C3.1 Algebraic Topology

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Sheet 1

Convention: all spaces are topological spaces, maps of spaces are always continuous.

1) A map $f: X \to Y$ of spaces is **homotopic** to $g: X \to Y$ if f can be continuously deformed into g, meaning: $\exists$ map $F: X \times [0,1] \to Y$ with $F(x,0) = f(x)$, $F(x,1) = g(x)$. We write $f \simeq g$.

a) Show that $\simeq$ is an equivalence relation on maps $X \to Y$.

Two spaces X, Y are **homotopy equivalent** if $\exists f: X \to Y$, $g: Y \to X$ such that $g \circ f \simeq id_X$, $f \circ g \simeq id_Y$.

b) Show that $\simeq$ is an equivalence relation on spaces.

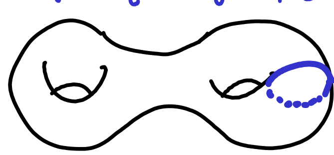
c) Show that point $\simeq \mathbb{R}^n$ ($\leftarrow$ Example of hpy equiv. spaces which are not homeo. $(n \geq 1)$)

d) Show that the solid torus $\simeq$ circle.

e) Let $A \subseteq X$ be subspace. We say "A can be contracted down to a point in X" if $\exists H: X \times [0,1] \to X$ with $H_t(A) \subseteq A$ all t, $H_0 = id$, $H_1(A) = $ some point $\in A$, where $H_t = H(\cdot, t): X \to X$. Deduce that $X \simeq X/A$ $\leftarrow$ (**quotient topological space**: $x \sim y \Leftrightarrow x = y$ or $x, y \in A$)

(Hints Let $f: X \to X/A$ quotient map. Construct a map $Q_t: X/A \to X/A$ such that $f \circ H_t = Q_t \circ f$. Build $g: X/A \to X$ with $g \circ f = H_1$.

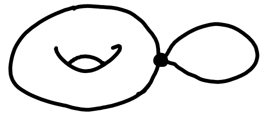
f) Let

$\Sigma_2 =$  $=$ genus 2 surface

Σ_2/A = Take Σ_2 and identify all the points of A

Show (by drawing convincing pictures) that:

$$X \overset{H(\cdot,1)}{\rightleftarrows} X/A$$

$\Sigma_2/A \simeq$  $= T^2 \vee S^1 = T^2 \sqcup S^1 / \text{identify one point of } T^2 \text{ with one point of } S^1$

($\vee$: wedge sum)

g) Prove (using pictures): $S^n \setminus \text{point} \simeq \mathbb{D}^n$

$S^n \setminus 2 \text{ points} \simeq S^{n-1}$

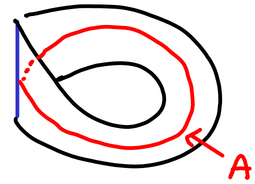
$S^n \setminus (k \geq 2 \text{ points}) \simeq \underbrace{S^{n-1} \vee \cdots \vee S^{n-1}}_{k-1 \text{ copies}}$

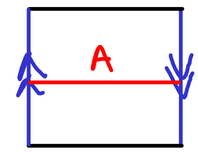
2) Draw an example of a loop in Σ_2 which is non-zero in $\pi_1(\Sigma_2)$ (i.e. not contractible: not homotopic to a constant map), but which is zero in $H_1(\Sigma_2)$ (i.e. it arises as the boundary of a "nice" 2-dimensional subspace of Σ_2).

Proof not required

3) A **retraction** of a space X onto a subspace A is a map $r: X \to X$ with $r(X) = A$ and $r(a) = a$ $\forall a \in A$.

a) Show that the Möbius band retracts onto the equator A.

$X =$ 

$=$ 

b) Assume we have a functor $F: \mathrm{Top} \to \mathrm{Gps}$ such that: $F(S^1) = \mathbb{Z}$,

$$F(S^1 \xrightarrow{z^2} S^1) = (\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z})$$

and if $(f: A \to X) \simeq (g: A \to X)$ then $F(f) = F(g)$ (for example $F = H_1$ = first homology has this property).

By considering the maps $A \xrightarrow[i]{\text{include}} X \xrightarrow[r]{\text{retract}} A$, show that $F(i)$ is injective and $F(r)$ is surjective.

Deduce that the Möbius band X does <u>not</u> retract onto the boundary circle $A_2 = \partial X$ (Hint. compare A from (a) with A_2).

Having seen the functorial proof, could you rephrase the proof into a topological argument for a Part A Topology undergraduate?

4) Given a functor $F_*: \mathrm{Top} \to$ **Graded Abelian Groups** with $F_*(\text{point}) = \begin{cases} \mathbb{Z} & *=0 \\ 0 & \text{else} \end{cases}$

Define $\widetilde{F}_*(X) = \mathrm{Ker}\,(F_*(X) \to F_*(\text{point}))$ ← induced by the constant map $X \to$ point

Prove $\widetilde{F}_*(X) \cong F_*(X)$ for $* \neq 0$ and $F_0(X) \cong \widetilde{F}_0(X) \oplus \mathbb{Z}$ ← (ASSUME $X \neq \emptyset$)

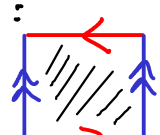
5) Draw a Δ-complex structure on:

S^2, Σ_2 and N_3

$\Sigma_2 = T^2 \# T^2$, $N_3 = \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$

see my B3.2 Geometry of Surfaces notes for a reminder about **connected sums** (Sec. 4.3) and what happens to standard polygon models with side-identifications (Sec. 5.1)

6) a) Compute the simplicial homology of:

S^2, $\mathbb{RP}^2$, K = Klein bottle = 

b) Compute their simplicial homology with $\mathbb{Z}/2$ coefficients.

i.e. replace every $\mathbb{Z}$ by $\mathbb{Z}/2$ in the chain cx

7) Prove that $\Delta^n \cong \mathbb{D}^n$ are homeomorphic.