C3.1 Algebraic Topology

Sheet 0 (not for hand in)

Oxford 2019
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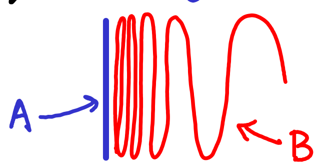
1) **Topologist's sine curve** $S = \underbrace{\{(x, \sin\frac{1}{x}) : x \in (0,1]\}}_{B} \cup \underbrace{0 \times [0,1]}_{A} \subseteq \mathbb{R}^2$



Show that S is connected, but not path-connected indeed: A, B are the path-components.

2) Let $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ be a **short exact sequence** of abelian groups, meaning: A, B, C are abelian groups, α, β group homs, and $\mathrm{Ker}(\text{map}) = \mathrm{Im}(\text{previous map})$ in the sequence.

Show that if C is a **free abelian group** $(C \cong \bigoplus_{i \in I} \mathbb{Z})$ then

$$B \cong A \oplus C.$$

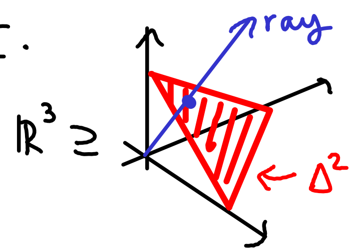
Give an example of A, B, C for which this fails when C is not free.

3) Recall the **Brower fixed point theorem**

Let $\mathbb{D}^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, $f : \mathbb{D}^n \to \mathbb{D}^n$ continuous $\Rightarrow \exists\, p \in \mathbb{D}^n$, $f(p) = p$.

Use this to prove that any real $n \times n$ matrix with (strictly) positive entries has a real eigenvalue $\lambda > 0$ with eigenvector $(v_1, \ldots, v_n)$ s.t. $v_i \geq 0\ \forall i$.

Hint.



Consider $X = \{\text{rays in "positive octant" of } \mathbb{R}^n\}$

$x \in \mathbb{R}^n : x_i > 0\ \forall i$

show $X \overset{\text{homeo.}}{\cong} \Delta^{n-1} = \{x \in \text{octant} : \sum x_i = 1\} \cong \mathbb{D}^{n-1}$

$\text{ray} \longmapsto \text{ray} \cap \Delta^n$

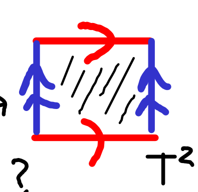
4) Let A = free abelian group generated by the intervals $[a,b] \subseteq \mathbb{R}$, with $a, b \in \mathbb{Q}$

$B = A / \langle \text{relations } [a,b] + [b,c] = [a,c] \rangle$

Show that B is a free abelian group by finding an explicit basis (in other words, a $\mathbb{Z}$-linearly independent and spanning set, if we view B as a $\mathbb{Z}$-module)

5) View the torus as $T^2 = \mathbb{R}^2 / (\mathbb{Z}^2\text{-translation})$ so $(x,y) \sim (x+n, y+m)$ for $n, m \in \mathbb{Z}$

Let $L \subseteq \mathbb{R}^2$ be a straight line through 0. Draw some pictures of L inside T^2 when you view T^2 as a square with side-identifications $\rightarrow$



How many path-components does $T^2 \setminus L$ have, in terms of the slope of L?

(Try slope $\in \mathbb{Q}$ first)