Infinite groups 2018: Sheet 1

October 8, 2018

Exercise 1. Let G be a finitely generated group and let S be an infinite set of generators of G. Show that there exists a finite subset F of S so that G is generated by F.

Recall that an exact sequence *is a sequence of groups and group homomorphisms*

$$\dots G_{n-1} \xrightarrow{\varphi_{n-1}} G_n \xrightarrow{\varphi_n} G_{n+1} \dots$$

such that $\operatorname{Im}\varphi_{n-1} = \operatorname{Ker}\varphi_n$ *for every* n. *A* short exact sequence *is an exact sequence of the form:*

$$\{1\} \longrightarrow N \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow \{1\}\,. \tag{1}$$

In other words, φ *is an isomorphism from* N *to a normal subgroup* $N' \lhd G$ *and* ψ *descends to an isomorphism* $G/N' \simeq H$.

Exercise 2. Suppose that we have a short exact sequence of groups

$$1 \to G_1 \xrightarrow{i} G_2 \xrightarrow{\pi} G_3 \to 1$$

such that the groups G_1, G_3 are finitely generated. Prove that G_2 is also finitely generated.

Exercise 3. Let H be the group of permutations of $\mathbb{Z}$ generated by the transposition $t = (01)$ and the translation map $s(i) = i + 1$. Let H_i be the group of permutations of $\mathbb{Z}$ supported on $[-i, i] = \{-i, -i+1, \dots, 0, 1, \dots, i-1, i\}$, and let H_ω be the group of finitely supported permutations of $\mathbb{Z}$ (i.e. the group of bijections $f : \mathbb{Z} \to \mathbb{Z}$ such that f is the identity outside a finite subset of $\mathbb{Z}$),

$$H_\omega = \bigcup_{i=0}^{\infty} H_i\,.$$

Prove that H_ω is a normal subgroup in H and that $H/H_\omega \simeq \mathbb{Z}$. Prove that H_ω is not finitely generated.

Recall that the wreath product $A \wr C$ *of two groups* A *and* C *is the semidirect product*

$$(\oplus_C A) \rtimes C$$

where C *acts on the direct sum by pre-compositions:* $f(x) \mapsto f(xc^{-1})$*. Thus, the elements of the wreath product* $A \wr C$ *are pairs* (f, c)*, where* $f : C \to A$ *is a map with finite support, and* $c \in C$*. The product structure on this set is given by the formula*

$$(f_1(x), c_1) \cdot (f_2(x), c_2)) = (f_1(xc_2^{-1})f_2(x), c_1c_2).$$

For each $a \in A$ *we define the map* $\delta_a : C \to A$ *so that the image of* $1 \in C$ *is* $a \in A$*, and all the other elements of* C *are mapped to* $1 \in A$*.*

Exercise 4.

1. Let $a_i, i \in I$, and $c_j, j \in J$, be sets of generators of A and C, respectively. Prove that the set of elements $(1, c_j), j \in J$, and $(\delta_{a_i}, 1), i \in I$, generate $G_A := A \wr C$. In particular, if A and C are finitely generated, so is $A \wr C$.

2. Let G be the wreath product $\mathbb{Z} \wr \mathbb{Z} \cong N \rtimes \mathbb{Z}$, where N is the (countably) infinite direct sum of copies of $\mathbb{Z}$. Prove that G is 2-generated, and that the normal subgroup N is not finitely generated.

Exercise 5. Prove that every short exact sequence

$$1 \to N \to G \xrightarrow{r} F(X) \to 1$$

splits.

Exercise 6. Suppose that $g \in Bij(X)$ is a bijection such that for some $A \subset X$,

$$g(A) \subsetneq A.$$

Prove that g has infinite order.

Exercise 7. Let F_2 be the free group of rank two.

- State and prove a generalization of the Ping-pong lemma to n elements $g_1, g_2, \ldots, g_n$.
- Prove that for every n, the group F_2 has a subgroup isomorphic to F_n.
- Prove that every free group of countable rank can be embedded as a subgroup of F_2.