C2.7 Category theory: problem sheet 3

Starred questions are optional.

1. Show that $f\colon X \to Y$ is a monomorphism iff the square

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ {\scriptstyle \mathrm{id}}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

is Cartesian (a pullback square). Similarly, show that it is an epimorphism iff the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f}\downarrow & \ulcorner & \downarrow{\scriptstyle \mathrm{id}} \\ Y & \xrightarrow[\mathrm{id}]{} & Y \end{array}$$

is coCartesian (a pushout square).

2. Prove that inductive limits commute with binary products in Set; i.e. for infinite sequences of sets $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ with maps $X_n \to X_{n+1}$ and $Y_n \to Y_{n+1}$ construct a natural map

$$\mathrm{colim}_n(X_n \times Y_n) \to (\mathrm{colim}_n\, X_n) \times (\mathrm{colim}_n\, Y_n)$$

and show it is an isomorphism.

3. Let k be a field. Construct a coproduct in the categories of unital and non-unital commutative k-algebras.

4. Observe that in the category of sets every morphism into the initial object is an isomorphism. Deduce that the category of sets is not equivalent to its opposite.

5. (*)
   (a) Observe that any non-zero vector space V has a monomorphism $k \to V$ from a one-dimensional vector space.
   (b) Deduce that under an equivalence $\mathrm{Vect} \cong \mathrm{Vect}^{op}$ the one-dimensional vector space k would be sent to itself.
   (c) Using the fact that an infinite-dimensional vector space has a smaller dimension (in the sense of cardinal arithmetic) than its dual, deduce that there is no equivalence $\mathrm{Vect} \cong \mathrm{Vect}^{op}$.