) = (\sigma(x),t)$

$$\partial P(\sigma) = \partial(\sigma\times id)_*(\Gamma_n) = (\sigma\times id)_*(\partial\Gamma_n)$$

$$= \sum_i \sum_{j\le i} (-1)^i(-1)^j [i_0\sigma e_0, \dots, \widehat{i_0\sigma e_j}, \dots, i_0\sigma e_i, i_1\sigma e_i, \dots, i_1\sigma e_n]$$

$$+ \sum_i \sum_{j\ge i} (-1)^i(-1)^{j+1}[i_0\sigma e_0, \dots, i_0\sigma e_i, i_1\sigma e_i, \dots, \widehat{i_1\sigma e_j}, \dots, i_1\sigma e_n]$$

$$= i_{1*}\sigma - i_{0*}\sigma - P\partial\sigma . \qquad \square$$

(this abbreviated notation means the map $(t_0,\dots,t_n)\mapsto\left(\sigma\begin{pmatrix}t_0e_0+\dots+\widehat{t_je_j}+t_je_{j+1}+\dots\\+t_{i-1}e_i+t_ie_i+\dots+t_ne_n\end{pmatrix}, t_i+\dots+t_n\right) \in X\times I$, with $(t_0,\dots,t_n)\in\Delta^n$)

$i=j=0$: 1st sum; $i=j=n$: 2nd sum; $P\partial\sigma$: $((\partial\sigma)\times id)_*\Gamma_{n-1}$, with $\Gamma_{n-1} = \sum_i (-1)^i [v_0,\dots,v_i, w_i,\dots,w_{n-1}]$

(★)
$(\sigma\times id)(v_i) = (\sigma\times id)(e_i,0) = (\sigma(e_i),0) = i_0(\sigma)(e_i)$

$(\sigma\times id)(w_i) = (\sigma(e_i),1) = i_1(\sigma)(e_i)$

now use (★) and

$\partial\sigma = \sum(-1)^k[\sigma e_0,\dots,\widehat{\sigma e_k},\dots,\sigma e_n]$

(Note for $k<i$ the $P$ operator on $(-1)^k[\sigma e_0,\dots,\widehat{\sigma e_k},\dots,\sigma e_n]$ gives $\sum \underline{\underline{(-1)^{i-1}}}(-1)^k[i_0\sigma e_0,\dots,\widehat{i_0\sigma e_k},\dots,i_0\sigma e_i, i_1\sigma e_i,\dots,i_1\sigma e_n]$)

## Homotopy invariance

Def $f_0, f_1 : X\longrightarrow Y$, $f_0\simeq f_1$ **homotopic** if $\exists$ continuous map $F: X\times[0,1]\longrightarrow Y$ called **homotopy** s.t. $\begin{cases} f_0 = F\circ i_0\\ f_1 = F\circ i_1\end{cases}$.

Idea Think of this as a continuous family of maps $f_t = F(\cdot,t): X\to Y$ from $f_0$ to $f_1$.

Exercise $\simeq$ is an equivalence relation.

Homotopic **relative to $A\subseteq X$** if $F(a,t) = f_0(a) = f_1(a)$ all $a\in A$, all $t$.
write "$f\simeq g$ **rel A**"

Def $X\simeq Y$ **homotopy equivalent** spaces if $\exists$ maps $X \overset{f}{\underset{g}{\rightleftarrows}} Y$ with $g\circ f\simeq id$, $f\circ g\simeq id$

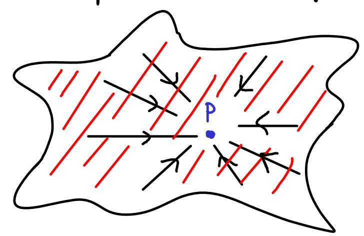
Rmk homeo $\Rightarrow$ hpy equivalent

Def $X$ **contractible** if $X\simeq pt$

equivalently $(X\xrightarrow{id}X) \simeq (X\xrightarrow{const} point\in X)$

Example • $\mathbb{R}^n\simeq pt$
$F(x,t) = tx$ then $f_0\equiv 0$, $f_1 = id$.

• (star-shaped subsets of $\mathbb{R}^n$) $\simeq pt$





contains line segments to a specific point $p$

WLOG $p=0$ & use same $F$ (↑ translate)

(examples: $\mathbb{D}^n$, convex sets, ...)

Theorem $f_0\simeq f_1 \Rightarrow f_{0*} = f_{1*} : H_*X\to H_*Y$

Pf
$$f_{1*} - f_{0*} = F_*i_{1*} - F_*i_{0*}$$
$$= F_*(i_{1*} - i_{0*})$$
$$\overset{\text{previous Thm}}{=} F_*(\partial P + P\partial)$$
$$\overset{F_* \text{ chain map}}{=} \partial\circ(F_*\circ P) + (F_*\circ P)\circ\partial$$
$\Rightarrow F_*\circ P$ is chain hpy from $f_{0*}$ to $f_{1*}$ $\square$

(where $F$ = homotopy, $i_0, i_1$ as in previous Thm)

Cor $X\simeq Y \Rightarrow H_*X\cong H_*Y$

Pf $f_*g_* = id_*$, $g_*f_* = id_*$ $\square$

Example $X$ contractible $\Rightarrow H_*X\cong H_*(pt)\cong\begin{cases}\mathbb{Z} & *=0\\ 0 & \text{else}\end{cases}$

Cultural Rmk (Whitehead's theorem) For nice topological spaces (CW complexes – see later in course) if $X,Y$ are simply connected and $\exists f: X\to Y$ inducing isomorphisms on $H_*$ then $X\simeq Y$ are homotopy equivalent.

## Relative homology

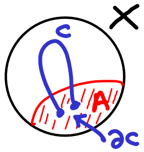
**Def** $(X,A)$ **pair of spaces** if $A \subseteq X$ topological subspace

$\Rightarrow i = \mathrm{incl} : A \hookrightarrow X$ induces a subcx $i_* : C_*A \to C_*X$

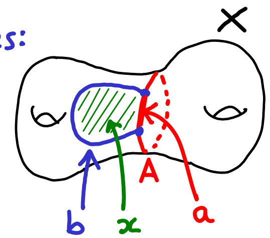
$\Rightarrow C_*X / C_*A$ quotient chain cx (recall $\partial[x] = [\partial x]$)

$$\boxed{H_*(X,A) = H_*(C_*X / C_*A)}$$

Idea: **relative cycles**: $c \in C_*X$ s.t. $\partial c \in C_*A$



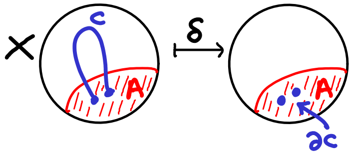

**relative boundaries**: $b \in C_*X$ s.t. $\exists\, x \in C_{*+1}X$ $\partial x = b + a$, $a \in C_*A$




$\Rightarrow 0 \to C_*A \to C_*X \to C_*X/C_*A \to 0$ SES

**Cor** $\boxed{\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \to H_n(X,A) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*} \cdots}$

↖ **LES for the pair**




$\delta c = \partial c \in C_*A$ ↖ Need not be $\partial a$ some $a \in C_*A$

## Reduced homology

$\tilde{H}_* X = \mathrm{Ker}\left(H_*X \to H_*(pt)\right)$ ← induced by $X \to pt$

For $X \neq \emptyset$, $\tilde{H}_*X = H_*$ of **augmented chain complex**:

$$\cdots \to C_n(X) \to \cdots \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

**augmentation** ↗ $\varepsilon(\Sigma n_i \cdot p_i) = \Sigma n_i$ ($n_i \in \mathbb{Z}$, points $p_i \in X$)

(can view $C_{-1}(X) = \mathbb{Z} \cdot (\text{map } \emptyset \to X)$ where allow the empty simplex $\emptyset$)

**Example** $\tilde{H}_*(\mathbb{R}^n) \cong \tilde{H}_*(\mathbb{D}^n) \cong \tilde{H}_*(pt) = 0$

**Check** · $H_*X = \tilde{H}_*X$ $* \neq 0$, and $H_0X \cong \tilde{H}_0X \oplus \mathbb{Z}$ for $X \neq \emptyset$

· $f : X \to Y \Rightarrow f_* : \tilde{H}_*X \to \tilde{H}_*Y$

**Lemma** $(X,A)$ pair $\Rightarrow \exists$ LES

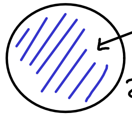
$$\boxed{\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \to H_n(X,A) \to \tilde{H}_{n-1}(A) \xrightarrow{i_*} \cdots \to H_0(X,A) \to 0}$$

**Pf** use augmented ch. cx. and $\mathbb{Z}/\mathbb{Z} = 0$ □

**Cor** $H_*(X, pt) \cong \tilde{H}_*(X)$ for $X \neq \emptyset$

**Pf** $\tilde{H}_*(pt) = 0$. □

**Example** LES: $\tilde{H}_*(S^{n-1}) \to \tilde{H}_*(\mathbb{D}^n) = 0$, $H_*(\mathbb{D}^n, S^{n-1}) \xrightarrow{[-1]} \tilde{H}_*(S^{n-1})$




$$\Rightarrow \boxed{H_*(\mathbb{D}^n, S^{n-1}) \cong \tilde{H}_{*-1}(S^{n-1})}$$

## Naturality of the LES for pairs

**Def** A **map of pairs of spaces** $(X,A) \xrightarrow{f} (Y,B)$ means $f : X \to Y$ and $f(A) \subseteq B$.

**Lemma**

$$\begin{array}{ccccccccc} \cdots \to & H_*A & \to & H_*X & \to & H_*(X,A) & \to & H_{*-1}A \to \cdots \\ & \downarrow f_* & & \downarrow f_* & & \downarrow & & \downarrow f_* \\ \cdots \to & H_*B & \to & H_*Y & \to & H_*(Y,B) & \to & H_{*-1}B \to \cdots \end{array}$$

and similarly for $\tilde{H}_*$.

**Pf**

$$\begin{array}{c} 0 \to C_*A \to C_*X \to C_*X/C_*A \to 0 \\ \quad f_*\downarrow \quad\; f_*\downarrow \quad\;\; f_*\downarrow \\ 0 \to C_*B \to C_*Y \to C_*Y/C_*B \to 0 \end{array}$$

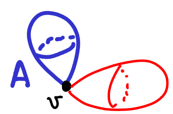
$\Rightarrow$ claim follows by naturality of LES induced by SESs of chain cxes. □

## 5. EXCISION THEOREM AND QUOTIENTS

$(X,A)$ pair

**Def** $r : X \to X$ **retraction** onto $A$ if $\begin{cases} r(X) = A \\ r|_A = \mathrm{id}_A \end{cases}$ (equivalently $r^2 = r$ then define $A = \mathrm{im}(r)$)

**Example**




$X = S^2 \vee S^2$ ($S^2 = A$) = two spheres glued at one point $v$ (**wedge sum**)

$r : X \to A$ map second sphere to $v$

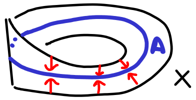
**Example** In Pf of Brower fixed pt thm we built a retraction $r$ by contradiction

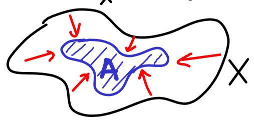
**Cor** $r$ retraction $\Rightarrow$ $r_* : H_*X \to H_*A$ surjective, $\mathrm{incl}_* : H_*A \to H_*X$ injective

**Pf** $A \xrightarrow{\mathrm{incl}} X \xrightarrow{r} A$ now use $H_*$ functorial □

**Def** $r : X \to X$ **deformation retraction** onto $A$ if $\begin{cases} r \text{ retraction} \\ r \simeq id_X \ \text{ rel } A \end{cases}$

**Example** $X$ = Möbius strip, $A$ = equator

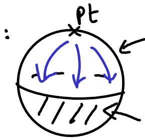



**Lemma** $r$ def. retr. $\Rightarrow$ • $A \xrightarrow[\simeq]{incl} X$ is a homotopy equivalence.
• $incl_*$ and $r_*$ are isos on $H_*$, so $H_*A \cong H_*X$

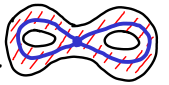
**Pf** $A \underset{r}{\overset{incl}{\rightleftarrows}} X$ $\quad incl \circ r = r \simeq id_X$, $\quad r \circ incl = r|_A = id_A$ □

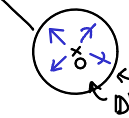
**Example** $S^n \setminus pt$ def. retracts to $\mathbb{D}^n \cong$ lower hemisphere:




$\Rightarrow S^n \setminus pt \simeq \mathbb{D}^n$

$\Rightarrow S^n \setminus \{2 \text{ points}\} \simeq \mathbb{D}^n \setminus pt \cong \mathbb{D}^n \setminus 0 \simeq S^{n-1}$

$\Rightarrow S^n \setminus \{3 \text{ points}\} \underset{def.ret.}{\simeq}$  $\underset{ret.}{\simeq}$  $S^{n-1} \vee S^{n-1}$

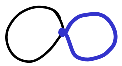
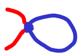



## Excision theorem

$E \subseteq A \subseteq X$ subspaces with $\overline{E} \subseteq A^\circ$ $\Longrightarrow$ $(X \setminus E, A \setminus E) \to (X, A)$ induces iso

$$\boxed{H_*(X \setminus E, A \setminus E) \xrightarrow{\cong} H_*(X, A)}$$

Proof later.

**Example** $X = S^1 \vee S^1 =$  $\supseteq A =$  $\supseteq E =$  $\cong S^1$

$\Rightarrow H_1(X, A) \underset{\text{exc. thm.}}{\cong} H_1($$) \underset{\text{hpy invce}}{\cong} H_1(\mathbb{D}^1, \partial \mathbb{D}^1) \cong \widetilde{H}_0(S^0) \cong \mathbb{Z}$ (where $\partial\mathbb{D}^1 = S^0$, $S^0$ = 2 points)

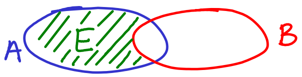
**Example** Invariance of dimension from chapter 0 also holds if replace $\mathbb{R}^n, \mathbb{R}^m$ by non-empty open sets $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ because for $p \in U$: (statement becomes: $U \cong V \Rightarrow n = m$)

$$H_*(U, U-p) \cong H_*(\mathbb{R}^n, \mathbb{R}^n - p) \cong H_{*-1}(\mathbb{R}^n - p) \cong H_{*-1}(S^{n-1})$$

(excise $\mathbb{R}^n \setminus U$; LES of pair using $H_*(\mathbb{R}^n) = 0$; deformation retract $\mathbb{R}^n - p \simeq S^{n-1}$)

**Rephrasing of Excision Thm**

$X = A^\circ \cup B^\circ$ $(A, B \subseteq X$ subspaces$)$ $\Longrightarrow$ $\boxed{H_*(X, A) \cong H_*(B, A \cap B)}$ induced by inclusion $(X, A) \leftarrow (B, A \cap B)$




**Pf** Take $E = X \setminus B$ so $X \setminus E = B$ and $A \cap B = A \setminus E$. □

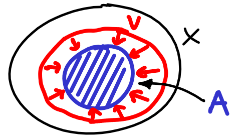
**Idea** why excision holds: $C_*(A) + C_*(B) \to C_*(X)$ is a homotopy equivalence and $C_*(A) \cap C_*(B) = C_*(A \cap B)$. Idea: can subdivide chains in $X$ many times, and small enough chains lie either in $A$ or in $B$ (or in both).

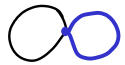
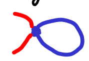
## Good pairs and quotients

For $(X, A)$ pair:

• **Quotient** $X/A = X/\sim$ ← equivalence relation $x \sim y \Leftrightarrow \begin{cases} x, y \in A \\ \text{or} \\ (x = y) \end{cases}$

• $(X, A)$ **good pair** if $\begin{cases} A \neq \emptyset \\ A \text{ closed} \\ A \text{ deformation retract of nbhd } V \text{ of } A \end{cases}$

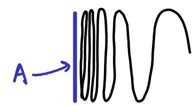



**Example** $X = S^1 \vee S^1 =$  $\supseteq V =$  $\supseteq A =$  $\cong S^1$

$X/A \cong$  ← (all points of $A$ are identified with the node)

**Non-example** **Topologist's sine curve**

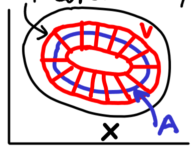
$\{(x, \sin\frac{1}{x}) : x \in (0, 1]\} \cup \underbrace{0 \times [0, 1]}_{A} \subseteq \mathbb{R}^2$




not a good pair.

connected
not path-connected
not locally connected
not locally path-connected

**Cultural Rmk**
Smooth submanifold $\subseteq$ smooth manifold is a good pair (tubular neighbourhood theorem)




**Cor** $(X, A)$ good $\Rightarrow (X, A) \to (X/A, pt)$ induces iso

$$\boxed{H_*(X, A) \longrightarrow H_*(X/A, pt) = \widetilde{H}_*(X/A)}$$

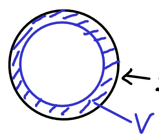
**Pf** good $\Rightarrow \exists$ nbhd $V$ of $A$, and $A \xrightarrow[incl]{\simeq} V$.

$$\begin{array}{ccccc} H_n(X, A) & \xrightarrow{\cong} & H_n(X, V) & \xleftarrow{\cong} & H_n(X \setminus A, V \setminus A) \\ \text{quot.} \downarrow & & \text{quot.} \downarrow & & \| \downarrow id_* = \text{identity} \\ H_n(X/A, A/A) & \xrightarrow{\cong} & H_n(X/A, V/A) & \xleftarrow{\cong} & H_n(X/A \setminus p, V/A \setminus p) \end{array}$$

(LES for pairs & 5-Lemma since $A \simeq V$, $A/A \simeq V/A$; excision; $A/A$: call this point $p$)

Hence all arrows are isos. □

**Example** $\mathbb{D}^n \supseteq S^{n-1}$ good:




$\xrightarrow{\text{quotient}}$


$\Longrightarrow H_*(\mathbb{D}^n, S^{n-1}) \underset{Cor}{\cong} \widetilde{H}_*(\mathbb{D}^n / S^{n-1}) \cong \widetilde{H}_*(S^n)$ $\quad \mathbb{D}^n/S^{n-1} \cong S^n$

**Exercise** Check that the iso in the Cor is natural (for a map $(X, A) \to (Y, B)$ of good pairs get comm. diagram...)

Recall we proved $H_*(\mathbb{D}^n, S^{n-1}) \cong \widetilde{H}_{*-1}(S^{n-1})$ (from LES & $\widetilde{H}_*(\mathbb{D}^n)=0$)

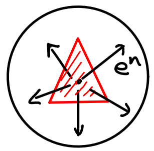
$\Rightarrow$ (inductively, using Example) $\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1}) \cong \dots \cong \widetilde{H}_{k-n}(S^0) = \begin{cases} \mathbb{Z} & n=k \\ 0 & \text{else} \end{cases}$

$S^0$ = 2 points, $H_0(2\text{pts}) = \mathbb{Z}\oplus\mathbb{Z}$

## Generator of $H_n(S^n) \cong \widetilde{H}_n(\mathbb{D}^n/S^{n-1}) \cong H_n(\mathbb{D}^n, S^{n-1}) \cong \mathbb{Z}$

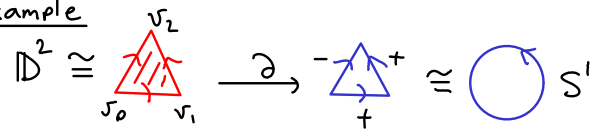
Observe $\exists$ homeo $e^n : \Delta^n \cong \mathbb{D}^n$ (homework) inducing $\Delta$-cx structure on $S^{n-1}$:

$$\boxed{\partial\Delta^n \cong \partial\mathbb{D}^n = S^{n-1}}$$




stretch ctsly outwards from barycentre($\Delta^n$)

Example




Upshot ($n \geq 2$)

$$\begin{array}{|lcl|} H_n(\mathbb{D}^n, S^{n-1}) & = & \mathbb{Z}\cdot e^n \\ H_{n-1}(S^{n-1}) & = & \mathbb{Z}\cdot\partial e^n \\ \widetilde{H}_n(\mathbb{D}^n/S^{n-1}) & = & \mathbb{Z}\cdot[e^n] \end{array}$$

LES — for $n-1\geq 1$, so $n \geq 2$

by Cor — $[e^n]$ really lives in $H_n(\mathbb{D}^n, S^{n-1}) \cong H_n(\mathbb{D}^n/S^{n-1}, S^{n-1}/S^{n-1})$

Exercise Recall another $\Delta$-cx structure on $S^n$:


$$\boxed{S^n = \Delta^n \sqcup \Delta^n \,/\, \text{glue along } \partial\Delta^n}$$

call this $\Delta_1$, this $\Delta_0$

then $\boxed{H_n S^n = \mathbb{Z}\cdot(\Delta_1 - \Delta_0)}$

because $\Delta_1 - \Delta_0$ is a cycle and

$H_n(S^n) \cong H_n(S^n, \Delta_0) \overset{\text{exc.}}{\cong} H_n(\Delta_1, \partial\Delta_1) \cong H_n(\mathbb{D}^n, \partial\mathbb{D}^n) \cong H_n(S^n)$

$\Delta_1 - \Delta_0 \longmapsto \Delta_1 \longmapsto e^n$

(by LES for $(S^n, \Delta_0)$ using $n \geq 1$)

## Another remark about orientations

Fact $\{\text{homeos } \Delta^n \to \mathbb{D}^n\}$ has 2 path-components

Above we chose a path-component by constructing $e^n$.

If $r$ is any reflection in $\mathbb{R}^{n+1}$ then $e^n \circ r$ is in the other path-component

e.g. swap 2 coordinates in $\Delta^n$

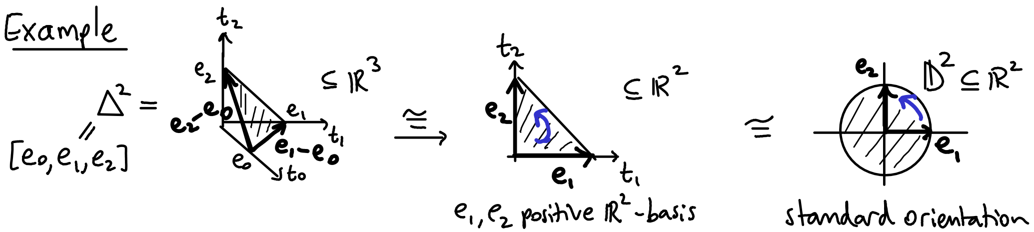
$$H_n(S^n) \cong H_n(\mathbb{D}^n, S^{n-1}) \xrightarrow{\cong} \mathbb{Z}$$
$$e^n \longmapsto +1$$
$$e^n \circ r \longmapsto -1$$

We will see later in the course that this corresponds to a choice of orientation of $\mathbb{D}^n$ and $S^n$.

Our choice is consistent with the inclusion $\mathbb{D}^n \subseteq \mathbb{R}^n$ (with the positive (canonical) orientation of $\mathbb{R}^n$) and the inclusion

$$(\Delta^n \subseteq \mathbb{R}^{n+1}) \xrightarrow{\cong} \{(t_1, \dots, t_n) \in \mathbb{R}^n : t_i \geq 0, \textstyle\sum_{i=1}^n t_i \leq 1\}$$
$$(t_0, \dots, t_n) \longmapsto (t_1, \dots, t_n)$$
$$t_i \geq 0, \textstyle\sum t_i = 1$$

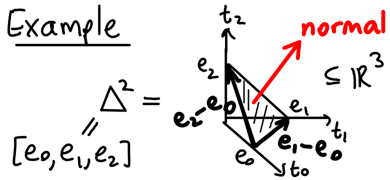
Example




Our choice is also consistent with the "normal first" convention for orienting hyperplanes with a given choice of normal:

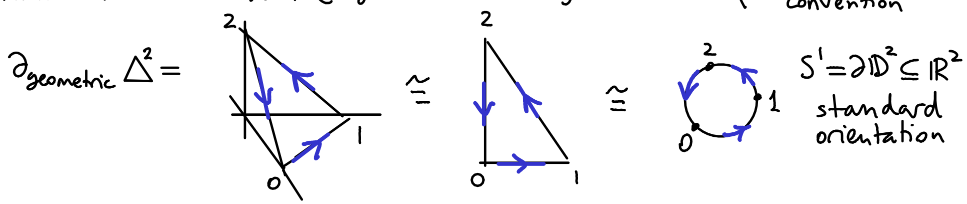
$\Delta^n \subseteq \text{hyperplane } \{(t_0, \dots, t_n) : \sum t_j = 1\} \subseteq \mathbb{R}^{n+1}$ normal $(1, 1, \dots, 1)$ (so pointing to $\infty$ in positive quadrant)

Example




normal, $e_1 - e_0$, $e_2 - e_0$ positive $\mathbb{R}^3$-basis

Consistent also with the geometric boundary orientation (outward normal first convention)




Compare $\partial\Delta = +[\hat{e}_0, e_1, e_2] - [e_0, \hat{e}_1, e_2] + [e_0, e_1, \hat{e}_2]$

This $-[e_0, e_2]$ is not equal to singular chain $[e_2, e_0]$ since they are different maps and we take free abelian group generated by maps.

But $[e_0, e_2] + [e_2, e_0]$ is homologous to $0$ (Homework).

## Locality (or "small simplices theorem")

$\mathcal{U} = \{\text{subspaces } U_i \subseteq X\}$ whose interiors cover $X$:
$X = \bigcup U_i^\circ$

**Def** $C_*^{\mathcal{U}}(X) \subseteq C_*(X)$ subcx generated by n-simplices $\sigma$ with $\sigma(\Delta^n) \subseteq U_i$ some $i$

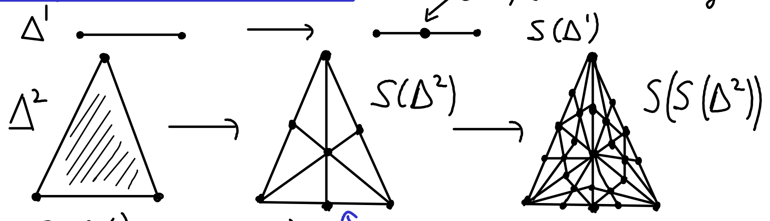
**Theorem**

$$H_*(C_*^{\mathcal{U}}(X)) \cong H_*(C_*(X)) = H_* X$$

**Sketch Pf** ① **Barycentric subdivision**

↑ Non-examinable

**barycentre** of $[v_0,\dots,v_n]$ is $\frac{1}{n+1}(v_0+\dots+v_n)$



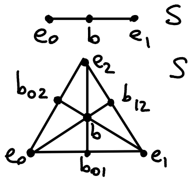

$\Rightarrow$ chain map $S: C_*(X) \longrightarrow C_*(X)$
$\sigma \longmapsto$ "$\sigma \circ S$"

and $S(C_*^{\mathcal{U}}) \subseteq C_*^{\mathcal{U}}$

subdivide the boundary (inductively by dimension) then draw the new faces obtained by convex combinations involving the new vertices and the barycentre

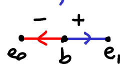
Construction of "$\sigma \circ S$" is inductive:

On linear simplices (then for maps $\sigma$ you restrict $\sigma|_{\dots}$)

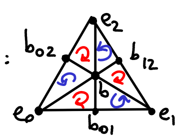



$S[e_0] = [e_0]$

$S[e_0,e_1] = [b,e_1] - [b,e_0]$ (= "$[b, S\partial[e_0,e_1]]$") ← geometrically




$S[e_0,e_1,e_2] =$ "$[b, S\partial[e_0,e_1,e_2]]$"
$=$ "$[b, S[e_1,e_2]] - [b, S[e_0,e_2]] + [b, S[e_0,e_1]]$"
$= ([b,b_{12},e_2] - [b,b_{12},e_1]) - ([b,b_{02},e_2] - [b,b_{02},e_0])$
$+ ([b,b_{01},e_1] - [b,b_{01},e_0])$ ← geometrically:




So for $\sigma: \Delta^2 \to X$ you take $S(\sigma) = \sigma|_{[b,b_{12},e_2]} - \sigma|_{[b,b_{12},e_1]} - \dots$

② $S$ chain hpic to id:

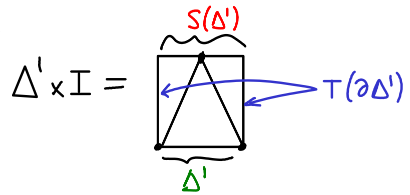
$T: C_n(X) \longrightarrow C_{n+1}(X)$

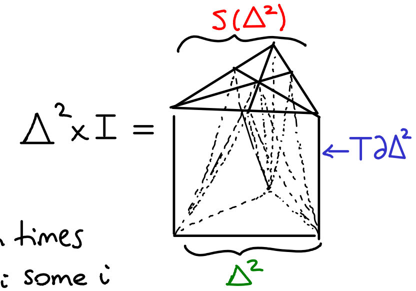
$T(\sigma): \Delta^n \times I \xrightarrow{\text{project}} \Delta^n \xrightarrow{\sigma} X$

exercise: $\partial T + T\partial = S - id$

$\Rightarrow$ $S_*: H_*(X) \xrightarrow{id} H_*(X)$

**Idea:**




③ $\forall$ n-simplex $\sigma: \Delta^n \to X$, apply $S(\cdot)$ enough times until $\sigma$(each n-simplex of subdivision) $\subseteq U_i$ some $i$

③ $\Rightarrow$ $\forall$ cycle $c$, $\exists n$ s.t. $S^n(c) \in C_*^{\mathcal{U}}(X)$ cycle
$\Rightarrow H_*^{\mathcal{U}}(C) \longrightarrow H_*(X)$ surjective
$[S^n(c)] \longmapsto S_*^n[c] = [c]$ by ②

③ $\Rightarrow$ $\forall$ bdry $c = \partial b$, $\exists n$ s.t. $S^n(b) \in C_*^{\mathcal{U}}(X)$

claim: $H_*^{\mathcal{U}}(C) \longrightarrow H_*(X)$ injective

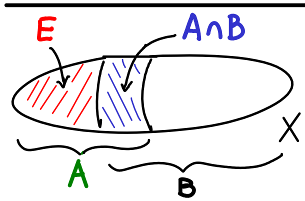
suppose $[c] \longmapsto 0$ then $c = \partial b$ for $b \in C_*(X)$

now $S^n c, S^n b \in C_*^{\mathcal{U}}(X)$ for large $n$

$\Rightarrow \partial S^n b = S^n \partial b = S^n c$ in $C_*^{\mathcal{U}}(X)$

$\Rightarrow [c] \overset{②}{=} S_*^n[c] = [S^n c] = [\partial S^n b] = 0$ in $H_*^{\mathcal{U}}(X)$ ✓ □

## Proof of excision theorem




Let $B = X \setminus E$

use $\mathcal{U} = \{A, B\}$

so $C_*^{\mathcal{U}}(X) = C_*(A) + C_*(B) \subseteq C_*(X)$

$$\Rightarrow \frac{C_*(X\setminus E)}{C_*(A\setminus E)} = \frac{C_*(B)}{C_*(A\cap B)} \cong \frac{C_*(B)}{C_*(A)\cap C_*(B)} \cong \frac{C_*^{\mathcal{U}}(X)}{C_*(A)}$$

↖ 2nd isomorphism theorem for groups

$\Rightarrow$ Compare LES's:

$H_*(X\setminus E, A\setminus E)$
$\cong$ ← by above isos

$$\begin{array}{ccccccccc} H_*(A) & \longrightarrow & H_*(C_*^{\mathcal{U}}X) & \longrightarrow & H_*(C_*^{\mathcal{U}}X/C_*A) & \longrightarrow & H_{*-1}(A) & \longrightarrow & H_{*-1}(C^{\mathcal{U}}X) \\ \| & & \downarrow \text{locality} \cong & & \downarrow \text{iso by 5-lemma} & & \| & & \downarrow \text{locality} \cong \\ H_*(A) & \longrightarrow & H_*(X) & \longrightarrow & H_*(C_*X/C_*A) & \longrightarrow & H_{*-1}(A) & \longrightarrow & H_{*-1}(X) \end{array}$$

$H_*(C_*X/C_*A) = H_*(X,A)$

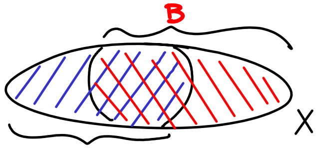
(we are using naturality of LES's induced by SES's)

□

# 6. MAYER-VIETORIS SEQUENCE ← Key computational tool

$X = A \cup B$ s.t. $X = A^\circ \cup B^\circ$

(A, B: any subspaces)




MV Theorem ∃ LES :

$$\cdots \longrightarrow H_*(A\cap B) \xrightarrow{i_*} H_*(A)\oplus H_*(B) \longrightarrow H_*(X) \longrightarrow H_{*-1}(A\cap B) \xrightarrow{i_*[-1]} \cdots$$

& same holds for $\widetilde{H}_*$ provided $A\cap B \neq \emptyset$.

Pf SES $0 \longrightarrow C_*(A\cap B) \longrightarrow C_*(A)\oplus C_*(B) \longrightarrow C_*^{\mathcal{U}}(X) \longrightarrow 0$ ($\mathcal{U}=\{A,B\}$)

$\sigma \longmapsto (\sigma, -\sigma)$

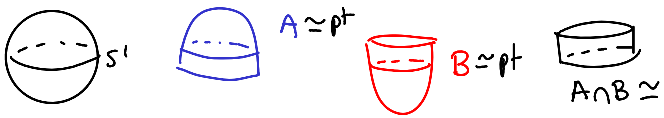
$(\alpha, \beta) \longmapsto \alpha+\beta$

⇒ induces the LES (using locality $H_*^{\mathcal{U}}X \cong H_*X$). □

Exercise connecting map is $\delta: H_*(X) \to H_{*-1}(A\cap B)$

$[\alpha+\beta] \longmapsto [\partial\alpha] = -[\partial\beta]$

Example $S^2$

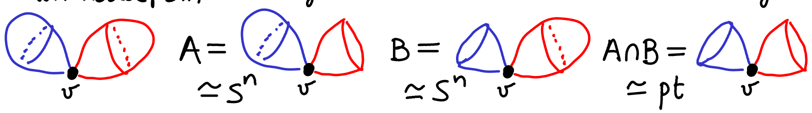



$$\cdots \to H_2(pt)\oplus H_2(pt) \to H_2S^2 \to H_1(S^1) \to H_1(pt)\oplus H_1(pt) \to \cdots$$

($H_2(pt)\oplus H_2(pt) = 0$, hence $H_2S^2 \cong \mathbb{Z}$, $H_1(S^1)\cong\mathbb{Z}$, $H_1(pt)\oplus H_1(pt) = 0$)

Exercise Compute $H_*S^n = \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z} & *=n \\ 0 & \text{else} \end{cases}$ using MV

Example **wedge sum** of $X, Y$ with basepoints $x \in X$, $y \in Y$: $X \vee Y = X \sqcup Y / x\sim y$

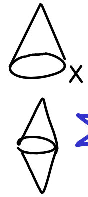



$$0 \to \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\cong} H_n(X) \to 0 \to \cdots \to 0 \to \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z} \to H_0(X) \to 0$$

($1 \mapsto (1,-1)$, $H_0(X) \cong \mathbb{Z}$)

Similarly $\boxed{H_*(X\vee Y) \cong H_*(X)\oplus H_*(Y) \text{ for } *\neq 0}$ if ∃ contractible nbhds of $x\in X$, of $y \in Y$.

## Cones and suspensions



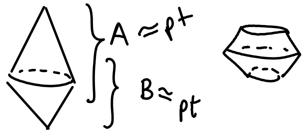
$CX = (X\times[0,1]) / (x,s)\sim(y,t)$ iff equal or $s=t=1$; $CX \simeq pt$

$\Sigma X = (X\times[0,1]) / (x,s)\sim(y,t)$ iff equal or $s=t=0$ or $s=t=1$

Example $CS^n \cong D^{n+1}$, $\Sigma S^n \cong S^{n+1}$.

Lemma $\boxed{\widetilde{H}_*(\Sigma X) \cong \widetilde{H}_{*-1}(X)}$

Pf




now apply MV. □

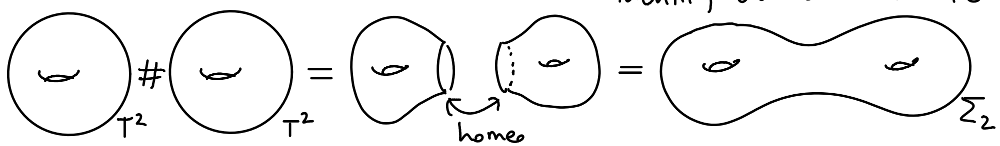
Rmk $\emptyset \neq A \subseteq X \Rightarrow \widetilde{H}_*(X \cup_A CA) \overset{LES}{\cong} H_*(X\cup_A CA, CA) \overset{exc.}{\cong} H_*(X,A)$

(identify $a \in A \subseteq X$ with $(a,0)\in CA$)

## Connected sum

$M, N$ connected n-manifolds ⇒ $M \# N = (M \setminus \text{open n-ball}) \cup (N\setminus \text{open n-ball})$

identify ∂balls via a homeo




Fact compact connected orientable surfaces are homeo to $S^2$ or $T^2\#\cdots\#T^2$ ($g = \#$ copies, called $\Sigma_g$; genus, $g=0$ for $S^2$)

and " " non-orientable ones: $\mathbb{RP}^2\#\cdots\#\mathbb{RP}^2$.

Exercise (Homework) For $M,N$ compact connected n-mfds:

By MV, $\boxed{H_*(M\#N) \cong H_*(M)\oplus H_*(N) \text{ for } 1\le *\le n-2}$

If $M$ or $N$ orientable: $*=n-1$ also works

If both non-orientable: $*=n-1$ one of $\mathbb{Z}/2$ summands becomes $\mathbb{Z}$

$H_0(M\#N)\cong\mathbb{Z}$ since connected

fact: $H_n(M\#N)$ is $\mathbb{Z}$ if $M,N$ both orientable, or $0$ else (see later in course)

Cor 1) $\chi(M\#N) = \chi(M) + \chi(N) - \begin{cases} 2 & \text{for } n \text{ even} \\ 0 & \text{" " odd} \end{cases}$ (the bracketed term $= \chi(S^n)$)

2) $H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}^{2g} & *=1 \\ \mathbb{Z} & *=2 \end{cases}$ (← genus g)

# 7. DEGREE OF MAPS OF SPHERES

$f: S^n \longrightarrow S^n \quad \Rightarrow \quad f_n : H_n S^n \longrightarrow H_n S^n$

$\mathbb{Z} \longrightarrow \mathbb{Z}$ (with $H_n S^n \cong \mathbb{Z}$)

$1 \longmapsto$ **deg(f)** $\in \mathbb{Z}$

$\Rightarrow f_* : \tilde{H}_* S^n \to \tilde{H}_* S^n$ is $\deg(f)\cdot id$

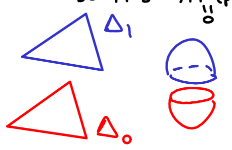
Properties
1) $\deg(id) = 1$
2) $\deg(f \circ g) = \deg f \cdot \deg g$
3) $f \simeq g \Rightarrow \deg f = \deg g$
4) $f \simeq const \Rightarrow \deg f = 0$
5) $f$ homeomorphism $\Rightarrow \deg f = \pm 1$ ← (sign depends on whether $f$ is orientation-preserving or reversing)

Pf
$id_* = id$, $(f\circ g)_* = f_* \circ g_*$, $f \simeq g \Rightarrow f_* = g_*$, $const_* = 0$, $f$ homeo $\Rightarrow f_n$ iso. □
(since $S^n \to pt \to S^n$ factors so $H^n S^n \to H^n(pt) \to H^n S^n$, with $H^n(pt) = 0$)

Examples

1) $S^n = \Delta^n \times 1 \cup \Delta^n \times 0$ ← call this $\Delta_0$ (call this $\Delta_1$ for $\Delta^n \times 1$)

$(b,1) \sim (b,0)$ if $b \in \partial\Delta$




recall $H_n S^n = \mathbb{Z}\cdot(\Delta_1 - \Delta_0)$

reflection: $r: S^n \longrightarrow S^n$, $r(x,t) = (x, 1-t)$

so $\Delta_0 \leftrightarrow \Delta_1$ swapped by $r$, so $r_*(\Delta_1 - \Delta_0) = -(\Delta_1 - \Delta_0)$

$\Rightarrow \deg(r) = -1$

2) antipodal map $-id: S^n \to S^n$ viewing $S^n \subseteq \mathbb{R}^{n+1}$

$\Rightarrow$ $\boxed{\deg(-id) = (-1)^{n+1}}$

Pf $-id = \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots \end{pmatrix} \circ \begin{pmatrix} 1 & & \\ & -1 & \\ & & \ddots \end{pmatrix} \circ \cdots \circ \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$ composition of $n+1$ reflections each homotopic to $r$. □

3) $A \in O(n) \Rightarrow A: S^{n-1} \to S^{n-1} \Rightarrow \deg A = \det A \in \{\pm 1\}$

Pf fact $SO(n)$ is path-connected so $A \in SO(n)$ is $\simeq id$ so $\deg A = \det A = +1$

The other path-component of $O(n)$ is $r \circ SO(n)$ where $r$ is any reflection. □

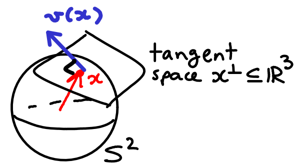
4) $\boxed{f \text{ not surjective} \Rightarrow \deg f = 0}$

Pf If $y \notin \operatorname{Im} f \Rightarrow H_n(S^n) \xrightarrow{f_*} H_n(S^n)$, factoring through $f_*$: $H_n(S^n \setminus y) \cong H_n(\mathbb{R}^n) = 0$ □

## Application to vector fields on $S^n$

$v: S^n \longrightarrow \mathbb{R}^{n+1}$ tangent vector field on $S^n$

so $v(x) \perp x$




Cor **Hairy ball theorem** $\boxed{\exists \text{ nowhere zero v.f. on } S^n \iff n \text{ odd}}$

(case $n=2$: "you cannot comb a ball of hair without creating a tuft")

Pf Suppose $v(x) \neq 0 \quad \forall x$

$\Rightarrow$ hpy $F: S^n \times [0,1] \longrightarrow S^n$

$F(x,t) = \cos(\pi t)\, x + \sin(\pi t) \dfrac{v(x)}{\|v(x)\|}$

$\Rightarrow F_0 = id, \quad F_1 = -id$

$\Rightarrow 1 = \deg F_0 = \deg F_1 = (-1)^{n+1}$

$\Rightarrow n$ odd

For $n$ odd $\exists$ $v(x) = (-x_2, x_1, \ldots, -x_{2k}, x_{2k-1}) \in \mathbb{R}^{2k}$ ($n = 2k-1$). □

Cultural Remark Adams in 1962 proved using alg. topology:

(max # pointwise linearly independent vector fields on $S^n$) $= 2^b + 8a - 1$

where $n+1 = 2^{4a+b}\cdot(\text{odd number})$, $0 \le b \le 3$, $a, b \in \mathbb{N}$, $n \ge 1$. ← get 0 if $n$ even $\Rightarrow$ Cor ✓

## Local degree

$f: S^n \longrightarrow S^n$, $x \longmapsto y = f(x)$

(★) Suppose points $\neq x$ near $x$ do not map to $y$:

$\exists$ nbhds $x \in U \cong D^n$, $y \in V \cong D^n$ s.t. $\boxed{(U, U\setminus x) \xrightarrow{f} (V, V \setminus y)}$ ← call this $f|_x$ **local map at x**

$\Rightarrow (f|_x)_* : H_n(U, U\setminus x) \xrightarrow{f_*} H_n(V, V\setminus y)$

$H_n(U, U\setminus x) \cong H_n(S^n, S^n \setminus x)$ (excise $S^n \setminus U$) $\cong \tilde{H}_n S^n$ (since $S^n \setminus x \cong \mathbb{R}^n \simeq$ point) $\cong \mathbb{Z}$, and $H_n(V, V\setminus y) \cong \mathbb{Z}$

$\mathbb{Z} \longrightarrow \mathbb{Z}$

$1 \longmapsto$ **$\deg_x f$**

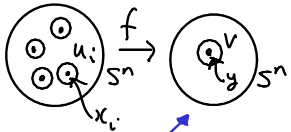
will use this again later: $\tilde{H}_n(S^n) \cong H_n(S^n, S^n \setminus pt)$ ← quotient

**Lemma** $f: S^n \to S^n$, $f^{-1}(y) = \{x_1, \dots, x_k\}$

$$\Rightarrow \boxed{\deg f = \sum \deg_{x_i} f}$$

**Pf**

$$\begin{array}{ccc} \tilde{H}_n S^n & \xrightarrow{f_*} & \tilde{H}_n(S^n) \\ \text{quotient} \downarrow & & \cong \downarrow \text{quotient} \\ H_n(S^n, S^n \setminus \{x_1,\dots,x_k\}) & \xrightarrow{f_*} & H_n(S^n, S^n \setminus y) \\ \text{exc. } S^n \setminus \cup U_i \downarrow \cong & & \cong \downarrow \text{exc. } S^n \setminus V \\ \oplus H_n(U_i, U_i \setminus x_i) & \xrightarrow{\oplus (f|_{x_i})_*} & H_n(V, V \setminus y) \end{array}$$




Rmk: can use same V for all i by taking $\tilde{V} = \cap V_i$, $\tilde{U}_i = f^{-1}(V) \cap U_i$

is: $1 \in \mathbb{Z} \xrightarrow{\deg f} \mathbb{Z}$, with vertical maps down to $(1, \dots, 1) \in \oplus \mathbb{Z} \xrightarrow{\oplus \deg_{x_i} f} \mathbb{Z}$ □

(map to each summand is exc. of $S^n \setminus U_i$ so iso.)

(the 2 squares commute: 1st: quotient is natural, 2nd: excision is natural)

**Example** $p: \mathbb{C} \to \mathbb{C}$ polynomial $a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$, $a_i \in \mathbb{C}$ ($a_n \neq 0$)

$\Rightarrow f: S^2 = \mathbb{C}P^1 \to \mathbb{C}P^1 = S^2$, $z \mapsto p(z)$, $\infty \mapsto \infty$ (where view $\mathbb{C}P^1 = \mathbb{C} \cup \infty \cong S^2$ stereographic projection)

$\Rightarrow$ hpy $F(z,t) = a_n z^n + t(a_{n-1} z^{n-1} + \dots + a_0)$

$F_0 = a_n z^n$ and $F_1 = f$

(hpy is continuous at $\infty$ since $a_n z^n$ dominates other terms: $F^{-1}(\mathbb{C}P^1 \setminus K) = \mathbb{C}P^1 \setminus (\text{some compact set})$ $\forall$ compact K.) (this would fail if you tried to homotope $t(a_n z^n) + a_{n-1} z^{n-1} + \dots$)

$\Rightarrow \deg f = \deg(a_n z^n) \overset{\text{Lemma}}{=} \sum_{k=0}^{n-1} \deg_{w_k} a_n z^n = n$ $\leftarrow w_k = e^{\frac{2\pi i k}{n}}$

(each $\deg_{w_k} a_n z^n = 1$: orient$^n$ preserving homeo near $w_k$; holomorphic maps are always orientation preserving)

$= n$

$=$ degree of the poly p.

**Cor** (Fundamental Thm of Algebra) $n \geq 1 \Rightarrow p$ has a root

**Pf** $p^{-1}(0) = \emptyset \Rightarrow f^{-1}(0) = \emptyset \Rightarrow \deg f = 0$ ↯ □

**Cultural Rmk** For smooth $f: S^n \to S^n$

$\deg f = $ (the number of preimages (counted with orientation signs) of a generic point)

(i.e. almost any point works)

**Example** $S^2 \to S^2$


$S^2 \setminus \text{North pole} \cong \mathbb{C}$

map $\mathbb{C} \to \mathbb{C}$, $z \mapsto z^d$ and North $\mapsto$ North

$\Rightarrow \deg = d = \#$ preimages of a point except if pick North/South pole ($z = \infty$) ($z = 0$)

# 8. CELLULAR HOMOLOGY

**Def** **CW complex** $X$ is sequence $\emptyset = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq X^2 \subseteq \dots$

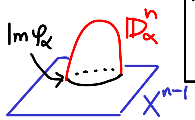
s.t. $X^0$ is any set

**n-skeleton**

$$\boxed{X^n = X^{n-1} \sqcup \bigsqcup_{\alpha \in I_n} \mathbb{D}^n_\alpha \Big/ x \sim \varphi_\alpha(x)}$$

(n-discs labelled by some index set $I_n$)

$\varphi_\alpha : \partial \mathbb{D}^n \to X^{n-1}$ **attaching map** (any continuous map, often not injective)



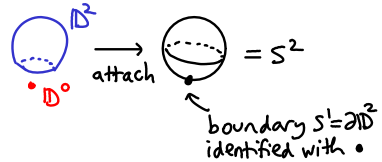

$\Rightarrow X = \bigcup_{n \geq 0} X^n$ top. space with **weak topology**:

$U \subseteq X$ open $\Leftrightarrow U \cap X^n \subseteq X^n$ open $\forall n$.

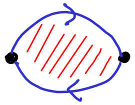
($\Leftrightarrow U \cap \mathbb{D}^n_\alpha \subseteq \mathbb{D}^n_\alpha$ open $\forall n, \alpha$)

Call $X$ **n-dimensional** if $X = X^n$ and this is the least such n.

**Example** $S^n = (\mathbb{D}^0 \sqcup \mathbb{D}^n) / (\mathbb{D}^0 \sim \partial \mathbb{D}^n)$




**Example** $X = \mathbb{R}P^2 =$



$X^0 = \bullet = \mathbb{D}^0$

$X^1 = $ (loop) $= S^1 = (\mathbb{D}^0 \sqcup \mathbb{D}^1) / (\bullet \sim \varphi_1(x))$, $\partial \mathbb{D}^1 = S^0 = \{0,1\} \xrightarrow{\varphi_1} X^0 = \bullet$

$X^2 = ($(loop) $\sqcup$ (disc)$) / ($wrap $\partial$ of (disc) twice around (loop)$)$

$= (X^1 \sqcup \mathbb{D}^2) / (\partial \mathbb{D}^2 = S^1 \ni z \sim z^2 \in X^1 = S^1)$   $\partial \mathbb{D}^2 = S^1 \xrightarrow{\varphi_2} S^1$, $\varphi_2(z) = z^2$

**Fact** If we homotope $\varphi_\alpha$, we get a homotopy equivalent space

**Example** If use another degree 2 map $\varphi_2$ above, get $X \simeq \mathbb{R}P^2$.

**Cultural Rmk** Every CW-cx X is hpy equivalent to a simplicial complex Y (so in particular a $\Delta$-cx). [If X finite/n-dim then can ensure Y is finite/n-dim]

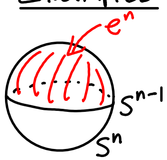
X is partitioned as a set by interiors of **n-cells** $\boxed{e^n_\alpha = \text{Image}(\mathbb{D}^n_\alpha \to X)}$

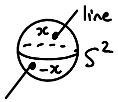
$$X^n = X^{n-1} \sqcup \bigsqcup_{\alpha \in I_\alpha} \mathring{e}^n_\alpha$$

$$= \Big(\bigsqcup_{\alpha \in I_0} e^0_\alpha\Big) \sqcup \Big(\bigsqcup_{\alpha \in I_1} \mathring{e}^1_\alpha\Big) \sqcup \Big(\bigsqcup_{\alpha \in I_2} \mathring{e}^2_\alpha\Big) \sqcup \dots$$

← Rmk: interior $\mathbb{D}^0 = \mathbb{D}^0$ so $\mathring{e}^0_\alpha = e^0_\alpha$

**Examples** **real projective space** $\mathbb{R}P^n = S^n / (\mathbb{Z}/_2\text{-action by } \pm id)$ so $x \sim -x$




$X^k = \mathbb{R}P^k$ inductively

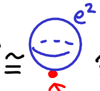
$X^n = X^{n-1} \cup e^n$ with $\varphi : S^{n-1} \to X^{n-1} = \mathbb{R}P^{n-1} = S^{n-1}/\pm id$

$x \longmapsto [x] = [-x]$

**complex projective space**

$\mathbb{C}P^n = (S^{2n+1} \subseteq \mathbb{C}^{n+1}) / (S^1\text{-action by } \lambda \cdot Id)$ ← $x \sim \lambda x$ for $\lambda \in S^1 \subseteq \mathbb{C}^*$

$X^0 = X^1 = pt = \mathbb{C}P^0$

$X^2 = X^3 = X^0 \cup e^2 = \mathbb{C}P^1$, $\varphi : S^1 \to pt$ ← $\mathbb{C}P^1 \cong$  $\cong S^2$ ($e^2$, $\mathbb{C}P^0$)

$X^4 = X^5 = X^2 \cup e^4 = \mathbb{C}P^2$, $\varphi : S^3 \to \mathbb{C}P^1 = S^3/(S^1\text{-action})$

$x \longmapsto [x] = [\lambda x], \forall \lambda \in S^1$

$X^{2n} = X^{2n-2} \cup e^{2n} = \mathbb{C}P^n$, $\varphi : S^{2n-1} \to X^{2n-2} = \mathbb{C}P^{n-1}$

$x \longmapsto [x]$

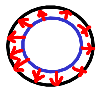
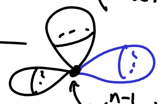
In coordinates: $\mathbb{C}P^n = \{[z_0 : \dots : z_n] : \text{not all } z_j \in \mathbb{C} \text{ are } 0\}$ and $[z] \sim [\lambda z]$, $\forall \lambda \in \mathbb{C}^*$

Can rescale so that $\sum |z_j|^2 = 1$ so $z \in S^{2n-1}$ and left with rescaling by $\lambda \in S^1 \subseteq \mathbb{C}^*$.

$\mathbb{C}P^{n-1} \cong X^{2n-2} = \{[z_0 : \dots : z_{n-1} : 0]\} \subseteq \mathbb{C}P^n = X^{2n}$ and

$e^{2n} : \mathbb{D}^{2n} = \{(w_0, \dots, w_{n-1}) : \sum |w_j|^2 \leq 1\} \longrightarrow X^{2n}$ via $[w_0 : \dots : w_{n-1} : \sqrt{1 - \sum |w_j|^2}]$ ← notice this $= 0$ if $w \in S^{2n-1} \cong \partial \mathbb{D}^{2n}$

**Observe**: For $X$ CW complex, for $n \geq 1$: ← (For $n=0$: $(X^0, X^{-1}) = (X^0, \emptyset)$, $X^0/X^{-1} = X^0$)

- $(X^n, X^{n-1})$ is a good pair ← (since $\exists$ nbhd of $\partial \mathbb{D}^n$ that deformation retracts to $\partial \mathbb{D}^n$) 
- $X^n / X^{n-1} \cong \bigvee_{\alpha \in I_n} S^n$ ←  $S^n_\alpha = \mathbb{D}^n_\alpha / \partial \mathbb{D}^n_\alpha$; $X^{n-1}$ identified to a point

**Def** **Cellular complex** for $X$ a CW cx,

$$C^{CW}_n(X) = H_n(X^n, X^{n-1}) \cong H_n(X^n/X^{n-1})$$
$$= \text{free abelian gp gen. by the } n\text{-cells } e^n_\alpha$$

since $\Delta^n \cong \mathbb{D}^n \longrightarrow (e^n \subseteq X^n) \longrightarrow \mathbb{D}^n_\alpha / \partial \mathbb{D}^n_\alpha = S^n_\alpha$ generate ← (as usual we use the standard orientations of $\Delta^n, \mathbb{D}^n, S^n$.)

Will build **cellular differential** $d : C^{CW}_* \to C^{CW}_{*-1}$, prove $d \circ d = 0$

$\Rightarrow$ get
$$H^{CW}_*(X) = H_*(C^{CW}_*(X), d)$$

**Example** $C^{CW}_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} \cdot e^k & \text{for } k = 0, 2, 4, \dots, 2n \\ 0 & \text{else} \end{cases}$ hence $d = 0$ so $H^{CW}_*(\mathbb{C}P^n) = C^{CW}_*(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & 0 \leq * \leq 2n \text{ even} \\ 0 & \text{else} \end{cases}$

**Cellular differential:**

$$d\, e^n_\alpha = \sum_{\beta \in I_{n-1}} d^n_{\alpha\beta} \cdot e^{n-1}_\beta$$

now describe the coefficients $d^n_{\alpha\beta} \in \mathbb{Z}$ and why that is a finite sum.

$$d : C^{CW}_n \longrightarrow C^{CW}_{n-1}$$
$$H_n(X^n, X^{n-1}) \xrightarrow[\text{connecting map of LES}]{\delta^n_*} H_{n-1}(X^{n-1}) \xrightarrow[\text{quotient map}]{q^{n-1}_*} H_{n-1}(X^{n-1}, X^{n-2})$$
$$e^n_\alpha \longmapsto (\partial \mathbb{D}^n_\alpha \to \varphi_\alpha(S^{n-1})) \longmapsto \sum d_{\alpha\beta}\, e^{n-1}_\beta$$


Recall LES

$$H_*(X^{n-1}) \longrightarrow H_*(X^n) \xrightarrow{q^n_*} H_*(X^n, X^{n-1}) \xrightarrow[{[-1]}]{\delta^n_*} H_*(X^{n-1})$$

here it is important that we chose identifications $\Delta^n \cong \mathbb{D}^n$, $S^n \cong \mathbb{D}^n/\partial \mathbb{D}^n$ compatibly with orientations.

Therefore:

$$d^n_{\alpha\beta} = \deg\big(S^{n-1} \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q} X^{n-1}/X^{n-2} \cong \bigvee_{I_{n-1}} S^{n-1} \longrightarrow S^{n-1}\big)$$

where $S^{n-1} = \partial \mathbb{D}^n_\alpha$, the last map is the quotient by $\bigvee_{I_{n-1} \setminus \beta} S^{n-1}$, and $S^{n-1} = \mathbb{D}^{n-1}_\beta / \partial \mathbb{D}^{n-1}_\beta$.

**Rmk** Only finitely many $d^n_{\alpha\beta} \neq 0$ (for fixed $\alpha$) because $\varphi_\alpha, q$ are continuous and $S^{n-1}$ compact, so get a compact image in $\bigvee_\beta S^{n-1}$, therefore cannot surject onto $\infty$ many $S^{n-1}_\beta$. ← recall if don't surject then deg=0

**Lemma** $d \circ d = 0$

**Pf** $d_n = q^{n-1}_{n-1} \circ \delta^n_n$

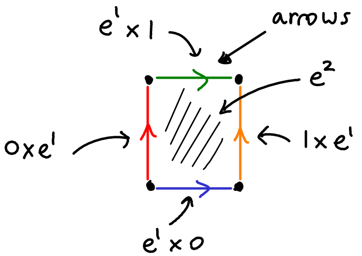
$d_{n-1} \circ d_n = q^{n-2}_{n-2} \circ \underbrace{\delta^{n-1}_{n-1} \circ q^{n-1}_{n-1}}_{=0 \text{ by LES}} \circ \delta^n_n$ $\square$

**Cor** $\text{rank } H^{CW}_n(X) \leq \#n\text{-cells}$

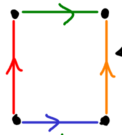
**Pf** $\#n\text{-cells} = \text{rank } C^{CW}_n(X) \geq \text{rank Ker } d^{CW}_n \geq \text{rank } H^{CW}_n(X)$ $\square$

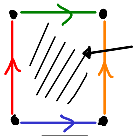
**Example** $I \times I$ $\quad I=[0,1]$ $\quad \mathbb{D}^1=[-1,1]$

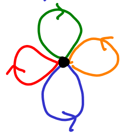
arrows here tell us how we map $[-1,1]\to$ edge (so orientation)



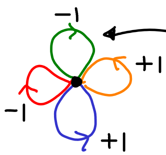

$X^0 =$ :: $= 4$ 0-cells

$X^1 =$  ← 4 1-cells $e'\times 0$, $1\times e'$, $e'\times 1$, $0\times e'$

$X^2 =$ 


$X^1/X^0 =$  orientations of cells tell us how to orient the circles

$e^2 : \mathbb{D}^2 \cong$ ▨ $\to X^1$

$\partial e^2 : S^1 \cong$ □ $\to X^1/X^0 =$ 


degree $-1$ because top edge of □ maps to ◯ by an orientation-reversing homeomorphism.

$\Rightarrow \partial e^2 = +e'\times 0 + 1\times e' - e'\times 1 - 0\times e'$

$(= (\partial e')\times e' - e'\times(\partial e')$ ← we come back to this later$)$

**Example** $\mathbb{RP}^n$ recall: 1 cell in each dim, $\varphi: S^k \to X^k = \mathbb{RP}^k$, $x \mapsto [\pm x]$

$S^{k-1} =$ $\Delta_1$ $\cup$ $\Delta_2$

$\Delta_1 \xrightarrow{\varphi} X^{k-1}/X^{k-2} = \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \cong S^{k-1}$ (via $\Delta_1/\partial\Delta_1$) $\quad \deg = +1$

$\Delta_2 \longrightarrow -\mathrm{id}(\Delta_1) \quad \deg = (-1)^k$

$$\Rightarrow d^k_{\alpha\beta} = \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd}\end{cases}$$

$C^{CW}_*(\mathbb{RP}^n)$ $\quad 0 \to \mathbb{Z} \to \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$

($k=n$ for the first $\mathbb{Z}$; $k=1$, $0$, $-1$ at the end; first map: 2 if $n$ even, 0 if odd; following maps: 0, 2 resp.)

$$H^{CW}_*(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & *=0, \text{ also } *=n \text{ if odd} \\ \mathbb{Z}/2 & *=1,3,5,\dots < n \\ 0 & \text{else}\end{cases}$$

**Example** $S^n$: $C^{CW}_*(S^n)$: $n\geq 2$: $0 \to \mathbb{Z}\cdot\mathbb{D}^n \xrightarrow{0} 0 \to \cdots \to 0 \to \mathbb{Z}\cdot\mathbb{D}^0 \to 0$

$n=1$: $0 \to \mathbb{Z}\cdot\mathbb{D}^1 \xrightarrow{0} \mathbb{Z}\mathbb{D}^0 \to 0$

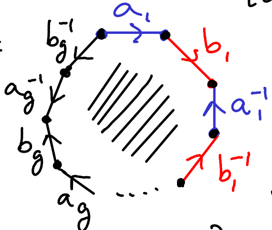
($\mathbb{D}^n$ attached to $\mathbb{D}^0$ via $\varphi$, $\deg\varphi = 0$)

$$\Rightarrow H^{CW}_*(S^n) \cong \begin{cases}\mathbb{Z} & *=0,n \\ 0 & \text{else}\end{cases}$$

$H_1(S^1,pt) \xrightarrow{\delta} H_0(pt) \xrightarrow{q} H_0(pt,\emptyset)$

$(\Delta^1 \cong [0,1] \to S^1) \longmapsto \partial\sigma = pt - pt = 0$, $\sigma$ = quotient map

if you work with degrees, need to remember orientations: $\partial\mathbb{D}^1 \cong \partial[0,1] = [1]-[0] \to$ point so degree $= +1-1=0$

**Example** $\Sigma_g =$  genus $g$ surface

boundary identifications $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$

← Notice all vertices are identified, call vertex $v$

$\partial a_i = v - v = 0$, $\partial b_i = v - v = 0$

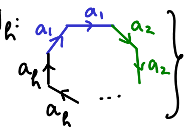
$0 \longrightarrow \mathbb{Z} \xrightarrow{\partial} \mathbb{Z}^{2g} \xrightarrow{\partial=0} \mathbb{Z} \longrightarrow 0$

$\mathbb{Z} \cong \mathbb{Z}\cdot\mathbb{D}$, $\mathbb{Z}^{2g} \cong \mathbb{Z}\langle a_1,b_1,\dots,a_g,b_g\rangle$, $\mathbb{Z} \cong \mathbb{Z}\cdot v$

$\mathbb{D} \mapsto -a_1 - b_1 + a_1 + b_1 + \dots - a_g - b_g + a_g + b_g = 0$

(signs: compare edge orientation with anticlockwise orientation of $\partial\mathbb{D}$)

$$\Rightarrow H_*(\Sigma_g) \cong \begin{cases}\mathbb{Z} & *=0,2 \\ \mathbb{Z}^{2g} & *=1 \\ 0 & \text{else}\end{cases}$$

**Example** Non-orientable surface $N_h$: 

$0 \to \mathbb{Z} \to \mathbb{Z}^h \xrightarrow{0} \mathbb{Z} \to 0$

$1 \mapsto (-2,\dots,-2)$

(Since $\mathbb{D} \mapsto -a_1 - a_1 - a_2 - a_2 - \dots - a_h - a_h$)

$$\Rightarrow H_*(N_h) \cong \begin{cases}\mathbb{Z} & *=0 \\ \mathbb{Z}^{h-1}\oplus\mathbb{Z}/2 & *=1 \\ 0 & \text{else}\end{cases}$$

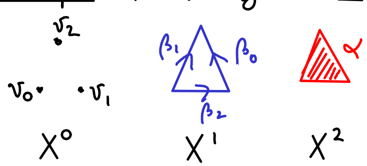
(use the standard basis for $\mathbb{Z}^h$ except replace $(0,\dots,0,1)$ by $(1,1,\dots,1)$.)

**Lemma** $X$ $\Delta$-cx structure $\Longrightarrow$ induces CW-cx structure on $X$ and $(C^{CW}_*(X), d^{CW}) \cong (C^\Delta_*(X), d^\Delta)$

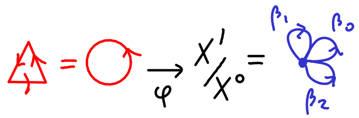
$$\Longrightarrow \boxed{H^{CW}_*(X) \cong H^\Delta_*(X)}$$

**Pf** $X^n = \bigcup n$-simplices of $X$ ($\cong \mathbb{D}^n$) and degrees are $\pm 1$ depending on orient$^n$ so can identify $d^{CW}$ and $d^\Delta$. □

**Example** $X = $ triangle $= \Delta^2$




$\Rightarrow d^\Delta\alpha = \beta_0 - \beta_1 + \beta_2$




$d_{\alpha\beta_2} = d_{\alpha\beta_0} = +1$, $\quad d_{\alpha\beta_1} = -1$

$\Rightarrow d^{CW}\alpha = d^\Delta\alpha$ ✓ □

**Theorem** $X$ CW cx (or $\Delta$-cx) $\Longrightarrow$ $\boxed{H_*^{CW}(X) \cong H_*(X)}$

$\Rightarrow H_*^{\Delta}, H_*^{CW}$ independent of choice of CW-cx/$\Delta$-cx structure.

**Pf** ① $H_*(X^n, X^{n-1}) \cong \tilde{H}_*(X^n/X^{n-1}) \cong \tilde{H}_*(\bigvee_\alpha S^n) \cong \bigoplus_\alpha \tilde{H}_* S^n$ (lives in degree $n$)

$= 0 \iff * \neq n$

LES for $(X^n, X^{n-1}) \Rightarrow H_*(X^{n-1}) \to H_*(X^n)$ iso for $* < n-1$, $* > n$

② for $* < n$: $H_*(X^n) \cong H_*(X^{n+1}) \cong H_*(X^{n+2}) \cong \dots \cong H_*(X)$ (the first $\cong$ by ①; the last by compactness each sing. chain lands in $X^N$, some $N$)

③ for $* > n$: $H_*(X^n) \cong H_*(X^{n-1}) \cong H_*(X^{n-2}) \cong \dots \cong H_*(X^{-1}) = 0$

④ LES: $\dots \to H_n(X^{n-1}) \to H_n(X^n) \xrightarrow{q_n^n} H_n(X^n, X^{n-1}) \to \dots$, where $H_n(X^{n-1}) = 0$ by ③

$\Rightarrow q_n^n$ injective $\forall n$

⑤ LES: $\dots \xrightarrow{q_{n+1}^{n+1}} H_{n+1}(X^{n+1}, X^n) \xrightarrow{\delta_{n+1}^{n+1}} H_n(X^n) \to H_n(X^{n+1}) \to 0$ (the 0 by ①)

UPSHOT $H_n(X) \overset{②}{\cong} H_n(X^{n+1}) \overset{⑤}{\cong} H_n(X^n) / \operatorname{im} \delta_{n+1}^{n+1} \overset{④}{\cong} (q_n^n H_n(X^n)) / \operatorname{im} q_n^n \circ \delta_{n+1}^{n+1} \cong H_n^{CW}(X)$ (the ④ step is the 1st iso thm)

where $q_n^n H_n(X^n) = \operatorname{im} q_n^n = \operatorname{Ker} \delta_n^n$ (by exactness of LES) $\overset{④}{=} \operatorname{Ker} q_{n-1}^{n-1} \circ \delta_n^n = \operatorname{Ker} d_n^{CW}$, and $q_n^n \circ \delta_{n+1}^{n+1} = d_{n+1}^{CW}$ $\square$

**Rmk** by ① $\boxed{H_k \text{ not affected if attach } (k+2)\text{-cells or higher}}$

by ② $\boxed{\text{Inclusion } X^n \to X \text{ induces iso } H_*(X^n) \to H_*(X) \text{ for } * < n}$

**Cor** $X$ $n$-dimensional cell cx $\Longrightarrow$ $H_*(X) = 0$ for $* > n$

## Axioms for homology: Eilenberg-Steenrod axioms

It was no accident that $H_*^{\Delta}, H_*^{CW}, H_*$ all agreed.

**Def** A **generalised homology theory (GHT)** is a **functor** $F$: TopPairs = (Category of pairs of spaces, and maps of pairs) $\longrightarrow$ Graded Abelian Gps

with a **natural transformation** $\delta : F_*(X,A) \longrightarrow F_{*-1}(X, \emptyset)$ (abbreviated: $F_{*-1}(X)$) satisfying:

1) **homotopy invariance**: $f \simeq g \Rightarrow F(f) = F(g)$

2) **exactness**: $\exists$ LES $\boxed{\dots \to F_*(A) \to F_*(X) \to F_*(X,A) \xrightarrow{\delta} F_{*-1}(A) \to \dots}$ (the maps being $F(\text{incl}: A \to X)$ and $F(\text{incl}: (X,\emptyset) \to (X,A))$)

3) **additivity**: $(X,A) = \bigsqcup (X_i, A_i)$, $\text{incl}_i : (X_i, A_i) \to (X,A)$

then $\boxed{\sum F(\text{incl}) : \bigoplus F(X_i, A_i) \xrightarrow{\cong} F(X,A)}$

4) **excision**: $\boxed{\overline{E} \subseteq A^\circ \subseteq X \Rightarrow F(X \setminus E, A \setminus E) \xrightarrow[F(\text{incl})]{\cong} F(X,A)}$

**Remark** (4) $\iff$ $X = A^\circ \cup B^\circ$, $\text{incl}: (B, A \cap B) \to (X,A)$

then $\boxed{F(\text{incl}) : F(B, A \cap B) \xrightarrow{\cong} F(X,A)}$

**Pf** $B = X \setminus E$, $E = X \setminus B$ noticing that $(X \setminus E)^\circ \cup A^\circ = X$

$E = A \setminus B$ noticing that $\overline{E} \subseteq \overline{A} \setminus B^\circ \subseteq A^\circ \setminus B^\circ \subseteq A^\circ$. $\square$ (since $X = A^\circ \cup B^\circ$ so $\partial B \subseteq A^\circ$)

**Rmk** In (3), the topology on the disjoint union $\bigsqcup (X_i, A_i)$ is defined by: $U \subseteq \bigsqcup (X_i, A_i)$ open $\iff$ $U \cap X_i \subseteq X_i$ open $\forall i$

FACT **Theorem**

a) $(F, \delta_F), (G, \delta_G)$ GHTs, $\alpha : F \to G$ a natural transformation commuting with $\delta_F, \delta_G$ such that $\alpha_{\text{point}} : F(\text{point}) \to G(\text{point})$ is an iso, then $\alpha$ is an iso.

b) If $(F, \delta_F)$ GHT satisfies (5) **dimension**: $\boxed{F_*(\text{point}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}}$

Then $\exists$ natural iso $F \cong H_*$ (such an $F$ is called a **homology theory**)

**Rmk** In (b) if require $F_0(\text{point}) = G$ an abelian group (instead of $\mathbb{Z}$)

$\Longrightarrow F(X,A) \cong H_*(X,A;G) =$ (homology with coefficients in $G$) $\leftarrow$ later in course

# 9. COHOMOLOGY

$(C_*, \partial_*)$ chain cx s.t. $C_*$ **free** $\mathbb{Z}$-module ($C_* \cong \oplus_\alpha \mathbb{Z}$)

**Def** **n-cochains** $\boxed{C^n = \mathrm{Hom}(C_n, \mathbb{Z})}$

**coboundary map** (this is the dual of $\partial$) $\partial^n : C^n \to C^{n+1}$, $\partial^n(\phi) = \phi \circ \partial_{n+1}$

$C_n \xleftarrow{\partial_{n+1}} C_{n+1}$, $\phi \downarrow$ to $\mathbb{Z}$, $\partial^*\phi = \phi \circ \partial$

Notice $\partial^*$ is degree $+1$ map (not $-1$)

$$\boxed{H^m(C_*, \partial_*) = \mathrm{Ker}\,\partial^m / \mathrm{Im}\,\partial^{m-1}}$$

($\mathrm{Ker}\,\partial^m$ ← **cocycles**, $\mathrm{Im}\,\partial^{m-1}$ ← **coboundaries**)

(Note $\partial^* \circ \partial^* = 0$: $\partial^*\partial^*\phi = \phi \circ \partial \circ \partial = 0$, since $\partial \circ \partial = 0$)

**Rmk** If use negative grading, $(C^{-*}, \partial^{-*})$ is a chain complex with homology so many results from $H_*$ carry over to $H^*$. So abusively write

$$\boxed{H^*(C_*, \partial_*) = H_*(\mathrm{Hom}(C_*, \mathbb{Z}), \partial^*)}$$

**Warning** A cochain $\varphi \in C^*$ takes values $\varphi(c) \in \mathbb{Z}$ on chains $c \in C_*$. However the cohomology class $\alpha = [\varphi] \in H^*$ does not have a well-defined value on $c$: $[\varphi] = [\varphi + \partial^*(\psi)]$ and $(\varphi + \partial^*\psi)(c) = \varphi(c) + \psi(\partial_* c)$. If $c$ is a cycle, so $\partial_* c = 0$ then $\alpha(c) = \varphi(c)$ is well-defined, so $\exists$ pairing $H^* \times H_* \to \mathbb{Z}$

**Remarks about dualisation**

$\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) \cong \mathbb{Z}^n$ generated by projection maps $\pi_i(x_1, \dots, x_n) = x_i$

(this is the dual of the standard basis: $\pi_i = e_i^* : e_i \to 1$, $e_k \to 0$, $k \neq i$)

$\alpha : \mathbb{Z}^n \to \mathbb{Z}^m$, $x \mapsto A \cdot x$ ($A$ = $m \times n$ matrix) $\Rightarrow$ $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) \xleftarrow[\alpha^*]{\text{dual}} \mathrm{Hom}(\mathbb{Z}^m, \mathbb{Z})$, $\alpha^*\phi = \phi \circ \alpha$

$\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) \cong \mathbb{Z}^n \xleftarrow{\text{transpose}(A)} \mathbb{Z}^m \cong \mathrm{Hom}(\mathbb{Z}^m, \mathbb{Z})$

**Def** $X$ space $\Rightarrow$ **singular cohomology** $\boxed{H^*(X) = H^*(C^*(X), \partial^*)}$ (dualise $C_* = C_*(X)$)

similarly define $H^*_\Delta$, $H^*_{CW}$

**Example** $\mathbb{RP}^3$: $C^{CW}_*(\mathbb{RP}^3)$: $0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$

dualise: $C^*_{CW}(\mathbb{RP}^3)$: $0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0$

$$H^*(\mathbb{RP}^3) \cong H^*_{CW}(\mathbb{RP}^3) = \begin{cases} \mathbb{Z} & * = 0, 3 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$$

← notice $H_1(\mathbb{RP}^3) \cong \mathbb{Z}/2$ has moved to grading 2.

## Functoriality

$f : X \to Y \Rightarrow f_* : C_* X \to C_* Y$

$\Rightarrow C^*X \xleftarrow{f^*} C^*Y$ is dual: $\boxed{f^*\phi = \phi \circ f_*}$ ← called **pull-back**

**Lemma** $f^*$ is a **cochain map** (meaning $\partial^* \circ f^* = f^* \circ \partial^*$)

$\Rightarrow \boxed{f^* : H^*Y \to H^*X}$

**Pf**

$$\begin{aligned}(\partial^* \circ f^*)(\phi) &= \partial^*(\phi \circ f_*) \\ &= (\phi \circ f_*) \circ \partial \\ &= (\phi \circ \partial) \circ f_* \quad \text{as } f_* \text{ chain map} \\ &= f^* \circ (\phi \circ \partial) \\ &= f^* \circ (\partial^* \phi) \\ &= (f^* \circ \partial^*)(\phi)\end{aligned}$$

**Properties**
- $\mathrm{id}^* = \mathrm{id}$
- $(f \circ g)^* = g^* \circ f^*$ notice order!

$\Rightarrow \boxed{H^* : \mathrm{Top} \to \mathrm{GradedAbGps}}$ contravariant functor

**Exercise** $H^0(X) = \prod_{\pi_0 X} \mathbb{Z}$ where $\pi_0 X = \{\text{path-components of } X\}$

**Homotopy invariance**

**Lemma** $f_*, g_* : C_* \to \tilde{C}_*$ chain h'pic (free) $\Rightarrow f^* = g^* : H^*\tilde{C} \to H^*C$

**Pf** $f_* - g_* = \tilde{\partial} \circ h + h \circ \partial$ some $h : C_* \to \tilde{C}_*[1]$

$f^* - g^* = h^* \circ \tilde{\partial}^* + \partial^* \circ h^*$ for dual $h^* : \tilde{C}^* \to C^*[-1]$. (notice degree $-1$, not $+1$) $\square$

**Def** $h^*$ called **cochain homotopy**

**Cor** $f \simeq g : X \to Y \Rightarrow f^* = g^* : H^*Y \to H^*X$ $\square$

## Algebra : dual of SES

**Lemma** $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ exact, $A, B, C$ free
$\Rightarrow 0 \leftarrow A^* \xleftarrow{i^*} B^* \xleftarrow{j^*} C^* \leftarrow 0$ exact

*Cultural Remark*: $\left(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}\right)^* = \prod_{n\in\mathbb{N}} \mathbb{Z}^*$ is not free. (Baer 1937) so $A^*, B^*, C^*$ are not free unless $A, B, C$ have finite ranks.

Pf $C$ free $\Rightarrow \exists$ splitting $B \underset{s}{\overset{j}{\rightleftarrows}} C$, $j\circ s = id$
(Pick preimages $b_i$ for basis $e_i$ of $C$, then $s(e_i) = b_i$)
$\Rightarrow A \oplus C \underset{i\oplus s}{\overset{\cong}{\longrightarrow}} B$

*Rmk* inverse is $B \cong A \oplus C$, $b \mapsto i^{-1}(b - s(b)) \oplus j(b)$

dual $\Rightarrow A^* \oplus C^* \underset{i^*\oplus s^*}{\overset{\cong}{\longleftarrow}} B^*$ and $s^* \circ j^* = id$
$\to$ so $i^*$ surj $\quad \to$ so $j^*$ inj

$\Rightarrow 0 \leftarrow A^* \xleftarrow{i^*} B^* \underset{s^*}{\overset{j^*}{\leftrightarrows}} C^* \leftarrow 0$

where $0 = (j\circ i)^* = i^* \circ j^*$ so $\operatorname{Im} j^* \subseteq \operatorname{Ker} i^*$

prove $\supseteq$: $i^* b = 0 \Rightarrow b - j^* s^* b \in \operatorname{Ker} i^* \cap \operatorname{Ker} s^* = \{0\}$ (since $s^* j^* = id$; since $i^* \oplus s^*$ is iso.)
$\Rightarrow b = j^* s^* b \in \operatorname{Im} j^*$
$\Rightarrow \operatorname{Ker} i^* = \operatorname{Im} j^*$ □

### Relative cohomology

$H^*(X,A) = H^*(\operatorname{Hom}(C_*(X,A), \mathbb{Z}))$

recall $C_*(X,A) = C_*(X)/C_*(A)$ and homs $C_*(X)/C_*(A) \to \mathbb{Z}$ correspond precisely to homs $C_*(X) \to \mathbb{Z}$ which vanish on $C_*(A)$.
So relative cocycles are cocycles on $X$ which vanish on chains in $A$.

### Excision, LES, Mayer–Vietoris

By previous Lemma get dual results:

Excision $\overline{E} \subseteq A^\circ \subseteq X \Rightarrow H^*(X\setminus E, A\setminus E) \underset{i^*}{\overset{\cong}{\longleftarrow}} H^*(X,A)$

LES for pair $(X,A)$: $\cdots \xleftarrow{q^*[+1]} H^{*+1}(X,A) \xleftarrow{\delta} H^*(A) \xleftarrow{i^*} H^*(X) \xleftarrow{q^*} H^*(X,A) \leftarrow \cdots$

M.V. $X = A^\circ \cup B^\circ \Rightarrow \cdots \leftarrow H^{*+1}(X) \leftarrow H^*(A\cap B) \xleftarrow{i_A^* \oplus -i_B^*} H^*(A) \oplus H^*(B) \xleftarrow{j_A^* \oplus j_B^*} H^*(X) \leftarrow \cdots$

where $A\cap B \overset{i_A}{\underset{i_B}{\rightrightarrows}} A, B \overset{j_A}{\underset{j_B}{\rightrightarrows}} X$ are the obvious maps

### Axioms for cohomology

These are analogous to the axioms for homology except we reverse all arrows, and we change axiom (3): $\prod$ instead of $\oplus$

**additivity**: $(X,A) = \coprod (X_i, A_i)$, $incl_i : (X_i, A_i) \to (X,A)$

then $\boxed{\prod F(incl_i) : \prod F(X_i, A_i) \xleftarrow{\cong} F(X,A)}$

## 10. CUP PRODUCT

**Theorem** $H^*(X)$ (space) is ① **unital** ② **graded-commutative** ring via
$\smile : H^k(X) \times H^\ell(X) \longrightarrow H^{k+\ell}(X)$ determined by

$$\boxed{\begin{aligned} \smile : C^k(X) \times C^\ell(X) &\longrightarrow C^{k+\ell}(X) \\ (\phi \smile \psi)(\sigma) &= \phi(\sigma|_{[e_0,\dots,e_k]}) \cdot \psi(\sigma|_{[e_k,\dots,e_{k+\ell}]}) \end{aligned}}$$

① $1 \in C^0(X)$ constant function $\Rightarrow 1 \smile \phi = \phi \smile 1 = \phi$
② $\phi \smile \psi = (-1)^{\deg\phi \cdot \deg\psi} \psi \smile \phi$

**Useful trick** If $X$ is $\Delta$-cx, $C_*^\Delta(X) \underset{\simeq}{\xrightarrow{\text{inclusion}}} C_*(X)$, so $C^*_\Delta(X) \underset{\simeq}{\xleftarrow{\text{restriction}}} C^*(X)$, $\phi\circ incl \longleftarrow\!\shortmid \phi$
and can define cup product on $C^*_\Delta(X)$ so that:

$H^*_\Delta(X) \times H^*_\Delta(X) \xrightarrow{\smile} H^*_\Delta(X)$ $\leftarrow$ at chain level
$\cong\uparrow \qquad\qquad \uparrow\cong$ $\qquad (\phi\smile\psi)([v_0,\dots,v_n]) = \phi([v_0,\dots,v_k]) \cdot \psi([v_k,\dots,v_n])$, $n = \ell + k$
$H^*(X) \times H^*(X) \xrightarrow{\smile} H^*(X)$

So you can compute cup products on $H^*(X)$ by picking simplicial cocycle representatives: so define values on the simplicial chains defining the $\Delta$-cx structure, and use ↑

(cannot do this for $C^{CW}_*(X)$ because there is no meaningful analogue for $\mathbb{D}^n$ of the "bottom face" $[e_0,\dots,e_k]$ and "top face" $[e_k,\dots,e_n]$)

**Proof of Theorem**

$$\begin{aligned} \partial^*(\phi\smile\psi)(\sigma) &= (\phi\smile\psi)(\partial\sigma) \\ &= (\phi\smile\psi) \sum (-1)^i \sigma|_{[e_0,\dots,\hat{e}_i,\dots,e_n]} \quad (n = k+\ell) \\ &= \sum_{i\le k} (-1)^i \phi(\sigma|_{[e_0,\dots,\hat{e}_i,\dots,e_{k+1}]}) \cdot \psi(\sigma|_{[e_{k+1},\dots,e_n]}) \\ &\quad + \sum_{i\ge k} (-1)^i \phi(\sigma|_{[e_0,\dots,e_k]}) \cdot \psi(\sigma|_{[e_k,\dots,\hat{e}_i,\dots,e_n]}) \cdot \underbrace{(-1)^{i-k}(-1)^{k-i}}_{1} \\ &= ((\partial^*\phi)\smile\psi(\sigma)) + (-1)^k \phi\smile\partial^*\psi \end{aligned}$$

induces $[\phi]\smile[\psi] = [\phi\smile\psi]$:

well-defined:
- cycles $\to$ cycle: $\partial(\phi\smile\psi) = \overset{=0}{(\partial\phi)}\smile\psi \pm \phi\smile\overset{=0}{(\partial\psi)} = 0$ ✓
- $[\phi] = [\phi + \partial\alpha]$ so need $[(\partial\alpha)\smile\psi] = 0$
  $(\partial\alpha)\smile\psi = \partial(\alpha\smile\psi)$ ✓ (using $\partial\psi = 0$)
- Similarly $[\phi]\smile[\partial\beta] = 0$

bilinear, associative, distributive: true at chain level

<u>unital</u>: $(\partial 1)(\sigma) = 1(\sigma|_{[e_1]}) - 1(\sigma|_{[e_0]}) = 1 - 1 = 0$

$(1 \cup \psi)(\sigma) = 1(\sigma|_{[e_0]})\cdot\psi(\sigma|_{[e_0,\dots,e_n]}) = \psi(\sigma)$ ($\phi \cup 1 = \phi$ similar)

<u>graded-comm.</u> sketch proof: ← **non-examinable**

Let $r: C_n(X) \to C_n(X)$, $r(\sigma) = \varepsilon_n \overline{\sigma}$ where: $\varepsilon_n = (-1)^{\frac{n(n+1)}{2}}$

and $\overline{\sigma}|_{[v_0,\dots,v_n]} = \sigma|_{[v_n,\dots,v_0]}$ ← reverse order of vertices: is product of $\underbrace{n+(n-1)+\dots+1}_{n(n+1)/2}$ transpositions

(<u>idea</u>: each transposition is a reflection in hyperplane, so reverses orientation of simplex, so insert $\varepsilon_n$ to compensate)

one checks:
- $r$ chain map
- $\underbrace{r^*\varphi \cup r^*\psi}_{\varepsilon_k\varepsilon_\ell} = \underbrace{r^*(\psi \cup \varphi)}_{\varepsilon_{k+\ell}}$ (differ by $(-1)^{k\ell}$)
- $r \simeq id$ so can drop $r^* = id$ on cohomology

$\left( \begin{array}{l} r - id = P\partial + \partial P \text{ with} \\ P\sigma = \sum (-1)^i \varepsilon_{n-i} (\sigma \circ \pi)|_{[v_0,\dots,v_i,w_n,\dots,w_i]} \end{array} \right)$ ($v_i, w_i$ like for prism operator; projection $\Delta^n \times I \xrightarrow{\pi} \Delta^n$) □

**<u>Naturality of cup product</u>**

<u>Lemma</u> $f: X \to Y \implies f^*: H^*Y \to H^*X$ hom of unital rings

<u>Pf</u> $f^*(\varphi \cup \psi)(\sigma) = (\varphi \cup \psi)(f_*\sigma)$

$= \varphi(f_*\sigma|_{[e_0,\dots,e_k]}) \cdot \psi(f_*\sigma|_{[e_k,\dots,e_n]})$

$= ((\varphi \circ f_*) \cup (\psi \circ f_*))(\sigma)$

$= (f^*\varphi \cup f^*\psi)(\sigma)$

unital: $f^*(1) = 1 \circ f_* = 1$ □

<u>UPSHOT</u>

| $H^*$ : Top $\longrightarrow$ {Graded-commutative unital rings with graded unital ring homs} contravariant functor. |
|---|

<u>Warning</u> An (iso)morphism $H^*(Y) \to H^*(X)$ of groups will also preserve the ring structure <u>if</u> $f^*$ is induced by a map of spaces $X \to Y$ (by above Lemma).

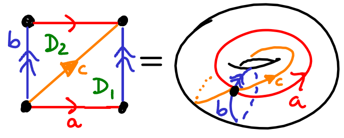
$\implies$ <u>Cor</u> The excision theorem iso on cohomology is an iso of rings.

However the connecting hom in M.V. or LES <u>cannot</u> possibly be a ring hom since it raises gradings by 1 ($\Rightarrow \delta(a \cup b)$ and $\delta(a) \cup \delta(b)$ have different grading!)

<u>Example</u> $H^1(T^2) \times H^1(T^2) \xrightarrow{\cup} H^2(T^2)$ is bilinear form $\mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}$ with matrix $\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$

<u>Pf</u> By the Useful Trick, it is enough to work with $H^*_\Delta$ instead of $H^*$.

Δ-complex structure for $T^2$:




$C^\Delta_*: 0 \to \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \to 0$ (gens: $D_1, D_2$; gens: a, b, c)

dualise: $\Downarrow$

$C^*_\Delta: 0 \leftarrow \mathbb{Z}^2 \xleftarrow{\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}} \mathbb{Z}^3 \xleftarrow{0} \mathbb{Z} \leftarrow 0$ (gens: $D_1^*, D_2^*$; gens: $a^*, b^*, c^*$ ← dual basis for basis a, b, c (e.g. $a^*(a) = 1$, $a^*(b) = 0$, $a^*(c) = 0$))

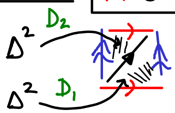
| * | | $H^\Delta_*(T^2)$ | $H^*_\Delta(T^2)$ |
|---|---|---|---|
| 0 | $\mathbb{Z}$ | $\mathbb{Z}\cdot pt$ | $\mathbb{Z}\cdot 1$ |
| 1 | $\mathbb{Z}^2$ | $\mathbb{Z}a \oplus \mathbb{Z}b$ | $\mathbb{Z}\cdot(a^*+c^*) + \mathbb{Z}\cdot(b^*+c^*)$ |
| 2 | $\mathbb{Z}$ | $\mathbb{Z}\cdot(D_1 - D_2)$ | $\mathbb{Z}\cdot D_1^*$ |

← abbreviate $\begin{cases} A = a^* + c^* \\ B = b^* + c^* \end{cases}$

←(Remark $[D_1^*] = -[D_2^*]$ in $H^2_\Delta(T^2)$)

<u>Claim</u> $\boxed{A \cup B = D_1^*}$

<u>Pf</u>



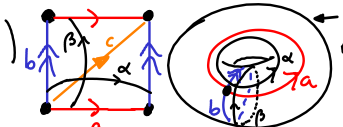

$(A \cup B)(D_1) = A(D_1|_{[e_0,e_1]}) \cdot B(D_1|_{[e_1,e_2]}) = A(a)B(b) = 1 \cdot 1 = 1$

$(A \cup B)(D_2) = A(b)B(a) = 0$. □ (can also check these by hand)

Graded-comm. $\Rightarrow B \cup A = -D_1^*$, $A \cup A = (-1)^{1\cdot 1} A \cup A$ so $= 0$, sim'ly $B \cup B = 0$. □

<u>Rmk</u> Recall that to specify a cochain in $C^k_\Delta(X)$ one needs to specify values on <u>all</u> generators of $C^\Delta_k(X)$ so not just on generators of $H^\Delta_k(X)$ (e.g. above A and $a^*$ agree on gens a, b of $H^\Delta_1(T^2)$ but disagree on $c \in C^\Delta_1(T^2)$, note $a^*$ is 1-cochain $\in C^1_\Delta(T^2)$ but is <u>not</u> a 1-cocycle)

Some (but not all) k-cochains $\varphi$ can be specified by drawing a "nice" $(n-k)$-dimensional subspace $\Sigma \subseteq X$ and defining $\boxed{\varphi(c) = \#(\text{times } \Sigma \text{ intersects } c)}$ for all $c \in C^\Delta_k(X)$ where one must explain with what sign ± an intersection point is counted and one has ensured that Σ intersects the generators of $C^\Delta_*(X)$ in a finite # points.




← We obtained the curves α, β by "pushing off" the curves a, b respectively away from themselves. Note the endpoints of α (and β) are the same so it is a loop (hence a 1-cycle in $T^2$).

$\Rightarrow$ get 1-cochains $\varphi_\alpha, \varphi_\beta \in C^1_\Delta(T^2)$:

$\varphi^*_\alpha(\underset{\in C^\Delta_1}{c}) = \#\, \alpha$ intersects c counted with orientation signs: (c ↑ α: +1; c ↑ α: −1) ← written $\alpha \cdot c$, called **intersection pairing**

Notice $\varphi_\alpha(a) = 0$, $\varphi_\alpha(b) = 1$, $\varphi_\alpha(c) = 1$ so $\varphi_\alpha = B$. Similarly, $\varphi_\beta = -A$.

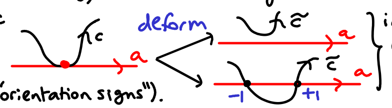
← **Non-examinable** comments about intersection numbers

<u>Fact</u> Since $T^2$ is an orientable manifold, $\varphi_\alpha \cup \varphi_\beta = (\alpha \cdot \beta)$ vol where vol is a generator of $H^2(T^2)$. Later in the course: vol is the "Poincaré dual" of the point class, and corresponds to the dual of the oriented sum of the top simplices. Above: vol $= D_1^*$ and

$\varphi_\alpha \cup \varphi_\beta = B \cup (-A) = A \cup B = (\alpha \cdot \beta)$ vol $=$ vol $= D_1^*$.

Defining intersection numbers rigorously is tricky, even when using smooth chains. ← ("tangencies" are the key issue)

One can calculate $\varphi_\Sigma(c)$ on a cycle c by first deforming c to a smooth homologous cycle $\tilde{c}$ which is "transverse" to Σ, and then we count intersection points $\Sigma \cap \tilde{c}$ (with "orientation signs").

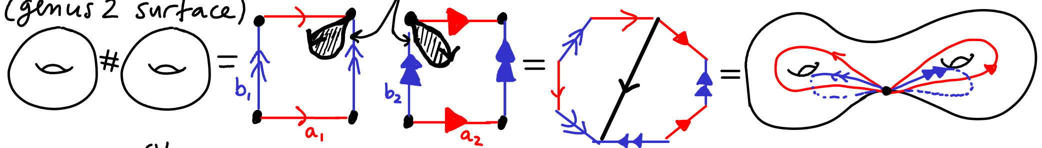



} in both cases $\varphi_a(c) := \varphi_a(\tilde{c}) = 0$

The fact that we consider the intersection number $a \cdot a = 0$ is because we can push a off itself:

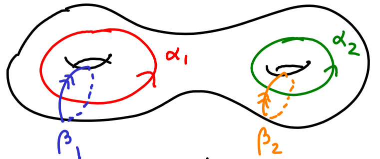
Exercise $\Sigma_2$ (genus 2 surface)

remove balls & glue bdries




| | | $H^{CW}_*(\Sigma_2)$ | $H^*_{cw}(\Sigma_2)$ | |
|---|---|---|---|---|
| 0 | $\mathbb{Z}$ | $\mathbb{Z}\cdot pt$ | $\mathbb{Z}\cdot 1$ | |
| 1 | $\mathbb{Z}^4$ | $\mathbb{Z}a_1+\mathbb{Z}b_1+\mathbb{Z}a_2+\mathbb{Z}b_2$ | $\mathbb{Z}\langle a_1^*, b_1^*, a_2^*, b_2^*\rangle$ | ← dual basis in $C^1_{cw}$ (easy to define on $C^1_{cw}(\Sigma_2)$ but not so obvious on $C^1_\Delta(\Sigma_2)$) |
| 2 | $\mathbb{Z}$ | $\mathbb{Z}\cdot D$ | $\mathbb{Z}\cdot D^*$ | |

Deform curves $a_1, b_1, a_2, b_2$ to get $\alpha_1, \beta_1, \alpha_2, \beta_2$:

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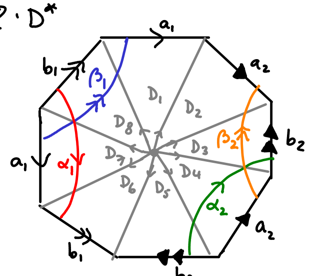



← $\Delta$-cx structure on $T^2$

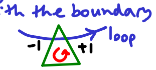
then notice for $c \in C_1^{CW}(\Sigma_2)$

$a_i^*(c) = -\#(\beta_i$ intersects $c)$ ← signed count

$b_i^*(c) = \#(\alpha_i$ " ")

so can extend this to a definition of $a_i^*, b_i^* \in C^1_\Delta(\Sigma_2)$ by allowing $c \in C_1^\Delta(\Sigma_2)$. Check that $a_i^*, b_i^*$ are 1-cocycles in $C^1_\Delta(\Sigma_2)$.

(key idea: a loop has total signed intersection # = 0 with the boundary of a triangle)




**Exercise:** $a_i^* \cup b_j^* = \delta_{ij} \cdot D^* = -b_j^* \cup a_i^*$ $\quad \delta_{ij} = \begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$

$a_i^* \cup a_i^* = b_i^* \cup b_i^* = 0$ ⟩ so same as geometric intersection numbers of corresponding curves.

Hint: represent D as a sum ± triangles in last picture. (orientation signs: $D = -D_1 - D_2 + D_3 + D_4 - D_5 + D_6 + D_7 - D_8$ using + if outer edge is oriented anticlockwise)

Cultural Rmk on general theory (Intersection Theory/Differential Topology)

$M^m$ oriented m-manifold

$N^n \subseteq M^m$ oriented compact n-dim submfd $\Rightarrow$ $H_n(N) \xrightarrow{incl_*} H_n(M)$ ← see later in course

$\exists$ generator $[N] \longmapsto [N]$

N, M also smooth (see Differential Geometry course) $\Rightarrow$ $\omega_N \in H^{m-n}(M)$ counts # intersections with N (with signs)

$N_1^{n_1}, N_2^{n_2} \subseteq M^m$ compact oriented smooth submfds and **transverse** (= at every $p \in N_1 \cap N_2$ the tangent spaces to $N_1, N_2$ at p span the tangent space to M at p) $\Rightarrow$

(can always "homotope" $N_1$ (or $N_2$) to achieve transversality, and class $\omega_{N_1}$ does not change if homotope) (homotope the inclusion map)

(tang. space means the "best" vector space approximation at p determined by the local smooth coordinates.)

> $N_1 \cap N_2$ is a compact orientable mfd of dim $= n_1+n_2-m$
> $\omega_{N_1} \cup \omega_{N_2} = \omega_{N_1\cap N_2} \in H^{2m-n_1-n_2}(M)$
> In particular if $n_1+n_2=m$, and M connected, then $H^m(M) \cong \mathbb{Z}$ s.t. $\omega_{N_1} \cup \omega_{N_2} \longmapsto \#(N_1 \cap N_2) \in \mathbb{Z}$.
> In non-orientable case, this all holds if work over $\mathbb{Z}/2$

($N_1 \cap N_2$ has an orientation induced by $N_1, N_2$. If $n_1+n_2=m$, get $N_1 \cap N_2$ is sum $\Sigma \pm$ points. Sign compares oriented bases of tang. spaces: $T_pM = T_pN_1 \oplus T_pN_2$.)

Fact (Thom 1954)

Not all $a \in H^j(M)$ arise as $\omega_N$ for connected compact oriented codim=j smooth submfd N

But $\exists N \in \mathbb{N}$ s.t. $N \cdot a$ does arise. They do arise for $H^*(M;\mathbb{Q}), H^*(M,\mathbb{R}), H^*(M;\mathbb{Z}/2)$

# 11. KÜNNETH FORMULA AND PRODUCT SPACES

## Algebra: tensor products

R ring (comm. with 1) ← e.g. abelian groups = $\mathbb{Z}$-mods, vector spaces/$\mathbb{F}$ = $\mathbb{F}$-mods

**Def** A, B R-modules $\Rightarrow$ **Tensor product** is R-module

$$A \otimes_R B = \langle (a,b) : a\in A, b \in B\rangle / \text{relations of bilinearity \& rescaling}$$

(or $A \otimes B$) ← R-mod generated; write $a \otimes b$ for its class

**bilinearity**: $(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$

$a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$

**rescaling**: $r \cdot (a \otimes b) = (ra) \otimes b = a \otimes (rb)$ $\quad \forall r \in R$ ("can move $r \in R$ across the $\otimes$ symbol")

- So general element looks like $\sum a_k \otimes b_k$ (finite sum) ← NOT UNIQUELY!
- Don't confuse with $A \times B$: e.g. $0 \otimes b = 0$ $\quad \forall b$

**Rmk** Can define $A \otimes_R B$ also by a universal property: for all R-mods C,

$$\boxed{\mathrm{Hom}_R(A \otimes_R B; C) \xrightarrow[\text{natural}]{\cong} \{R\text{-bilinear maps } A\times B \to C\}}$$

Using above description of $A\otimes B$: $\varphi \longmapsto (A\times B \to C, (a,b) \mapsto \varphi(a\otimes b))$

**Example** $(R = \mathbb{F})$ V, W v.s./$\mathbb{F}$ (basis $v_i$, basis $w_j$) $\Rightarrow$ $V\otimes W$ v.s./$\mathbb{F}$ basis $v_i \otimes w_j$

$\dim_{\mathbb{F}} V\otimes W = \dim V \cdot \dim W$

**Exercise** V, W finite $\dim/_{\mathbb{F}} \Rightarrow V^* \otimes W \cong \mathrm{Hom}_{\mathbb{F}}(V,W)$

Hint $f: V \to \mathbb{F}$, $w \in W$, $f \otimes w \mapsto (V \to W, v \mapsto f(v)\cdot w)$

(e.g. $(\mathbb{R}^n)^* \otimes \mathbb{R}^m \cong \mathrm{Mat}_{m\times n}(\mathbb{R}) \cong \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$; $e_i^* \otimes e_j \longleftrightarrow$ matrix A with $A_{ji}=1$, 0 else.)

**Examples** $(R=\mathbb{Z})$

- $\mathbb{Z}^n \otimes \mathbb{Z}^m \cong \mathbb{Z}^{n\cdot m}$
- $\mathbb{Z}/n \otimes \mathbb{Z} \cong \mathbb{Z}/n$ ⟵ $\sum a_i \otimes b_i = (\sum a_i b_i) \otimes 1$
- $\mathbb{Z}/2 \otimes \mathbb{Z}/3 = 0$ ⟵ $1 \otimes x = 3 \otimes x = 1 \otimes 3x = 0$
- $\mathbb{Z}/2 \otimes \mathbb{Z}/4 \cong \mathbb{Z}/2$ ⟵ $\{1\otimes 1 = 3\otimes 1 = 1\otimes 3$; $1 \otimes 2 = 2\otimes 1 = 0$

**Examples**

- $A \otimes B \cong B \otimes A$
- $(\bigoplus_i A_i) \otimes (\bigoplus_j B_j) \cong \bigoplus_i \bigoplus_j (A_i \otimes B_j)$
- $A \otimes R \cong A$ (so "$\cdot \otimes_R R$ does nothing")
- $A \otimes R/d \cong A/d\cdot A$

} hence now know $A\otimes B$ for any f.g. R-mods A, B.

for example $\mathbb{Z}/2 \otimes \mathbb{Z}/4 \cong (\mathbb{Z}/4)/2\cdot\mathbb{Z}/4 \cong \mathbb{Z}/2$ ⟵ (Rmk $(\mathbb{Z}/n)/m\cdot\mathbb{Z}/n \cong \mathbb{Z}/\gcd(m,n)$)

More generally: $\begin{cases} R/I \otimes_R R/J \cong R/(I+J) \text{ for ideals } I,J \subseteq R. \\ A \otimes_R R/J \cong A/J\cdot A \end{cases}$

**Warning** $\cdot \otimes A$ often not an exact functor, i.e. does not preserve exact sequences

indeed it can ruin injectivity: $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$ now take $\cdot \otimes \mathbb{Z}/2$ get $0 \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2$.

**Fact** $\boxed{\cdot \otimes_{\mathbb{Z}} \mathbb{Q} \text{ and } \cdot \otimes_{\mathbb{Z}} \mathbb{R} \text{ are exact functors on } \mathbb{Z}\text{-mods}}$ ← More generally $\cdot \otimes_R \mathrm{Frac}(R)$ is exact on $R$-mods where Frac R is fraction field, and R is an integral domain "localisation is an exact functor"

**example** A f.g. $\mathbb{Z}$-mod $\Rightarrow A \cong \mathbb{Z}^r \oplus \mathbb{Z}/_{d_1} \oplus \cdots \oplus \mathbb{Z}/_{d_k}$ some $d_i \neq 0$

$\Rightarrow A \otimes \mathbb{Q} \cong \mathbb{Q}^r \Rightarrow \boxed{r = \operatorname{rank} A = \dim_{\mathbb{Q}}(A \otimes \mathbb{Q})}$

**Corollary** Rank-nullity thm holds for $\mathbb{Z}$-modules if use rank instead of dim.

**Pf** $0 \to A \to B \xrightarrow{\varphi} C \to 0$ exact ($A = \ker\varphi$, $C = \operatorname{Im}\varphi$) $\Rightarrow 0 \to A \otimes \mathbb{Q} \to B \otimes \mathbb{Q} \to C \otimes \mathbb{Q} \to 0$ exact

$\Rightarrow \dim(C \otimes \mathbb{Q}) + \dim(A \otimes \mathbb{Q}) = \dim(B \otimes \mathbb{Q})$. □ ← rank-nullity for $\mathbb{Q}$-vector spaces.

## Tensor product of chain cxes

$C_*$, $\tilde{C}_*$ chain cxes of $R$-mods $\Rightarrow (C_* \otimes \tilde{C}_*)_n = \bigoplus_{i+j=n} C_i \otimes \tilde{C}_j$

$$\boxed{\partial(x \otimes y) = (\partial x) \otimes y + (-1)^{\deg x} x \otimes \tilde{\partial} y}$$ **"Leibniz rule"**

Think of $\partial$ as an operator of deg $= -1$ acting from left; since $\partial$ "jumps over $x$" get $(-1)^{\deg \partial \cdot \deg x}$

**Exercise** $\partial \circ \partial = 0$ ← would fail without sign

recall $Z_* = \ker\partial$ = cycles, $B_* = \operatorname{im}\partial$ = boundaries

$Z_i \otimes \tilde{Z}_j \subseteq Z_{i+j}(C_* \otimes \tilde{C}_*)$ and $\left.\begin{matrix} Z_i \otimes \tilde{B}_j \\ B_i \otimes \tilde{Z}_j \end{matrix}\right\} \subseteq B_{i+j}(C_* \otimes \tilde{C}_*)$

**Cor** $\exists$ natural maps $\boxed{\begin{matrix} H_i(C_*) \otimes H_j(\tilde{C}_*) \longrightarrow H_{i+j}(C_* \otimes \tilde{C}_*) \\ \sum [c_k] \otimes [\tilde{c}_k] \longmapsto \sum [c_k \otimes \tilde{c}_k] \end{matrix}}$

FACT: **Algebraic Künneth Thm**

$C_*, H_*(C_*)$ f.g. free $R$-mods (PID (principal ideal domain)) (no assumption on $\tilde{C}_*$)

$\Rightarrow \boxed{\bigoplus_{i+j=n} H_i(C_*) \otimes H_j(\tilde{C}_*) \cong H_n(C_* \otimes \tilde{C}_*)}$ via

## Algebra: Euler characteristic

$C$ finitely generated graded abelian gp (so $\mathbb{Z}$-mod) | more generally: $R$-mod for PID $R$

**Def** **Euler characteristic** $\chi(C) = \sum (-1)^i \operatorname{rank} C_i$

**Example/Motivation** $X$ finite CW-cx then take $C = C_*^{CW}(X)$ to get

$$\boxed{\chi(X) = \#(0\text{-cells}) - \#(1\text{-cells}) + \#(2\text{-cells}) - \cdots}$$

**Lemma** If $C_*$ f.g. chain cx $\Rightarrow \boxed{\chi(C_*) = \chi(H_*(C_*))}$ $\left(= \sum (-1)^i \operatorname{rank} H_i(C)\right)$

**Pf** Abbreviate $|C_i| = \operatorname{rank} C_i$ $\left(= \dim_{\mathbb{Q}}(C_i \otimes_{\mathbb{Z}} \mathbb{Q})\right)$ ← for $R$-mods, do $\dim_{\mathbb{F}}(C_i \otimes_R \mathbb{F})$ with $\mathbb{F} = \mathrm{Frac}(R)$ ($R$ integral domain) [Corollary still holds, same proof]

By previous Corollary about rank-nullity:

$0 \to Z_i \to C_i \xrightarrow{\partial} B_{i-1} \to 0 \Rightarrow |C_i| = |Z_i| + |B_{i-1}|$

$0 \to B_i \to Z_i \to H_i \to 0 \Rightarrow |H_i| = |Z_i| - |B_i|$

$\Rightarrow |C_i| - |H_i| = |B_{i-1}| + |B_i|$

$\Rightarrow \chi(C_*) - \chi(H_*) = \sum (-1)^i |B_{i-1}| + \sum (-1)^i |B_i| = \sum (-1)^i (-|B_i| + |B_i|) = 0$. □

**Cor** $X$ space $\Rightarrow$ $\boxed{\begin{aligned} \chi(X) &:= \sum (-1)^i \operatorname{rank} H_i(X) \\ &= \sum (-1)^i \operatorname{rank} C_i(X) \end{aligned}}$ ← if finite rank $H_*(X)$ ← if finite rank $C_*(X)$

So $\chi(X)$ is invariant up to hpy equivalence! **Example** $\chi\left(\begin{smallmatrix}\text{platonic} \\ \text{solid}\end{smallmatrix}\right) = \chi(S^2) = 2$

## Product spaces

$X, Y$ CW-cxes (cells $e_\alpha$, $e_\beta$) $\Rightarrow X \times Y$ CW-cx with cells $e_\alpha \times e_\beta$ attaching maps

$\partial(\mathbb{D}^i_\alpha \times \mathbb{D}^j_\beta) = (\partial \mathbb{D}^i_\alpha) \times \mathbb{D}^j_\beta \cup \mathbb{D}^i_\alpha \times (\partial \mathbb{D}^j_\beta)$

$\varphi_\alpha \times id \downarrow X^{i-1} \times Y^j$, $\quad id \times \varphi_\beta \downarrow X^i \times Y^{j-1}$, both $\to (X \times Y)^{i+j-1}$

**Cor** $\boxed{\chi(X \times Y) = \chi(X) \cdot \chi(Y)}$ $\forall$ finite CW-cxes $X, Y$

**Pf** $\sum (-1)^k \operatorname{rank} H_k^{CW}(X \times Y)$

$= \sum (-1)^k \operatorname{rank} C_k^{CW}(X \times Y) = \sum_k \sum_{i+j=k} (-1)^{i+j} \operatorname{rank} C_i^{CW}(X) \cdot \operatorname{rank} C_j^{CW}(Y)$ □

**Lemma** (proof later) $d(e^i_\alpha \times e^j_\beta) = (d e^i_\alpha) \times e^j_\beta + (-1)^i e^i_\alpha \times (d e^j_\beta)$

hence $\boxed{C_*^{CW}(X \times Y) \cong C_*^{CW}(X) \otimes C_*^{CW}(Y)}$

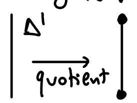
Hence if $H_*(Y)$ free then by Künneth $H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$.

**Example**

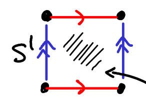
| $*$ | $H_*(S^1)$ |
|---|---|
| 0 | $A \cong \mathbb{Z}$ |
| 1 | $B \cong \mathbb{Z}$ |
| 2 | 0 |

| $*$ | $H_*(S^1) \otimes H_*(S^1) \cong H_*(S^1 \times S^1)$ ← torus | |
|---|---|---|
| 0 | $A \otimes A$ | $\cong \mathbb{Z}$ |
| 1 | $(A \otimes B) \oplus (B \otimes A)$ | $\cong \mathbb{Z}^2$ |
| 2 | $B \otimes B$ | $\cong \mathbb{Z}$ |
| 3 | 0 | |

B generated by

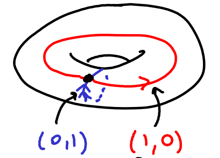



A generated by $\bullet$ ($\Delta^0 \to \bullet$)




$H_0$ generator is $\bullet$; square $\cong \Delta^2$ generator of $H_2$

$H_1$ generators:



Pf $(\partial \mathbb{D}^i_\alpha)\times\mathbb{D}^j_\gamma \xrightarrow{\varphi_\alpha\times id} X^{i-1}\times Y^j \longrightarrow X^{i-1}\times Y^j \Big/ \underbrace{(X\times Y)^{i+j-2}\cap(X^{i-1}\times Y^j)}_{\parallel\leftarrow \text{easy check}}$

This proof is Non-examinable

$\text{(star)} :=$ the above quotient, where $(X\times Y)^{i+j-2}\cap(X^{i-1}\times Y^j) = X^{i-2}\times Y^j \cup X^{i-1}\times Y^{j-1}$

$X^{i-1} = X^{i-2}\cup(\mathbb{D}^{i-1}_\beta \cup \dots)/\sim$

$Y^j = Y^{j-1}\cup(\mathbb{D}^j_\gamma\cup\dots)/\sim$

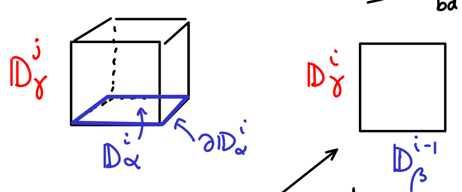
} get $\sim$ from attaching maps

$X^{i-1}\times Y^j = X^{i-2}\times Y^j \cup X^{i-1}\times Y^{j-1} \cup (\mathbb{D}^{i-1}_\beta\times\mathbb{D}^j_\gamma\cup\dots)/\sim$

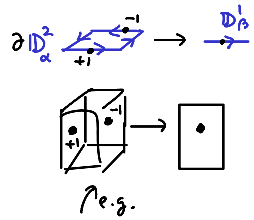
$\Rightarrow \text{(star)} = (\mathbb{D}^{i-1}_\beta\times\mathbb{D}^j_\gamma\cup\dots)\,/\,\text{boundaries}$

$= \mathbb{D}^{i-1}_\beta\times\mathbb{D}^j_\gamma \Big/ \partial(\mathbb{D}^{i-1}_\beta\times\mathbb{D}^j_\gamma) \quad\vee\quad \dots$

$(\partial\mathbb{D}^i_\alpha)\times\mathbb{D}^j_\gamma \xrightarrow{\varphi_\alpha\times id} \mathbb{D}^{i-1}_\beta\times\mathbb{D}^j_\gamma\big/\text{bdry} \quad\vee\quad\dots$




$(\partial\mathbb{D}^i_\alpha)\times\mathbb{D}^j_\gamma \xrightarrow{\varphi_\alpha\times id}$ / bdry




By considering local degrees now we see we get degree $= d_{\alpha\beta}$ for this.

$\Rightarrow$ get contribution $(d e^i_\alpha)\times e^j_\beta$ ✓

similarly

$\mathbb{D}^i_\alpha\times\partial\mathbb{D}^j_\gamma \xrightarrow{id\times\varphi_\gamma} \mathbb{D}^i_\alpha\times\mathbb{D}^{j-1}_\delta\big/\text{bdry}$

$\Rightarrow$ degree $(-1)^i d_{\gamma\delta}$

so get $(-1)^i e^i_\alpha\times d e^j_\beta$

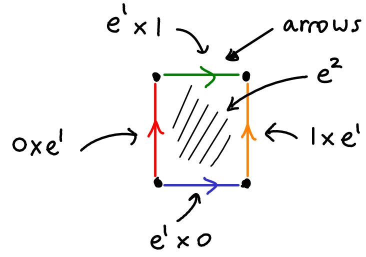
$(-1)^i$ caused by orientations:

Could reorder factors: $\mathbb{D}^i_\alpha\times\mathbb{D}^j_\gamma \cong \mathbb{D}^j_\gamma\times\mathbb{D}^i_\alpha$ by $\begin{pmatrix}0 & Id_j\\ Id_i & 0\end{pmatrix}$

whose $\det = (-1)^{ij}$. Then $\partial\mathbb{D}^j_\gamma\times\mathbb{D}^i_\alpha \to \mathbb{D}^{j-1}_\delta\times\mathbb{D}^i_\alpha/\text{bdry}$ gives degree $d_{\gamma\delta}$.

Swap factors $\mathbb{D}^{j-1}_\delta\times\mathbb{D}^i_\alpha/\text{bdry}$ by $\begin{pmatrix}0 & Id_i\\ Id_{j-1} & 0\end{pmatrix}$, $\det = (-1)^{i(j-1)}$. Total sign $= (-1)^i$.

<u>Example</u> Recall after definition of $H^{CW}_*$ we had example $I\times I$:




$\partial e^2 = +e^1\times 0 + 1\times e^1 - e^1\times 1 - 0\times e^1$

$= (\partial e^1)\times e^1 - e^1\times(\partial e^1)$ ✓

($(-1)^{\dim e^1}$)

## A further comment on orientation sign $(-1)^i$

$\mathbb{D}^i\times\mathbb{D}^j \cong \Delta^i\times\Delta^j$, $\Delta^i = [v_0,\dots,v_i]$, $\Delta^j = [w_0,\dots,w_j]$ $\leftarrow$ viewed in $\mathbb{R}^i, \mathbb{R}^j$

project $\mathbb{R}^{i+1}\to\mathbb{R}^i$, $(t_0,\dots,t_i)\mapsto(t_1,\dots,t_i)$

$\partial(\mathbb{D}^i\times\mathbb{D}^j) \cong \partial\Delta^i\times\Delta^j \cup \Delta^i\times\partial\Delta^j$

$\partial\Delta^i = \sum_k (-1)^k[v_0,\dots,\hat{v}_k,\dots,v_i]$, $\partial\Delta^j = \sum(-1)^k[w_0,\dots,\hat{w}_k,\dots,w_j]$

would be correct orientation sign for basis $w_1-w_0,\dots,\widehat{w_k-w_0},\dots,w_j-w_0$ but actually we have $[v_0,\dots,v_i]\times[w_0,\dots,\hat{w}_k,\dots,w_j]\subseteq\mathbb{R}^i\times\mathbb{R}^j$

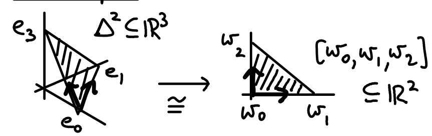
and $(-1)^{i+k}$ is the orientation sign for the basis

$v_1-v_0,\dots,v_i-v_0,w_1-w_0,\dots,\widehat{w_k-w_0},\dots,w_j-w_0$

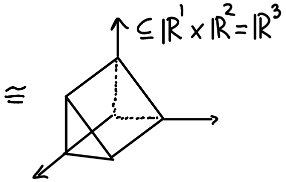
for the hyperplane in $\mathbb{R}^{i+j+1}$ containing

$\Rightarrow$ need $(-1)^i$ to fix orientation sign.

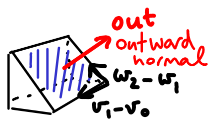
<u>Example</u> $\Delta^1\times\Delta^2$



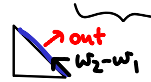

$\Delta^1\times\Delta^2 \cong$




$[v_0,v_1]\times[\hat{w}_0,w_1,w_2]$




out, $v_1-v_0$, $w_2-w_1$ is negative $\mathbb{R}^3$-basis




out, $w_2-w_1$ is positive $\mathbb{R}^2$-basis

differ due to $(-1)^i$, $i=1$.

Projections $X \times Y \xrightarrow{p_X} X$, $X \times Y \xrightarrow{p_Y} Y$

FACT:

**Künneth Theorem** If $H_n(Y)$ finitely generated, free $\forall n$ (no conditions on $X$; e.g. $Y \simeq$ finite CW cx; free is automatic if use field coefficients)

| | |
|---|---|
| $H_n(X \times Y) \cong \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y)$ | $H^n(X \times Y) \cong \bigoplus_{i+j=n} H^i(X) \otimes H^j(Y)$ <br> $p_X^* a \cup p_Y^* b \longleftarrow a \otimes b$ (★) <br> and extend linearly |

Recall for cellular homology this on generators is (chain level): $e^i_\alpha \times e^j_\beta \longleftarrow e^i_\alpha \otimes e^j_\beta$

This is hom of rings if use following product
$$(a \otimes b)\cdot(\tilde{a} \otimes \tilde{b}) = (-1)^{|b|\cdot|\tilde{a}|}(a \cup \tilde{a}) \otimes (b \cup \tilde{b})$$
($|b|, |\tilde{a}|$: gradings; think of it as "exchanging order of $b, \tilde{a}$")

Rmk
An indirect proof of the Thm is to write down two generalised cohomology theories $F(X,A) = H^*(X,A) \otimes H^*(Y)$ and $G(X,A) = H^*(X \times Y, A \times Y)$, and consider the natural transformation $\alpha : F \to G$ given by (★), notice for $X = pt$, $A = \emptyset$ both $F, G$ give $H^*(Y)$.

Example $X = S^n$, $Y = S^m$, $n \neq m$

$$H_*(S^n \times S^m) \cong \begin{cases} \mathbb{Z} & * = 0, n, m, n+m \\ 0 & \text{else} \end{cases} \cong H^*(S^n \times S^m)$$
$a_n \cup a_m = a_{n+m}$ (where $a_i = \text{dual}(e^i)$)

$$H_*(S^n \times S^n) \cong \begin{cases} \mathbb{Z} & * = 0, 2n \\ \mathbb{Z}^2 & * = n \\ 0 & \text{else} \end{cases} \cong H^*(S^n \times S^n)$$
$\leftarrow$ gens: $a_n^{(1)} = a_n \otimes 1$, $a_n^{(2)} = 1 \otimes a_n$

$a_n^{(1)} \cup a_n^{(2)} = a_{2n}$ (but $a_n^{(i)} \cup a_n^{(i)} = 0$)

**Cor** $H^k(T^n) \cong \Lambda^k[x_1, \ldots, x_n]$ ($T^n$ = n-torus $S^1 \times \cdots \times S^1$; $\deg x_i = 1$)

where $x_i = p_i^*(\text{gen. of } H^1(S^1))$, $p_i : T^n \to S^1$ projections to factors.

exterior algebra = free abelian gp. on gens. $\{x_{i_1} \wedge \cdots \wedge x_{i_k} : i_1 < \cdots < i_k\}$ so rank $= \binom{n}{k}$

product is "$\wedge$" using the rule $x_i \wedge x_j = -x_j \wedge x_i$ (compare graded-commutativity of cup product)

Pf Künneth & induction ($T^n = T^{n-1} \times S^1$) □

FACT cup product equals composition

$$\cup : H^i(X) \otimes H^j(X) \longrightarrow H^{i+j}(X \times X) \xrightarrow{\Delta^*} H^{i+j}(X)$$
$$(\Delta^i \xrightarrow{\sigma_1} X) \otimes (\Delta^j \xrightarrow{\sigma_2} X) \longmapsto (\Delta^i \times \Delta^j \xrightarrow{\sigma_1 \times \sigma_2} X \times X)$$
(exterior product; $\Delta^i \times \Delta^j \simeq \Delta^{i+j}$)

$\Delta$ = diagonal map $X \longrightarrow X \times X$, $x \longmapsto (x, x)$

## 12. UNIVERSAL COEFFICIENTS THEOREM

MOTIVATION: what is difference between $H^*(\text{Hom}(C_*, \mathbb{Z}))$ and $\text{Hom}(H_*(C_*), \mathbb{Z})$. Similarly: $H_*(C_* \otimes G)$ vs. $H_*(C_*) \otimes G$.

**Proof is non-examinable.** For $(C_*, \partial_*)$ chain cx:

$$\Longrightarrow 0 \longrightarrow Z_* = \text{Ker}\,\partial_* \xrightarrow{\text{incl}} C_* \xrightarrow{\partial_*} B_{*-1} = \text{Im}\,\partial_{*-1} \longrightarrow 0 \quad \text{is SES}$$
($\partial = 0$ on $Z_*$; $\partial = 0$ on $B_{*-1}$)

| |
|---|
| FACT: Submodules of a free $\mathbb{Z}$-module are free <br> Rmk The same holds for R-mods if R is PID |

($\mathbb{Z}$-module ≡ abelian gp; free means: $\bigoplus_{\text{indexing set}} \mathbb{Z}$)

(PID = principal ideal domain = integral domain R s.t. every ideal = R·a some a)

**Assume $C_*$ free $\mathbb{Z}$-mod**

FACT $\Rightarrow$ $Z_*, B_*$ free (as $\text{Ker}\,\partial^*, \text{Im}\,\partial^*$ are submods of $C_*$)

$\Longrightarrow$ SES splits, choose splitting $C_* \underset{s}{\overset{\partial_*}{\rightleftarrows}} B_{*-1}$ so $\partial_* \circ s = id$ (recall just pick preimages under $\partial_*$ of a basis for $B_*$)

dual SES $\Longrightarrow$ $0 \leftarrow Z^* \xleftarrow{\text{incl}^*} C^* \xleftarrow{\partial^*} B^{*-1} \leftarrow 0$

note: $\text{incl}^*$ = restrict to $Z_*$ since $\text{incl}^* \circ \phi : Z_* \xrightarrow{\text{incl}} B_* \xrightarrow{\phi} \mathbb{Z}$

$$\begin{array}{ccccccccc} 0 & \leftarrow & Z^n & \longleftarrow & C^n & \xleftarrow{\partial^*} & B^{n-1} & \longleftarrow & 0 \\ & & \uparrow \partial = 0 & & \uparrow \partial & & \uparrow \partial = 0 & & \\ 0 & \leftarrow & Z^{n-1} & \longleftarrow & C^{n-1} & \xleftarrow{\partial^*} & B^{n-2} & \longleftarrow & 0 \end{array}$$

(Rmk Although $\partial^* = 0 : B^* \to B^{*+1}$ the map $\partial^* : B^{n-1} \to C^n$ need not $= 0$: $\psi : B_{n-1} \to \mathbb{Z}$ $\Rightarrow \partial^*\psi = \psi \circ \partial : C_n \xrightarrow{\partial} B_{n-1} \xrightarrow{\psi} \mathbb{Z}$)

Connecting map $\delta : Z^{n-1} \to B^{n-1}$ of LES:

$$\begin{array}{ccccc} 0 & \longleftarrow & \partial^*\varphi & \xleftarrow{\partial^*} & \varphi|_{B_*} = \phi|_{B_*} \\ \updownarrow & & \uparrow & & \\ \varphi|_{Z_*} = \phi & \longleftarrow & \exists\varphi & & \end{array}$$
($B_* \subseteq Z_*$) $\Longrightarrow$ $\boxed{\delta(\phi) = \phi|_{B_*}}$

LES $\Longrightarrow$ $\cdots \xleftarrow{\delta^n} Z^n \longleftarrow H^n C \xleftarrow{\partial^*} B^{n-1} \xleftarrow{\delta^{n-1}} Z^{n-1}$
$\varphi \longleftarrow [\varphi]$, $\phi|_{B_{n-1}} \longleftarrow \phi$

($H^*B = B^*$, $H^*Z = Z^*$ since $\partial^* = 0$)

$\Longrightarrow$ $0 \leftarrow \text{Ker}\,\delta^n \leftarrow H^n C \leftarrow B^{n-1}/\text{Im}\,\delta^{n-1} \leftarrow 0$

$\text{Ker}\,\delta^n = \{\phi \in Z^n : \phi(B_n) = 0\} \Longrightarrow$ so: $\phi : Z_n \to \mathbb{Z}$ factors through $Z_n/B_n = H_n(C_*)$
$= \text{Hom}(H_n(C_*), \mathbb{Z})$

(★) **Universal Coefficients Thm:**

$$0 \longrightarrow B^{n-1}/\text{Im}\,\delta^{n-1} \longrightarrow H^n(C) \longrightarrow \text{Hom}(H_n(C_*), \mathbb{Z}) \longrightarrow 0 \quad \text{is SES and natural}$$
$$[\varphi] \longmapsto (\varphi : H_n(C_*) \to \mathbb{Z})$$
(evaluation of a cohomology class on cycles)

$B^{n-1}/\text{Im}\,\delta^{n-1} = \textbf{Ext}^1(H_{n-1}(C); \mathbb{Z})$ (see next Lemma)

and SES splits (but not naturally): $B^{n-1}/\text{Im}\,\delta^{n-1} \underset{s^*}{\overset{\partial^*}{\rightleftarrows}} H^n(C)$, $s^* \circ \partial^* = id$ (since $\partial \circ s = id \Rightarrow id = (\partial \circ s)^* = s^* \circ \partial^*$)

$$\Rightarrow H^n(C) \cong \text{Hom}(H_n(C), \mathbb{Z}) \oplus \text{Ext}^1(H_{n-1}(C); \mathbb{Z})$$

**Lemma** $\mathrm{Ext}^1(H_{n-1}(C);\mathbb{Z}) \cong B^{n-1}/\mathrm{Im}\,\delta^{n-1}$ canonically

## Algebra background on Extension groups $\mathrm{Ext}^i(M;R)$ $(=\mathrm{Ext}^i_R(M,R))$

**general case**

$M$ R-module, $R$ ring (comm. with 1)

$\Rightarrow \exists$ **free resolution**:

$\cdots \to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} M \to 0$ exact, $P_i$ free R-mods

(pick gens $x_\alpha$ for $M$ $\Rightarrow P_0 = \bigoplus_\alpha R \xrightarrow{\varphi_0} M$, $e_\alpha \mapsto x_\alpha$

" " $y_\beta$ for $\mathrm{Ker}\,\varphi_0 \Rightarrow P_1 = \bigoplus_\beta R \xrightarrow{\varphi_1} \mathrm{Ker}\,\varphi_0$, $e_\beta \mapsto y_\beta$

continue inductively)

**our case**

$H_{n-1}(C_*)$ $\mathbb{Z}$-mod $(R=\mathbb{Z})$

$0 \to B_{n-1} \hookrightarrow Z_{n-1} \to H_{n-1}(C) \to 0$

with $B_{n-1} = P_1$, $Z_{n-1} = P_0$, $H_{n-1}(C) = M$

Take $\mathrm{Hom}(\cdot\,;R)$ and drop $\mathrm{Hom}(M;R)$

$0 \to \mathrm{Hom}(P_0;R) \xrightarrow{\varphi_1^*} \mathrm{Hom}(P_1;R) \xrightarrow{\varphi_2^*} \cdots$

Is cochain complex but not exact

$\Rightarrow$ take cohomology groups:

**Def** $\mathrm{Ext}^0(M;R) = \mathrm{Ker}\,\varphi_1^*$

$\mathrm{Ext}^1(M;R) = \mathrm{Ker}\,\varphi_2^* / \mathrm{Im}\,\varphi_1^*$

$\cdots$

Fact: independent of choices $P_i, \varphi_i$

$0 \to Z^{n-1} \to B^{n-1} \to 0$

**Proof of Lemma**

By **Example 2**,

$\mathrm{Ext}^1(H_{n-1}(C_*);\mathbb{Z}) =$

$= \left\{ \begin{matrix} 0 \to B_{n-1} \to Z_{n-1} \\ \phi \downarrow \\ \mathbb{Z} \end{matrix} \right\}$ modulo

those arising from restriction

$\left\{ \begin{matrix} 0 \to B_{n-1} \to Z_{n-1} \\ \phi|_{B_{n-1}} \searrow \quad \downarrow \phi \\ \mathbb{Z} \end{matrix} \right\}$

Thus $B^{n-1}/\mathrm{Im}\,\delta^{n-1}$. $\square$

**Example 1** $\mathrm{Ext}^0(M;R) \cong \mathrm{Hom}(M,R)$

$P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} M$, $\phi: P_0 \to R$ with $\phi\circ\varphi_1 = 0$

descends: $m \mapsto \phi(\varphi_0^{-1}m)$

well defined since $\phi(\mathrm{Ker}\,\varphi_0) = 0$ ($\mathrm{Ker}\,\varphi_0 = \mathrm{Im}\,\varphi_1$)

**Example 2** $\mathrm{Ext}^1(M;R) =$

$\left\{ \phi : P_2 \xrightarrow{\varphi_2} P_1 \to P_0,\ \phi: P_1 \to R,\ \phi\circ\varphi_2 = 0 \right\} \Big/ \left\{ \phi = \varphi \circ \varphi_1 : P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi} R \right\}$

**Rmk** If $R$ PID, then $\mathrm{Ker}(P_0 \to M)$ is free (since submod of free mod $P_0$)

$\Rightarrow$ can pick $P_1 = \mathrm{Ker}(P_0 \to M)$, $P_k = 0$ for $k \geq 2$ $\Rightarrow \mathrm{Ext}^k(M;R) = 0$ $k \geq 2$

## (Co)homology with coefficients in a ring/field/module

**Motivation**

So far we had $(C_*, \partial_*)$ (group hom) chain cx of abelian groups } in graded sense

$\Rightarrow H_*(C_*) = \mathrm{Ker}\,\partial_* / \mathrm{Im}\,\partial_*$ abelian group (since $\mathrm{Ker}\,\partial$, $\mathrm{Im}\,\partial$ are)

We cannot use a chain cx of (non-abelian) groups, because $\mathrm{Im}\,\partial_*$ need not be a normal subgroup of $\mathrm{Ker}\,\partial_*$.

However, abelian groups can be thought of as $\mathbb{Z}$-modules, then given any **abelian group $G$**, define **homology with coeffs in $G$**

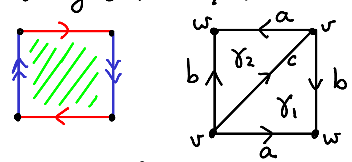
$$\boxed{H_*(C_*;G) = H_*(C_* \otimes_{\mathbb{Z}} G)}$$ ← with differential $\partial_* \otimes \mathrm{id}$

**Def** $X$ space $\Rightarrow$ $\boxed{H_*(X;G) = H_*(C_*(X) \otimes_{\mathbb{Z}} G)}$

Explanation:

$C_k(X)$ free $\mathbb{Z}$-mod $\cong \bigoplus_{I_k} \mathbb{Z} \Rightarrow C_k(X) \otimes_{\mathbb{Z}} G \cong \bigoplus_{I_k} G$ : just replace $\mathbb{Z}$ by $G$ (as $\mathbb{Z} \otimes_{\mathbb{Z}} \cdot \cong \cdot$)

**Why care?** We hope to get more/new invariants of spaces

**Example** $X = \mathbb{RP}^2 =$ 


| $*$ | $C^\Delta_*(\mathbb{RP}^2;G)$ |
|---|---|
| 0 | $Gv \oplus Gw$ |
| 1 | $Ga \oplus Gb \oplus Gc$ |
| 2 | $G\gamma_1 \oplus G\gamma_2$ |

for $G = \mathbb{Z}/2$: $0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{\partial_2} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{\partial_1} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 0$

$\partial_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\partial_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

$\Rightarrow H_*(\mathbb{RP}^2;\mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$

compare: $H_*(\mathbb{RP}^2) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ 0 & \text{else} \end{cases}$ ($G = \mathbb{Z}$ case)

Form cochain complex using $\mathrm{Hom}_{\mathbb{Z}}(\cdot, G)$ (= group homs to $G$) in place of $\mathrm{Hom}(\cdot,\mathbb{Z})$ with differential $\partial^*$: $\partial^*\phi = \phi \circ \partial_*$

$$\boxed{\begin{aligned} H^*(C_*;G) &= H_*(\mathrm{Hom}_{\mathbb{Z}}(C_*,G)) \\ H^*(X;G) &= H_*(\mathrm{Hom}_{\mathbb{Z}}(C_*(X);G)) \end{aligned}}$$

$X$ space; so: $H^*(C_*(X);G)$

Universal coefficients thm (Same proof using $\mathrm{Hom}_{\mathbb{Z}}(\cdot,G)$)

$$\boxed{\begin{aligned} 0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(C_*);G) \to H^n(C_*;G) &\longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_n(C_*),G) \to 0 \\ [\varphi] &\longmapsto (\varphi: H_n(C_*) \to G) \end{aligned}}$$

**Example** $X = \mathbb{RP}^2$, $G = \mathbb{Z}/2$, apply $\mathrm{Hom}_{\mathbb{Z}}(\cdot, G)$ to the complex in previous example

$$0 \leftarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xleftarrow{\partial^1 = \begin{pmatrix}1&1&1\\1&1&1\end{pmatrix}} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xleftarrow{\partial^0 = \begin{pmatrix}1&1\\1&1\\0&0\end{pmatrix}} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \leftarrow 0$$

$$\Rightarrow H^*(\mathbb{RP}^2; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$$

compare: $H^*(\mathbb{RP}^2) \cong \begin{cases} \mathbb{Z} & *=0 \\ 0 & *=1 \\ \mathbb{Z}/2 & *=2 \\ 0 & \text{else} \end{cases}$ ($G = \mathbb{Z}$ case)

(compare $\mathrm{Hom}(H_*(\mathbb{RP}^2), \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & *=0 \\ \mathbb{Z}/2 & *=1 \\ 0 & *\geq 2 \end{cases}$; caused by $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2$, where $\mathbb{Z}/2 = H_1(\mathbb{RP}^2)$, $G = \mathbb{Z}/2$)

**Can generalise further:**

| $C_*$ = chain cx of ... | coefficients in: | |
|---|---|---|
| abelian gps ($\mathbb{Z}$-mods) | abelian gp $G$ ($\mathbb{Z}$-mod) | $H_*(C_*; G) = H_*(C_* \otimes_{\mathbb{Z}} G)$ |
| $R$-modules ($R$ ← ring (comm. with 1)) | $R$-module $M$ | $H_*(C_*; M) = H_*(C_* \otimes_R M)$ |

**Rmk** $H_*(C; M)$ will be an $R$-module since $\ker\partial, \mathrm{Im}\,\partial$ are ($\partial_*$ is $R$-linear hom by assumption)

$X$ space $\Rightarrow C_k(X;R) = C_k(X) \otimes_{\mathbb{Z}} R \cong \bigoplus_{I_k} R$ : just replace $\mathbb{Z}$ by $R$ (as $\mathbb{Z} \otimes_{\mathbb{Z}} \cdot \cong \cdot$)

$$\Rightarrow \boxed{H_*(X; M) = H_*(C_*(X;R); M) = H_*(C_*(X;R) \otimes_R M)}$$

so just replace each $\mathbb{Z}$ by $M$ in $C_*(X)$

Form cochain complex using $\mathrm{Hom}_R(\cdot, M)$ (= $R$-linear homs to $M$) in place of $\mathrm{Hom}(\cdot, \mathbb{Z})$ with differential $\partial^*$: $\partial^*\phi = \phi \circ \partial_*$

$$\boxed{\begin{aligned} H^*(C_*; M) &= H_*(\mathrm{Hom}_R(C_*, M)) \\ H^*(X; M) &= H_*(\mathrm{Hom}_R(C_*(X;R); M)) \end{aligned}}$$

($X$ space) so: $H^*(C_*(X;R); M)$

**Rmk** These are $R$-mods. If we use $M = R$, then they are also rings via cup product

> **Universal Coefficients Thm** For $R$ any PID, $C_*$ chain cx of $R$-mods,
>
> $$0 \to \mathrm{Ext}^1_R(H_{n-1}(C_*); M) \to H^n(C_*; M) \to \mathrm{Hom}_R(H_n(C_*), M) \to 0$$ is SES and natural.
>
> ($\mathrm{Ext}^1_R \cong B^{n-1}/\mathrm{im}\,\delta^{n-1}$ working over $R$ using homs to $M$); $[\varphi] \longmapsto (\varphi : H_n(C_*) \to M)$
>
> and the SES splits but the splitting is not natural. (Same proof using $\mathrm{Hom}_R(\cdot, M)$)

**Example** $R = \mathbb{F}$ **field** $\Rightarrow C_*, H_*, H^*$ are vector spaces$/\mathbb{F}$.

**Rmk** all $\mathbb{F}$-mods (i.e. vector spaces$/\mathbb{F}$) are free $\mathbb{F}$-mods $\cong \bigoplus \mathbb{F}b_i$ (basis $b_i$); up to iso they are determined by $\dim_{\mathbb{F}}$ = cardinality of basis.

**Cor** $C_*$ = chain cx of $\mathbb{F}$-vector spaces $\Rightarrow \boxed{H^n(C_*; \mathbb{F}) \cong (H_n(C_*))^*}$ ← dual v.s.: $\mathrm{Hom}_{\mathbb{F}}(H_n(C_*); \mathbb{F})$

**Pf** Pick any basis $v_i$ for $\mathbb{F}$-v.s. $B_{n-1}$, extend it to a basis $v_i, w_j$ of $Z_{n-1}$ (also works in $\infty$ dim case).

$\Rightarrow$ can extend any $\mathbb{F}$-linear map $\psi : B_{n-1} \to \mathbb{F}$ to $\phi : Z_{n-1} \to \mathbb{F}$ just pick any values $\phi(w_j) \in \mathbb{F}$ e.g. $\phi(w_j) = 0$.

$\Rightarrow B^{n-1}/\mathrm{im}\,\delta^{n-1} = 0$ so $H^n(C_*; \mathbb{F}) \to \mathrm{Hom}(H_n(C_*); \mathbb{F})$ iso □

**Cor** $\boxed{H^n(X; \mathbb{F}) \cong H_n(X; \mathbb{F})^*}$ ← dual v.s. for any field $\mathbb{F}$.

**Cor** $\boxed{H^n(X;M) \cong H^n_{CW}(X;M) \cong H^n_{\Delta}(X;M)}$ (first iso: if $X$ is CW-cx; second: if $X$ is $\Delta$-cx)

**Pf** Cor holds for homology and the isos are natural ← i.e. functorial w.r.t. maps. The universal coeff. thm SES is natural. So result holds by 5-Lemma. □

## Algebra: Structure thm for f.g. abelian groups and R-mods

**Fact 1** $A$ f.g. abelian gp $\Rightarrow A \cong \underbrace{\mathbb{Z}^r}_{\text{free part } F} \oplus \underbrace{\mathbb{Z}/p_1^{n_1} \oplus \cdots \oplus \mathbb{Z}/p_a^{n_a}}_{\text{torsion part } T}$

where $p_i \in \mathbb{Z}$ prime (need not be distinct)
Also $r, a, p_i, n_i$ are unique (up to reordering)

**Example** $\mathbb{Z}/4 = \mathbb{Z}/2^2 \not\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$
$\mathbb{Z}/6 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3$ (Chinese Remainder Thm)

**Fact 2** $T \cong \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_k$ with $d_1 | d_2 | \ldots | d_k$ ($d_i \overset{\neq 0}{\in} \mathbb{N}$ unique)

**Example** $\mathbb{Z}/2 \oplus \mathbb{Z}/2^2 \oplus \mathbb{Z}/3 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3\cdot 2^2$   $d_1 = 2$, $d_2 = 12$

**Fact 3** $M$ f.g. $R$-mod, $R$ PID, then:

$$\boxed{\begin{aligned} M &\cong F \oplus T \\ F &\cong R^r \\ T &\cong R/p_1^{n_1} \oplus \cdots \oplus R/p_a^{n_a} \\ &\cong R/d_1 \oplus R/d_2 \oplus \cdots \oplus R/d_k \end{aligned}}$$

- $r \in \mathbb{N}$ unique, called **rank** of $M$
- $p_i \in R$ primes, $p_i^{n_i}$ unique up to ordering & mult$^n$ by
- $d_1 | \cdots | d_k$ non-zero, not invertible; $d_i$ called **invariant factors**, unique up to mult$^n$ by invertible elements (e.g. $\pm 1$ if $R = \mathbb{Z}$)

**Rmk** $T = \{m \in M : rm = 0 \text{ some } r \neq 0 \in R\}$ = **torsion elements**
$F \cong M/T$

## Torsion shift

**Easy Exercise** $\boxed{\mathrm{Ext}^*_R(\bigoplus_i M_i ; \prod_j N_j) \cong \prod_i \prod_j \mathrm{Ext}^*_R(M_i;N_j)}$ ← any R-mods $M_i, N_j$

↓

**Upshot** To compute $\mathrm{Ext}^1_R(M,R)$ for $M = R^r \oplus R/d_1 \oplus \cdots$ just need:

$$\boxed{\begin{array}{ll} \mathrm{Ext}^1_R(R;R) = 0 \\ \mathrm{Ext}^1_R(R/d;R) \cong R/d \end{array}}$$

- $\mathrm{Ext}^1_R(R;R)=0$ since $0 \to R \xrightarrow{id} R \to 0$ ($P_1$, $P_0$)
- $\mathrm{Ext}^1_R(R/d;R)\cong R/d$ since $0 \to R \xrightarrow{d} R \to R/d \to 0$, $\phi: R \to R$; so choice of $\phi(1) \in R$ modulo $\phi$ coming from $R \xrightarrow{d} R \xrightarrow{\varphi} R$, so $\phi(1) = d\cdot\varphi(1) \in d\cdot R$

$$\Rightarrow \boxed{\mathrm{Ext}^1_R(M;R) \cong \mathrm{Torsion}(M)}$$

**Exercises**

- $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n ; \mathbb{Z}/m) \cong \mathbb{Z}/\gcd(m,n)$
- $G$ abelian gp $\Rightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z};G) = 0$ and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/d;G) \cong G/d\cdot G$ ($d \neq 0$)
- $R$ any ring (comm. with 1), $x \in R$ not zero divisor $\Rightarrow \mathrm{Ext}^*_R(R/(x);N) \cong \begin{cases} \{n \in N : x\cdot n = 0\} & *=0 \\ N/x\cdot N & *=1 \\ 0 & \text{else} \end{cases}$ ($N$ R-mod)

**Cor** If $H_n(X;R)$ f.g. R-mod $\forall n$, $R$ PID,

$\Rightarrow H_n(X;R) = R^{r_n} \oplus T_n$ (free & torsion parts)

$\Rightarrow \boxed{H^n(X;R) \cong R^{r_n} \oplus T_{n-1}}$ ← torsion moves up!

(↑ not natural)

**Pf** $0 \to T_{n-1} \to H^n(X;R) \to \mathrm{Hom}(R^{r_n} \oplus T_{n-1}; R) \to 0$

$\mathrm{Hom}(R^{r_n} \oplus T_{n-1}; R) \cong (\mathrm{Hom}(R;R))^{r_n} \oplus \mathrm{Hom}(T_{n-1};R)$

- $(\mathrm{Hom}(R;R))^{r_n} \cong R^{r_n}$: $R \to R$, $1 \mapsto x$; $x$ determines the hom
- $\mathrm{Hom}(T_{n-1};R) = 0$ since $T_{n-1} \to R$, $1 \mapsto 0$ ($R$ is integral domain, so no torsion elts $\neq 0$)

$\Rightarrow 0 \to T_{n-1} \to H^n(X;R) \to R^{r_n} \to 0$

$R^{r_n}$ free, so can split the SES (pick lifts of basis ← so not canonical). □

**Example**

| $*$ | $H_*(\mathbb{RP}^3)$ | $H^*(\mathbb{RP}^3)$ |
|---|---|---|
| 0 | $\mathbb{Z}$ | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}/2$ | 0 |
| 2 | 0 | $\mathbb{Z}/2$ |
| 3 | $\mathbb{Z}$ | $\mathbb{Z}$ |

($\mathbb{Z}/2$ in $H_1$ → $\mathbb{Z}/2$ in $H^2$) torsion moves up

## Universal coefficients Theorem in homology

(recall $H_*(C_* \otimes_R M) = H_*(C_*;M)$)

**FACT** **Theorem** $C_*$ chain cx of free $R$-mods ($R$ PID), $M$ $R$-module

$$\Rightarrow \boxed{\begin{array}{l} \text{natural SES } 0 \to H_*(C_*) \otimes_R M \to H_*(C_* \otimes_R M) \to \mathrm{Tor}^R_1(H_{*-1}(C_*); M) \to 0 \\ \qquad [c] \otimes m \mapsto [c \otimes m] \\ \text{The SES splits, but the splitting is not natural.} \end{array}}$$

($\mathrm{Tor}$ defined below.)

**Torsion groups**: $A, B$ R-mods ($R$ comm. ring with 1)

pick $\cdots \to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} A \to 0$ free resolution (exact sequence, $P_i$ free R-mods)

take $\cdot \otimes B$, omit $A \otimes B$

$\Rightarrow \cdots \to P_2 \otimes B \xrightarrow{\varphi_2 \otimes id} P_1 \otimes B \xrightarrow{\varphi_1 \otimes id} P_0 \otimes B \to 0$ not exact but is chain cx

$\mathrm{Tor}^R_k(A,B) = H_k$ (this complex) ← fact: independent of choices of $P_i, \varphi_i$

**Rmk** $R$ PID $\Rightarrow \ker\varphi_0$ free $\Rightarrow$ pick $\begin{cases} P_1 = \ker\varphi_0 \\ P_k = 0 \text{ for } k \geq 2 \end{cases}$ $\Rightarrow$ only $\mathrm{Tor}^R_0, \mathrm{Tor}^R_1$ can be non-zero

**Example** $\mathrm{Tor}^{\mathbb{Z}}_*(\mathbb{Z}/a, \mathbb{Z}/b) = ?$

$0 \to \mathbb{Z} \xrightarrow{\cdot a} \mathbb{Z} \xrightarrow[\text{quotient}]{\varphi_0} \mathbb{Z}/a \to 0$ free resolution

take $\cdot \otimes \mathbb{Z}/b$, drop $\mathbb{Z}/a \otimes \mathbb{Z}/b$

$\Rightarrow 0 \to \mathbb{Z}/b \xrightarrow{\cdot a} \mathbb{Z}/b \to 0$ (since $\mathbb{Z} \otimes_{\mathbb{Z}} G \cong G$ any $G$)

$\mathrm{Tor}^{\mathbb{Z}}_0(\mathbb{Z}/a, \mathbb{Z}/b) = (\mathbb{Z}/b)/a\cdot\mathbb{Z}/b \cong \mathbb{Z}/\langle a,b\rangle \cong \mathbb{Z}/\gcd(a,b) \cong \mathbb{Z}/a \otimes \mathbb{Z}/b$

$\mathrm{Tor}^{\mathbb{Z}}_1(\mathbb{Z}/a; \mathbb{Z}/b) = \{x \in \mathbb{Z}/b : a\cdot x = 0\} \cong \mathbb{Z}/\gcd(a,b)$ via: $\frac{b}{\gcd(a,b)} \leftarrow 1$

**Facts** $\mathrm{Tor}^R_0(A,B) = P_0 \otimes B / \mathrm{Im}(\varphi_1 \otimes id) \cong A \otimes B$

$\mathrm{Tor}^R_*(A,B) \cong \mathrm{Tor}^R_*(B,A)$

**Exercise** $\mathrm{Tor}^R_*(\bigoplus A_i, \bigoplus B_j) \cong \bigoplus_i \bigoplus_j \mathrm{Tor}^R_*(A_i, B_j)$

$\mathrm{Tor}^R_*(A,B) = 0$ for $* \geq 1$ if $A$ or $B$ is free (use $M \otimes_R R \cong M$)

$\mathrm{Tor}^R_*(R/u; M) \cong \begin{cases} M/u\cdot M & *=0 \\ u\text{-torsion}(M) = \{x \in M : u\cdot x = 0\} & *=1 \\ 0 & \text{else} \end{cases}$

($u \in R$ not zero divisor, $R$ any ring (comm. with 1))

⇓ deduce $\mathrm{Tor}^R_1(A,M)$ for f.g. R-mods $A$ ($R$ PID)

**Example** $H_*(\mathbb{RP}^2; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & *=0 \\ \mathbb{Z}/2 & *=1 \\ \mathbb{Z}/2 & *=2 \end{cases}$ $\quad H_*(\mathbb{RP}^2) \otimes \mathbb{Z}/2 \cong \begin{cases} \mathbb{Z} \otimes \mathbb{Z}/2 \\ \mathbb{Z}/2 \otimes \mathbb{Z}/2 \\ 0 \end{cases} \cong \begin{cases} \mathbb{Z}/2 \\ \mathbb{Z}/2 \\ 0 \end{cases}$

caused by $\mathrm{Tor}^{\mathbb{Z}}_1(H_1(\mathbb{RP}^2); \mathbb{Z}/2) = \mathrm{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$

**Künneth Thm**

$R$ PID ($C_*$ free ch. cx. R-mods, $D_*$ any ch. cx. R-mods) $\Rightarrow$ natural SES:

$$\boxed{0 \to \bigoplus_{i+j=n} H_i(C_*) \otimes H_j(D_*) \to H_n(C_* \otimes D_*) \to \bigoplus_{i+j=n-1} \mathrm{Tor}^R_1(H_i(C), H_j(D)) \to 0}$$

and the SES splits but the splitting is not natural.

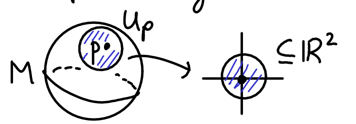
**Example** $R = $ field $\Rightarrow$ that $\mathrm{Tor}_1 = 0$

**Example** (take $C_* := C^{-*}(X)$, $D_* := C^{-*}(Y)$): $0 \to H^*(X) \otimes H^*(Y) \to H^*(X \times Y) \to \mathrm{Tor}_1(H^*(X), H^*(Y)) \to 0$ exact (degree increases by 1)

(if $C_*(X)$ has $\infty$ rank then $C^{-*}(X)$ may not be free but it will be "flat" and Thm holds if $C_*$ is flat R-mod)

# 13. MANIFOLDS : POINCARÉ–LEFSCHETZ DUALITY

- $M$ **n-mfd** is Hausdorff topological space s.t. $\forall p \in M$ $\exists$ open neighbourhood $U_p \subseteq M$ homeomorphic to $\mathbb{R}^n$ (equivalently: to an open ball, or any open set in $\mathbb{R}^n$)

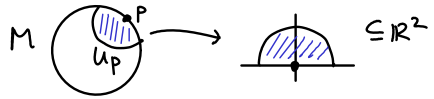



One also requires $M$ **second countable** i.e. $\exists$ countable basis of open sets
$\Longleftrightarrow$ $M$ is covered by countably many such $U_{p_i}$ (exercise)

A **submanifold** $N \subseteq M$ is a mfd s.t. inclusion $N \to M$ is an embedding (i.e. a homeomorphism onto its image)

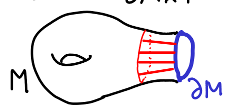
- $M$ **n-mfd with boundary** if also allow $U_p \cong$ **upper half space** $\mathbb{H}^n$ $= \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0} = \{x \in \mathbb{R}^n : x_n \geq 0\}$, $p \longmapsto 0$ (equivalently: any open nbhd of $0 \in \mathbb{H}^n$)

Such $p$ are called **boundary points**; they form the **boundary $\partial M$** which is an $(n-1)$-mfd without boundary.




FACT (collar nbhd thm) $\partial M \subseteq M$ has an open neighbourhood $\cong \partial M \times (0,1]$, $\partial M \to \partial M \times 1$

$M$ is **closed** if compact without boundary.




Rmk For manifolds, connected components = path components.
(since locally $\cong$ disc, so locally path-connected, so conn. $\Leftrightarrow$ path-conn.)

Examples

closed mfds : $S^n$, $\mathbb{RP}^n$, $T^n = S^1 \times \cdots \times S^1$ (**n-torus**), $\mathbb{CP}^n$, $O(n)$, $SU(n)$

non-compact mfds: $\mathbb{R}^n$, $Mat_{m\times n} \cong \mathbb{R}^{mn}$, $GL(n,\mathbb{R})$

mfds with bdry: $\mathbb{D}^n$, $\mathbb{D}^1 \times S^1 =$, Möbius band $=$ (picture), $T^2 \setminus$ open disc $=$ (picture)

FACT (Milnor 1959) Any mfd is homotopy equivalent to a CW-cx

fact If $M$ is a compact manifold then $H_*(M)$ are finitely generated

Rmk $M$ **triangulable** if $M \cong$ simplicial cx.
Not all mfds are triangulable, but most of those we encounter are.

## Compact manifolds have f.g. homology ← Non-examinable proof

① $X$ space is a **Euclidean neighbourhood retract** if $\exists$ **embedding** $j: X \to \mathbb{R}^N$ (homeo onto image) some $N$, s.t. $i(X)$ is a retract of a nbhd $V \subseteq \mathbb{R}^N$.

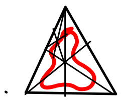
② $X$ is **weakly locally contractible** if $\forall$ nbhd $x \in U \subseteq X$, $\exists$ nbhd $x \in V \subseteq U$ s.t. $V$ is contractible inside $U$.

FACT compact $X \subseteq \mathbb{R}^n$ is ① $\Longleftrightarrow$ $X$ is ②

Rmk If we find nbhd $V$ as in ① with retraction $V \xrightarrow{f} X$ then any smaller nbhd $V'$ also retracts using $f|_{V'}: V' \to X$. Similarly in ② $V' \subseteq V$ is contractible: restrict the hey.

Lemma A $X$ compact & ① $\Rightarrow$ $X$ is the retract of a finite simplicial cx

pf $i(X) \subseteq \mathbb{R}^n$ compact $\Rightarrow$ lies inside some large n-simplex $\Delta^n \to \mathbb{R}^n$



Apply barycentric subdivision until simplices have diameter $<$ dist$(X, \partial V)$.
Simpl.cx. $= \bigcup\{$subsimplices which intersect $X\}$ using the restriction of retraction $V \to X$. □

Rmk Also deduce $X$ has f.g. homology since retractions are surjective on $H_*$.
($\oplus \mathbb{Z} \twoheadrightarrow H_*$(finite simpl. cx) $\xrightarrow{\text{retract}}$ $H_*(X)$ so get surjection from free $\mathbb{Z}$-mod, so f.g.)

Lemma B $M$ compact mfd $\Rightarrow$ $M$ embeds into $\mathbb{R}^N$, some $N$.

Pf "Just do it proof":
$\forall p \in M$, $\exists$ homeo $\mathbb{D}^n \xrightarrow{\varphi_p}$ nbhd$(p \in M)$
Pick finite subcover of $\varphi_{p_i}$ of $M = \bigcup_{p \in M} \varphi_p(\mathbb{D}^n)$. Say $i = 1, \ldots, k$
$\psi_{p_i} : M \xrightarrow{\varphi_{p_i}^{-1}} \mathbb{D}^n / \partial\mathbb{D}^n \cong S^n \subseteq \mathbb{R}^{n+1}$ define embedding $(\psi_{p_1}, \ldots, \psi_{p_k}): M \to \mathbb{R}^{k\cdot(n+1)}$
(send $M \setminus Im(\varphi_{p_i})$ to the point corresponding to $\partial\mathbb{D}^n \in \mathbb{D}^n/\partial\mathbb{D}^n$).
Finally use: a continuous bijection from a compact space to a Hausdorff space is $\cong$ □

Rmk Same works if $M$ has boundary, just consider its **double** $M \cup M$ (identify along $\partial M$) and apply the Lemma to the double.

Cor $M$ compact mfd (possibly with bdry) $\Rightarrow$ $M$ has f.g. homology

Pf Mfds satisfy ② since locally ball $\simeq$ pt. $M$ embeds in $\mathbb{R}^N$ by Lemma B.
① holds by FACT. Done by Lemma A. □

## Local orientations and orientability

**Def** A **local orientation** of $M$ at $x \in M$ is a choice of generator

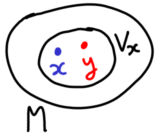
$$\mu_x \in H_n(M, M\setminus x) \cong H_n(\mathbb{D}^n, \mathbb{D}^n \setminus 0) \cong \tilde{H}_n(S^n) \cong \mathbb{Z}$$

(excise complement of nbhd $V_x \cong \mathbb{D}^n$ ← choice of homeo is not canonical!; $\mathbb{D}^n \setminus 0 \simeq \partial\mathbb{D}^n = S^{n-1}$)

(see section 5 of these notes) → $\cong \tilde{H}_n(S^n)$

**Def** An **orientation** of $M$ is a **locally consistent** choice $x \mapsto \mu_x$

meaning:




$$H_n(M, M\setminus V_x) \quad (V_x \cong \mathbb{D}^n \simeq pt)$$

$\swarrow \cong$ $\quad$ $\cong \searrow$ ← quotient maps

$$H_n(M, M\setminus x) \qquad H_n(M, M\setminus y)$$

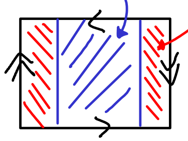
$\exists\, a \mapsto \mu_x$, $a \mapsto \mu_y$

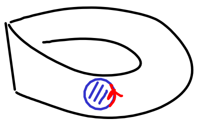
**Def** $M$ **orientable** if $\exists$ orientation on $M$

**oriented** if we chose an orientation

**Examples** $S^n, \mathbb{R}^n, \mathbb{C}P^n$, orientable surfaces $\Sigma_g$, $\mathbb{R}P^n$ ← odd $n$

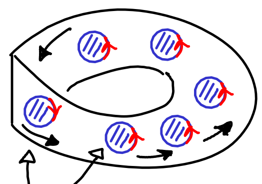
**Non-example** $\mathbb{R}P^2 =$ Möbius band $\cup\ \mathbb{D}^2$





choice of $\mu_x$ is choice of orientation of boundary circle of small disc containing $x$

by local consistency can move disc continuously and preserves orientation



discs are differently oriented
$\Rightarrow$ contradicts local consistency.

$\Longrightarrow \mathbb{R}P^2$ not orientable

## The fundamental class $[M]$

~~FACT~~ **Theorem** For $M$ closed $n$-mfd:

1) $M$ orientable connected $\Longrightarrow$ $H_n(M) \cong H_n(M, M\setminus x) = \mathbb{Z}\cdot\mu_x$ (← natural map from the LES; ← once we choose an orientation)

$\Longrightarrow$ (once we choose an orientation $(\mu_x)_{x\in M}$) $\exists\, [M] \longleftarrow \mu_x$

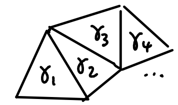
↑ called **fundamental class**

(if swap orientation: for $-\mu_x$ get $-[M]$)

2) $M$ not orientable connected $\Longrightarrow$ $H_n(M) = 0$

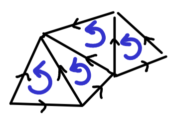
$H_n(M; \mathbb{F}_2) = \mathbb{F}_2 \cdot [M] \cong \mathbb{F}_2$ ← (or any field of characteristic 2)

**Construction of $[M]$ if $M$ has $\Delta$-complex structure**




$M$ compact $\Longrightarrow$ finite # $n$-simplices $\gamma_1, \dots, \gamma_N$

$M$ oriented $\Longrightarrow$ pick orientations of $\gamma_1, \dots, \gamma_N$ to agree with given orientation of $M$: (for $x \in \mathrm{Int}(\gamma_i)$)



$$\mathbb{Z}\cdot\mu_x = H_n(M, M\setminus x) \underset{exc.}{\cong} H_n(\gamma_i, \gamma_i \setminus x) = \mathbb{Z}\cdot\gamma_i$$

$$\mu_x \longmapsto \gamma_i$$

$\Longrightarrow$ $\boxed{[M] := \sum \gamma_i}$ satisfies $\partial[M] = 0$ ✓
(each facet arises twice with opposite signs)

$$H_n(M) \longrightarrow H_n(M, M\setminus x) \cong H_n(\gamma_i, \gamma_i\setminus x)$$

$$[M] \qquad \mu_x \longmapsto \gamma_i$$

More generally: $[M] := \sum \pm \gamma_i$ where signs come from $H_n(M, M\setminus x) \cong H_n(\gamma_i, \gamma_i\setminus x)$, $\mu_x \longmapsto \pm\gamma_i$ (so compare orientation $\mu_x$ with orientation of $\gamma_i$)

Not difficult to see that $H_n^{\Delta}(M) = \mathbb{Z}\cdot[M]$, so $\Rightarrow H_n(M) \cong H_n(M, M\setminus x)$, $[M] \longmapsto \mu_x$

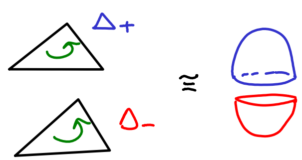
Also $\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}\cdot[M]$ since $C_{n+1}(M) = 0$ ($\nexists (n+1)$-simplices since $\dim M = n$)

$M$ non-orientble $\Rightarrow$ each facet of $\gamma_i$ appears twice in $\partial \sum \gamma_i$

$\Rightarrow$ $\partial \sum \gamma_i = 0$ over $\mathbb{F}_2$ independently of choices of orientations of $\gamma_i$. ✓

# Examples

1) $S^n = \Delta^n_{+} \cup \Delta^n_{-} \,/\,$ glue bdries

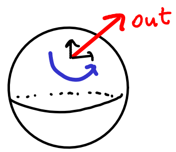
$S^2$: 

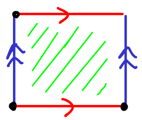

$[S^n] = \Delta_+ - \Delta_-$ if use canonical orientation we discussed

hence $\partial[S^n] = \partial\Delta - \partial\Delta = 0$

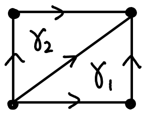
$\mathbb{D}^n \subseteq \mathbb{R}^n$ canonical orientation

$\Rightarrow S^{n-1} = \partial \mathbb{D}^n$ " using outward normal first rule

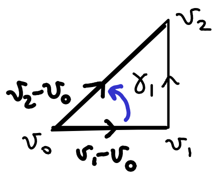



2) $T^2 =$ 

$\Delta$-complex structure (compatibly with side identifications!)

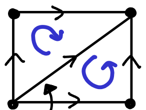



Want orientation induced by square $\subseteq \mathbb{R}^2$

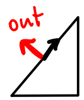



$v_1 - v_0$, $v_2 - v_0$ positive $\mathbb{R}^2$-basis

$\Rightarrow \gamma_1$ agrees with orientation

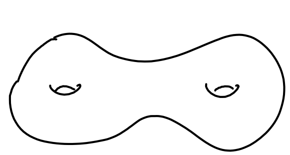
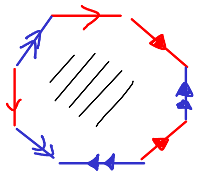
$\Longrightarrow$  $[T^2] = +\gamma_1 - \gamma_2$

($\gamma_2$ orientation disagrees)

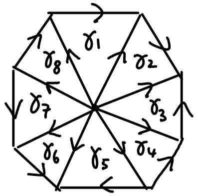



Rmk general feature: if two simplices are identified along a facet then the 2 respective outward normals are related by reflection.

So consistency $\Rightarrow$
- either simplices are compatibly oriented and the two induced orientations on facet are opposite
- or not compatibly oriented but facet orient$^n$ is same, then need sign like in example when build $[T^2]$

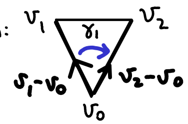
3) Recall $\Sigma_2 =$  $=$ 

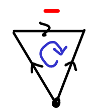
$\Delta$-cx structure (compatible with side identifications!):




Use the orientation induced by polygon $\subseteq \mathbb{R}^2$

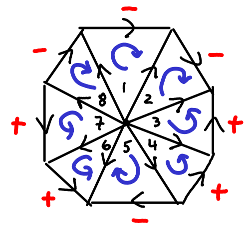
$\Rightarrow [\Sigma_2] = -\gamma_1 - \gamma_2 + \gamma_3 + \gamma_4 - \gamma_5 + \gamma_6 + \gamma_7 - \gamma_8$

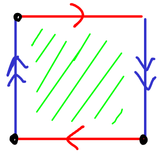
Explanation: 


$v_1 - v_0, v_2 - v_0$ is negative $\mathbb{R}^2$-basis $\Rightarrow$ 

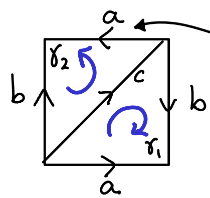

All simplices $\gamma_i$ have $v_0 =$ centre of polygon

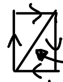
$\Rightarrow$ sign $<$ − if outer edge clockwise; + if outer edge anti-clockwise




3) $\mathbb{RP}^2 =$ 

(non-orientable example)




won't get $\Delta$-cx structure if you try  since get issue here

Use the orientation induced by square $\subseteq \mathbb{R}^2$

$\Rightarrow [\mathbb{RP}^2] = -\gamma_1 + \gamma_2$

$\partial[\mathbb{RP}^2] = -(b - a + c) + (a - b + c)$

$= -2b + 2a$

$\neq 0$ so not cycle in $C^{CW}_*(\mathbb{RP}^2)$

However, working modulo 2:

$\partial[\mathbb{RP}^2] = 0 \in C^{CW}_*(\mathbb{RP}^2; \mathbb{F}_2)$ since $2 = 0$ in $\mathbb{F}_2$

$\Rightarrow [\mathbb{RP}^2] \in H^2(\mathbb{RP}^2; \mathbb{F}_2)$

# Degree

<u>Def</u> $M, N$ oriented closed connected n-mfds, $f: M \to N$

$f_*: H_n(M) \longrightarrow H_n(N)$

$[M] \mapsto \underbrace{\deg(f)}_{\in \mathbb{Z}} \cdot [N]$

<u>Lemma</u> If $f^{-1}(y)$ finite,

then $\deg(f) = \sum_{x \in f^{-1}(y)} \deg(f_y)_*$ ← <u>local degree</u>, <u>local map</u> like in chapter 7

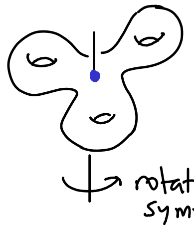
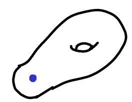
<u>Pf</u>

$$\begin{array}{ccccc} [M] \in & H_n(M) & \xrightarrow{f_*} & H_n(N) & \ni [N] \\ \downarrow & \downarrow & & \wr\wr & \uparrow \\ & \bigoplus_{x \in f^{-1}(y)} H_n(M, M\setminus x) & \xrightarrow{(f_x)_*} & H_n(N, N\setminus y) & \ni \mu_y^N \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \\ \oplus \mu_x^M & & \longmapsto & (\sum \deg(f_x)_*) \cdot \mu_y^N & \end{array}$$

□

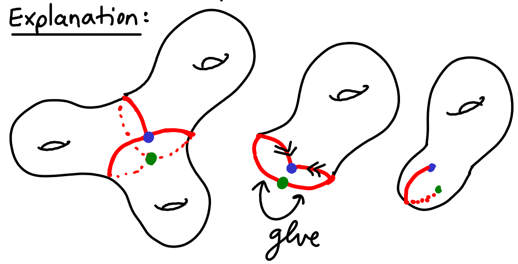
<u>Examples</u>

1) $S^1 \longrightarrow S^1$, $z \longmapsto z^n$, $[S^1] \longmapsto n \cdot [S^1]$ so $\deg = n$

2) $\Sigma_3 =$  $\xrightarrow[q]{\text{quotient}}$ $\Sigma_3 / \mathbb{Z}/_3$-rotation action $=$  $= \Sigma_1$ torus



<u>Explanation:</u>




Easy check: $\deg(q) = 3$
(e.g. use local degrees)

<u>Cultural Rmk</u>

For $M, N, f$ <u>smooth</u>, the $\deg f = \#$(preimages of a <u>generic</u> point of N)

Idea: $\deg f$ tells you how many times you cover N. (almost all points work)

# Poincaré duality

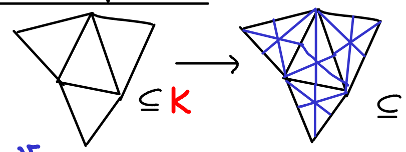
<u>FACT</u> <u>Theorem</u> For M closed n-mfd

M oriented $\Longrightarrow$ $\boxed{H^k(M) \cong H_{n-k}(M)}$ s.t. $1 \leftrightarrow [M]$, $H^0(M) \cong H_n(M)$

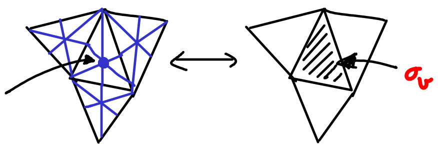
M non-oriented $\Longrightarrow$ same holds with $\mathbb{F}_2$ coefficients

<u>Sketch proof when M is a simplicial complex K</u> (Non-examinable)

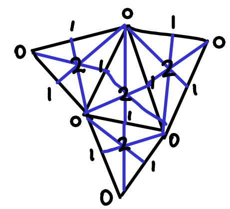
$S(K)$ = barycentric subdivision




1) simplex $\sigma = \sigma_v$ of K with barycentre $v$ $\longleftrightarrow$ $v = v_\sigma$ vertex of $S(K)$




2) $ht(v) = (\underline{\text{height}} \text{ of } v) = \dim \sigma_v$

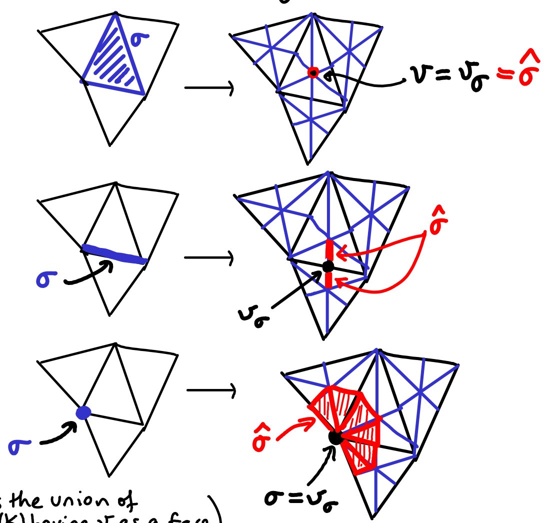



3) $\sigma$ k-simplex of K

↓

<u>dual simplex</u>

$\hat{\sigma} = \bigcup_{\substack{\tau \in S(K),\ v_\sigma \in \tau, \\ ht(v_\sigma) \text{ is min} \\ \text{of heights of} \\ \text{vertices of } \tau}} \tau$




<u>Rmk</u>: $\bigcup \tau$ with $ht(v_\sigma)$ max will give back $\sigma$.

Thus $\hat{\sigma}, \sigma$ intersect transversely at $v_\sigma$.

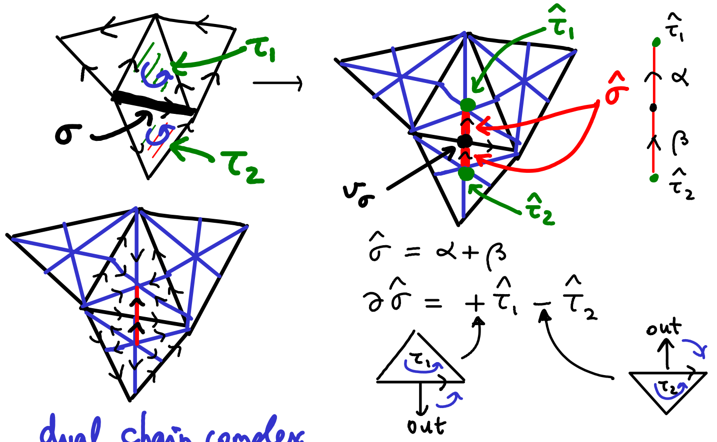
One can also describe $\hat{\sigma}$ as

$\hat{\sigma} = \bigcap_{\text{vertices } v \in \sigma} \overline{\text{Star}}_{S(K)}(v)$ ← (closed star is the union of simplices of S(K) having $v$ as a face)

FACTS • $\dim \hat\sigma = n - \dim \sigma$

• dual cells $\hat\sigma$ give a cell decomposition of $M$ ("polygonal" complex rather than Δ-cx)

(★) • $\partial\hat\sigma = \sum_{\substack{\text{facet } \sigma \subsetneq \tau \\ \tau \in K}} \pm \hat\tau$ ← need compare orientations of $\sigma,\tau$ (+ if $\sigma$ as a facet of $\tau$ has boundary orientation)




$\hat\sigma = \alpha + \beta$

$\partial\hat\sigma = +\hat\tau_1 - \hat\tau_2$

4) **dual chain complex**

**$D_{n-k}$** = free abelian group on dual chains $\hat\sigma$

$H_*(M) \cong H_*(D_*, \partial_*)$ (since $\hat\sigma$ give a cell decomp. of $M$)

5) $\varphi: D_{n-k} \longrightarrow C^k(M)$
$\hat\sigma \longmapsto \sigma^*$ where $\sigma^*(\alpha) = \begin{cases} 0 & \text{for } k\text{-cells } \alpha \neq \sigma \\ 1 & \text{for } \alpha = \sigma \end{cases}$

Rmk notice that $\sigma^*(\alpha) = \#\alpha$ intersects $\hat\sigma$ counted with orientation signs.

• $\varphi$ linear bijection ✓

• chain map:

$\varphi(\partial\hat\sigma) = \varphi \sum \pm \hat\tau = \sum \pm \tau^*$ (see ★)

$\partial^*\varphi(\hat\sigma) = \partial^*\sigma^* = \left(\sigma^* \circ \partial : \tau \overset{\partial}{\longmapsto} \sum_{\text{facets } \sigma_i \text{ of } \tau} \pm\sigma_i \overset{\sigma^*}{\longmapsto} \begin{cases} \pm 1 & \text{if one of } \sigma_i = \sigma \\ 0 & \text{else} \end{cases}\right)$

$= \sum \pm \tau^* = \varphi(\partial\hat\sigma)$ ✓

UPSHOT $\varphi$ is chain iso so get iso:
$H_*(M) \underset{(4)}{\cong} H_*(D_*, \partial_*) \xrightarrow[\varphi]{\cong} H^{n-*}(M)$

Cor $\chi$(odd dimensional closed orientable mfd) $= 0$

Pf **Betti numbers** $b_i$ = rank $H_i(M)$ = rank $H^i(M)$ = rank $H_{n-i}(M)$ (universal coeff. thm.; Poincaré duality)

$\chi(M) = b_0 - b_1 + \dots + b_{\dim M - 1} - b_{\dim M}$ (equal pairs) equal. □

## (Poincaré–)Lefschetz duality

Theorem

$M$ compact oriented $n$-mfd with boundary ⟹

$H^k(M) \cong H_{n-k}(M, \partial M)$

$1 \in H^0(M) \longleftarrow [M, \partial M] \in H_n(M, \partial M)$ **relative fundamental class**

$H_k(M) \cong H^{n-k}(M, \partial M)$

Non-oriented ⟹ same holds with $\mathbb{F}_2$ coefficients.

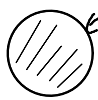
Pf basically same as Poincaré duality. □

Cor $M$ compact, connected, $\partial M \neq \emptyset$ ⟹ $\begin{cases} H^n(M) \cong H_0(M, \partial M) = 0 \\ H_n(M) \cong H^0(M, \partial M) = 0 \end{cases}$

either by universal coefficient thm since $H_0(M,\partial M) = 0$ or by hand since given $p \in M$, $\varphi \in C^0(M, \partial M)$ consider $(\partial\varphi)(\gamma)$ for $\gamma$ path from $p$ to any $q \in \partial M$.

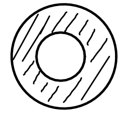
Examples

1) $\mathbb{D}^n$




$\mathbb{Z} \cong H^0 \mathbb{D}^n \cong H_n(\mathbb{D}^n, S^{n-1})$

2)



$A$ = annulus $\subseteq \mathbb{R}^2$ $\simeq S^1$

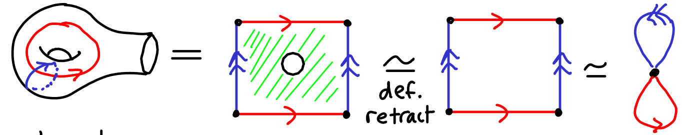
$\mathbb{Z} \cong H^0A \cong H_2(A, \partial A)$ ← generator $D_1 - D_2$

$\mathbb{Z} \cong H^1A \cong H_1(A, \partial A)$ ← generator (copy of $\mathbb{D}^1$) (notice $\partial\mathbb{D}^1 \to \partial A$)

$0 \cong H^2A \cong H_0(A, \partial A)$

Rmk notice gen. of $H_1(A)$ is ◎ which intersects gen. of $H_1(A, \partial A)$ once transversely.

3) $M = T^2 \setminus$ open ball $=$




$\simeq S^1 \vee S^1$

$\Rightarrow H_*(M, \partial M) \cong H^{2-*}(S^1 \vee S^1) = \begin{cases} \mathbb{Z}^2 & * = 1 \\ \mathbb{Z} & * = 2 \\ 0 & \text{else} \end{cases}$ ← gen. by 2 loops; ← gen. by $[M, \partial M]$

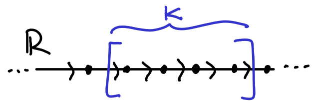
## What happens in the non-compact case?

**Locally finite homology (Borel-Moore)**

$C_*^{lf}(X)$ allow infinite sums $\sum n_i \sigma_i$ ($n_i \in \mathbb{Z}$, $\sigma_i$ generators of $C_*(X)$) s.t. given any compact subset $K \subseteq X$, $\#\{n_i \neq 0 : K \cap \mathrm{Im}\, \sigma_i \neq \emptyset\} < \infty$.

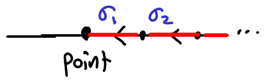
Examples

- $C_1^{lf}(\mathbb{R}) \ni \sum_{m \in \mathbb{Z}} \sigma_m$, $\sigma_m : I \cong [m, m+1] \subseteq \mathbb{R}$




$\Rightarrow$ get cycle $[\mathbb{R}] \in H_1^{lf}(\mathbb{R})$

- $C_0^{lf}(\mathbb{R}) \ni [\text{point}] \ni \partial(\sigma_1 + \sigma_2 + \cdots)$ is a boundary:




exercise $H_*^{lf}(\mathbb{R}) = \begin{cases} \mathbb{Z} & *=1 \\ 0 & \text{else} \end{cases}$ $(\cong H^{1-*}(\mathbb{R}))$

FACT Theorem M orientable n-mfd (possibly not compact) $\Rightarrow$ $\boxed{H^*(M) \cong H_{n-*}^{lf}(M)}$

**cohomology with compact supports** $H_c^*(X)$

$C_c^*(X)$: only allow cochains $\phi : C_* X \to \mathbb{Z}$ s.t. $\exists$ compact $K \subseteq X$ (depends on $\phi$) with

$\phi(C_*(X \setminus K)) = 0$ (vanish on chains in $X \setminus K$)

Example $c \in C_*(X)$ cycle $\Rightarrow$ $\phi(\alpha) =$ signed # intersections of $c$ with $\alpha$ (geometric intersection #)

$\Rightarrow \phi \in C_c^*(X)$ since $\phi(\alpha) = 0$ if $\alpha \subseteq X \setminus \mathrm{Im}(c)$

Thm M orientable n-mfd (possibly not compact) $\Rightarrow$ $\boxed{H_*(M) \cong H_c^{n-*}(M)}$

Warning $H_*^{lf}, H_c^*$ are not homotopy invariant (indeed non-trivial for $\mathbb{R}^n$)

Caused because they are not functorial. They are however functorial for **proper maps** (preimages of compact sets are compact)

Mayer-Vietoris holds for $H_c^*$ but not for $H_*^{lf}$.

Fact $H_c^*(X) \cong \varinjlim H^*(X, X \setminus K)$ where compacts $K_1 \subseteq K_2$ give $H^*(M, M\setminus K_1) \to H^*(M, M \setminus K_2)$

**Direct limit** $\varinjlim G_i$ via maps $G_i \to G_j$ means $\bigsqcup G_i /$ identify $g \in G_i$ with its images under those maps

(The indices are partially ordered & directed: $\forall i,j, \exists k \geq i,j$ so can compare $G_i, G_j$ inside $G_k$ (via $G_i \to G_k$, $G_j \to G_k$))

Fact $\varinjlim$ is an exact functor.

## Cap product and Poincaré duality revisited

X space, $k \geq \ell$

$\cap : C_k(X) \times C^\ell(X) \longrightarrow C_{k-\ell}(X)$ **cap product** (sometimes write $\phi \cap \sigma$)

$(\sigma : \Delta^k \to X) \cap (\phi : C_\ell(X) \to \mathbb{Z}) = \phi(\sigma|_{[e_0,\dots,e_\ell]}) \cdot \sigma|_{[e_\ell,\dots,e_k]}$

where $\sigma|_{[e_0,\dots,e_\ell]}$ is the "bottom face" ($\phi(\cdot) \in \mathbb{Z}$) and $\sigma|_{[e_\ell,\dots,e_k]}$ is the "top face" $\cong \Delta^{k-\ell}$, $\in C_{k-\ell}(X)$

(easy) Properties

- $\cap$ bilinear
- $\partial(\sigma \cap \phi) = (-1)^\ell (\partial\sigma \cap \phi - \sigma \cap \partial^* \phi)$
- cycle $\cap$ cocycle is cycle
- boundary $\cap$ cocycle, cycle $\cap$ coboundary are boundaries

$\Rightarrow$ $\boxed{\cap : H_k(X) \times H^\ell(X) \longrightarrow H_{k-\ell}(X)}$ bilinear

**Theorem (Poincaré duality)** The map $\boxed{\phi \longmapsto [M] \cap \phi}$ gives following isos

① For M closed oriented n-mfd

$[M] \cap \cdot : H^*(M) \xrightarrow{\cong} H_{n-*}(M)$

② For M non-compact oriented n-mfd,

$[M] \cap \cdot : H_c^*(M) \xrightarrow{\cong} H_{n-*}(M)$ ⊛

$[M] \cap \cdot : H^*(M) \xrightarrow{\cong} H_{n-*}^{lf}(M)$

**Sketch Pf of ① for smooth mfds** (Non-examinable)

If M smooth $\Rightarrow$ $\exists$ "good cover" $U_i$ of M meaning open cover s.t.

FACT from Riemannian geometry ("convex neighbourhoods")

$U_i \cong \mathbb{R}^n$

$U_{i_1} \cap \dots \cap U_{i_k} \cong \mathbb{R}^n$ or $\emptyset$

Then compute $H_c^*(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z} & *=n \\ 0 & \text{else} \end{cases}$ and ⊛ holds for $\mathbb{R}^n$.

$\Rightarrow$ ⊛ holds $\forall U_i$

$\Rightarrow$ by naturality of ⊛ and of Mayer-Vietoris get ⊛ for $\bigcup_{\text{finite}} U_i$ (use 5-lemma)

$\Rightarrow$ ⊛ for M, which is ①. □

## General Pf of Poincaré duality ← Non-examinable

**Step 1** : holds for $\mathbb{R}^n$

Pf $H^k_c(\mathbb{R}^n) \cong H^k(D^n, S^{n-1}) \cong \begin{cases} 0 & k \neq n \\ \mathbb{Z} & k = n \end{cases} \cong H_{n-k}(\mathbb{R}^n)$

(recall Fact: $H^*_c(X) \cong \varinjlim H^*(X, X\setminus K)$ ← can make $K$ larger by picking $K$ = large ball then $H^k(\mathbb{R}^n, \mathbb{R}^n \setminus K) \cong H^k(K, \partial K)$)

Pick $\Delta$-cx structure for $\mathbb{R}^n$. So $[\mathbb{R}^n] = \sum \pm \sigma_i$ ← sum over $n$-simplices.
Say $\exists$ simplex $\sigma_0 : \Delta^n \to \mathbb{R}^n$. Define $\phi : C^{cw}_c(\mathbb{R}^n) \to \mathbb{Z}$, $\phi(\sigma_0) = \pm 1$, $\phi(\text{other simplices}) = 0$ ⊛
$\Rightarrow \delta\phi = 0$ for dim reasons
$[\mathbb{R}^n] \cap \phi = \sum \phi(\pm\sigma_i) = \phi(\pm\sigma_0) = 1$ (pick sign in ⊛)

**Step 2** holds for $A, B, A\cap B \Rightarrow$ holds for $A\cup B$

Pf Mayer-Vietoris for $H^*_c$, naturality, 5-lemma ✓

**Step 3** holds for $A_i$, and $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \Rightarrow$ holds for $\cup A_i$

Pf By applying $\varinjlim$ : both sides of P.D. iso commute with limits ✓

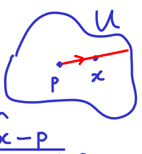
**Step 4** holds for open subsets in $\mathbb{R}^n$

Pf Every such set is a union of <u>convex</u> open sets (e.g. balls)

By Step 3 enough to consider case of finite union.

By <u>induction</u> on # convex open sets:

1 convex set $U \cong \mathbb{R}^n$ via a <u>proper</u> homeomorphism, (fix $p \in U$; $U \to \mathbb{R}^n$, $x \mapsto \frac{x-p}{\text{dist } p \text{ to } \partial U \text{ along ray to } x}$)




now use Step 1 ✓

2 convex sets : <u>KEY TRICK</u> convex set ∩ convex set is convex in $\mathbb{R}^n$!
⇒ use Step 2 & previous case

k+1 convex sets: $A = \bigcup\{$first $k$ convex sets$\}$, $B$ = last convex set
⇒ $A\cap B \subseteq B$ is a union of $k$ convex sets } ⇒ use step 2 & Inductive hypothesis ✓

**Step 5** holds for mfd $M$

Consider open sets in $M$ for which it holds.
By a <u>Zorn's Lemma</u> argument we get a maximal open subset $U$ where holds.
If $U \neq M$ pick $p \in M\setminus U$ and nbhd $V \cong \mathbb{R}^n$ of $p$. Then holds for $U, V, U\cap V$
(note $U\cap V \subseteq V \cong \mathbb{R}^n$ open, so Step 4 applies) so by step 2 holds for $U\cup V$
Contradicts maximality. ✓ □

Recall there is a well-defined evaluation of $H^*$-classes on $H_*$:

$\langle\cdot,\cdot\rangle : H_k(M;R) \otimes H^k(M;R) \longrightarrow R$
$c \otimes \alpha \longmapsto \langle c, \alpha\rangle = \varphi(c)$ ← any representative cocycle $\varphi$ for $\alpha$

<u>Easy exercise</u> $\boxed{\langle c, \alpha\cup\beta\rangle = \langle c\cap\alpha, \beta\rangle}$ any $\alpha, \beta \in H^*$, $c \in H_*$

## Corollary of Poincaré duality

$M$ compact oriented n-mfd, $\mathbb{F}$ field.

$\Rightarrow$ $\boxed{\begin{array}{l} H^k(M;\mathbb{F}) \otimes H^{n-k}(M;\mathbb{F}) \xrightarrow{⊛} \mathbb{F} \\ \quad \alpha \otimes \beta \longmapsto \langle [M], \alpha\cup\beta\rangle \\ \text{is a } \underline{\text{non-singular}} \text{ bilinear form.} \end{array}}$ ← hence: $H^*(M;\mathbb{F}) \cong (H^{n-*}(M;\mathbb{F}))^*$ (dual)

Pf. By exercise, $\langle [M], \alpha\cup\beta\rangle = \langle [M]\cap\alpha, \beta\rangle = \langle PD(\alpha), \beta\rangle$ ← definition of Poincaré duality PD
So the following diagram commutes:

$$\begin{array}{ccc} H^k(M;\mathbb{F}) \otimes H^{n-k}(M;\mathbb{F}) & \xrightarrow{\text{pairing } ⊛} & \\ \cong \downarrow PD\otimes id & & \mathbb{F} \\ H_{n-k}(M;\mathbb{F}) \otimes H^{n-k}(M;\mathbb{F}) & \xrightarrow{\langle\cdot,\cdot\rangle \leftarrow ⊛⊛} & \end{array}$$

PD is iso over $\mathbb{Z}$, hence iso over $\mathbb{F}$ by universal coefficients.

By universal coefficients, $H^*(M;\mathbb{F}) \cong \mathrm{Hom}(H_*(M;\mathbb{F}), \mathbb{F})$ via $\beta \mapsto \langle\beta,\cdot\rangle$
Hence ⊛⊛ is a non-degenerate bilinear pairing ← (using that for any $\mathbb{F}$-vector space $V$, $V\otimes V^* \to \mathbb{F}$, $v\otimes\varphi \mapsto \varphi(v)$ is non-deg. pairing; $V^* = \mathrm{Hom}(V,\mathbb{F})$)
Hence so is the pairing ⊛ in the diagram. □

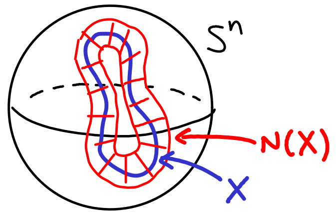
<u>Remark</u> For $M$ non-orientable, the same holds for $\mathbb{F}$ of characteristic 2, e.g. $\mathbb{Z}/2$
For $\mathbb{Z}$ coefficients it can fail if $H^*(M) \ncong \mathrm{Hom}(H_*M, \mathbb{Z})$. So we define: (has no torsion)

**<u>Betti group</u>** $B^k(M) = H^k(M)/\text{torsion}(H^k(M))$
$B_k(M) = H_k(M)/\text{torsion}(H_k(M))$

By what we proved in the section on universal coefficients, $B^q(M) \cong \mathrm{Hom}(B_q(M), \mathbb{Z})$
whenever $H_{q-1}(M)$ is finitely generated (which we know holds for compact mfds)
The iso is given by $\langle\cdot,\cdot\rangle$ again: this descends to quotients since $\langle c,\alpha\rangle = 0 \in \mathbb{Z}$ if $c$ or $\alpha$ has finite order (i.e. torsion). The same proof as above yields:
$M$ compact oriented n-mfd $\Rightarrow$ $\boxed{\begin{array}{l} B^k(M) \otimes B^{n-k}(M) \longrightarrow \mathbb{Z}, \ \alpha\otimes\beta \mapsto \langle [M], \alpha\cup\beta\rangle \\ \text{is non-degenerate bilinear form.} \end{array}}$
Also the <u>Remark</u> holds.

<u>Example</u> Use this to prove ex. 4(c) sheet 3. (<u>Hint</u>: $H^{2k}(\mathbb{C}P^n) \cup H^{2n-2k}(\mathbb{C}P^n) = H^{2n}(\mathbb{C}P^n)$)

## Alexander duality




(in fact, enough to assume X is locally contractible) s.t.

$\emptyset \neq X \subsetneq S^n$ compact subset s.t.

$\exists$ open neighbourhood **N(X)** which deformation retracts to X such that $\overline{N(X)} \subseteq S^n$ is an n-mfd with boundary.

**Theorem** $\boxed{\tilde H_*(X) \cong \tilde H^{n-*-1}(S^n \setminus X)}$

Pf later

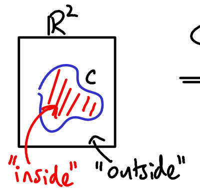
**Example** $X \subseteq S^3$ **knot** (i.e. $X = \text{image}(S^1 \xrightarrow{\text{homeomorphism onto the image}} S^3)$) ← **embedding**

$\Rightarrow N(X) \cong$ solid torus $\simeq S^1$

$$\Rightarrow \quad \begin{aligned} \tilde H_0(X) &= 0 = \tilde H^2(S^3 \setminus X) \\ \tilde H_1(X) &= \mathbb{Z} = \tilde H^1(S^3 \setminus X) \\ \tilde H_2(X) &= 0 = \tilde H^0(S^3 \setminus X) \end{aligned}$$

So the homology of a **knot complement** does not tell knots apart (always same)

**Theorem (Jordan curve theorem)**




$C \cong S^1$ closed curve in $\mathbb{R}^2 \subseteq S^2$ (e.g. by stereographic projection $S^2 \cong \mathbb{C} \cup \infty$)

$\Rightarrow \mathbb{R}^2 \setminus C$ has 2 path-components (= connected components)

Similarly for $S^n \cong C \subseteq \mathbb{R}^{n+1} \subseteq S^{n+1}$.

Pf $S^n \cong C \subseteq \mathbb{R}^{n+1} \subseteq S^{n+1} \Rightarrow \mathbb{Z} \cong \tilde H_n(S^n) \overset{\text{Alexander duality}}{\cong} \tilde H^0(S^{n+1} \setminus C)$

$\Rightarrow H^0(S^{n+1} \setminus C) \cong \mathbb{Z}^2$

$\Rightarrow S^{n+1} \setminus C$ has 2 path components. □

## Proof Alexander duality

Abbreviate $N = N(X)$ (nbhd of X which is $\simeq X$)

$$Y := S^n \setminus N \simeq S^n \setminus X$$

for $* < n-1$

$$\begin{aligned} \tilde H^{n-*-1}(Y) &= H^{n-*-1}(Y) \\ &\underset{\text{Lefschetz}}{\cong} H_{*+1}(Y, \partial Y) \\ &\underset{\text{exc.}}{\cong} H_{*+1}(S^n, \overline N) \\ &\underset{\text{LES using } * < n-1}{\cong} \tilde H_*(\overline N) \quad (\overline N \simeq X) \end{aligned}$$

for $* = n-1$

$$\begin{aligned} \tilde H^0(Y) \oplus \mathbb{Z} &\cong H^0(Y) \\ &\underset{\text{Lef.}}{\cong} H_n(Y, \partial Y) \\ &\underset{\text{exc.}}{\cong} H_n(S^n, \overline N) \\ &\cong \tilde H_{n-1}(\overline N) \oplus \mathbb{Z} \quad (\overline N \simeq X) \end{aligned}$$

Explanation of $\cong$:

LES: $0 \to \tilde H_n(S^n) \to H_n(S^n, \overline N) \to \tilde H_{n-1}(\overline N) \to 0$ ($= \tilde H_{n-1}(S^n)$) is SES

$\tilde H_n(S^n) \xrightarrow{\cong} H_n(S^n, S^n \setminus \infty) \cong \mathbb{Z}$ (via quotient $H_n(S^n, \overline N) \to H_n(S^n, S^n \setminus \infty)$), $S^n = \mathbb{R}^n \cup \infty$

$\Rightarrow$ Hence that quotient map gives a splitting of the SES.

(*) $\tilde H_n(\overline N) = H_n(\overline N) = 0$ (see Cor. to Poincaré-Lefschetz, using: each (path-)connected component of the manifold $\overline N$ has non-empty boundary)

for $* = n$ $\quad H^{n-*-1}(Y) = H^{-1}(Y) = 0$

$$H_n(X) \cong H_n(N) \underset{\text{Lefschetz duality}}{\cong} H^0(N, \partial N) = 0. \ \square$$

(see (*))