Infinite groups 2018: Sheet 2

October 8, 2018

Exercise 1.

1. Prove that if S and $\bar{S}$ are two finite generating sets of G, then the word metrics dist_S and $\mathrm{dist}_{\bar{S}}$ on G are bi-Lipschitz equivalent, i.e. there exists $L > 0$ such that
$$\frac{1}{L}\mathrm{dist}_S(g, g') \leqslant \mathrm{dist}_{\bar{S}}(g, g') \leqslant L\mathrm{dist}_S(g, g')\,, \forall g, g' \in G\,. \qquad (1)$$
2. Prove that an isomorphism between two finitely generated groups is a bi-Lipschitz map when the two groups are endowed with word metrics.

Exercise 2. Consider the *integer Heisenberg group*

$$H_{2n+1}(\mathbb{K}) = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & \dots & \dots & x_n & z \\ 0 & 1 & 0 & \dots & \dots & 0 & y_n \\ 0 & 0 & 1 & \dots & \dots & 0 & y_{n-1} \\ \vdots & \vdots & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 1 & 0 & y_2 \\ 0 & 0 & \dots & \dots & 0 & 1 & y_1 \\ 0 & 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix} ; x_1, \dots, x_n, y_1, \dots, y_n, z \in \mathbb{Z} \right\}.$$

Prove that $H_{2n+1}(\mathbb{Z})$ is nilpotent of class 2.

Exercise 3. The goal of this exercise is to prove that, given an arbitrary field $\mathbb{K}$, the group $\mathcal{U}_n(\mathbb{K})$ is nilpotent of class $n-1$.

Let $\mathcal{U}_{n,k}(\mathbb{K})$ be the subset of $\mathcal{U}_n(\mathbb{K})$ formed by matrices (a_{ij}) such that $a_{ij} = \delta_{ij}$ for $j < i + k$. Note that $\mathcal{U}_{n,1}(\mathbb{K}) = \mathcal{U}_n(\mathbb{K})$.

1. Prove that for every $k \geqslant 1$ the map
$$\begin{array}{rcc} \varphi_k : \mathcal{U}_{n,k}(\mathbb{K}) & \to & \left(\mathbb{K}^{n-k}\,, +\right) \\ A = (a_{i,j}) & \mapsto & (a_{1,k+1}, a_{2,k+2}, \dots, a_{n-k,n}) \end{array}$$
is a homomorphism. Deduce that $(\mathcal{U}_{n,k}(\mathbb{K}))' \subset \mathcal{U}_{n,k+1}(\mathbb{K})$ and that $\mathcal{U}_{n,k+1}(\mathbb{K}) \lhd \mathcal{U}_{n,k}(\mathbb{K})$ for every $k \geqslant 1$.

2. Let E_{ij} be the matrix with all entries 0 except the (i,j)–entry, which is equal to 1. Consider the triangular matrix $T_{ij}(a) = I + aE_{ij}$.

   Deduce from (1), using induction, that $\mathcal{U}_{n,k}$ is generated by the set

$$\{T_{ij}(a) \mid j \geqslant i+k, a \in \mathbb{R}\}\,.$$

3. Prove that for every three distinct numbers i,j,k in $\{1,2,\dots,n\}$

$$[T_{ij}(a), T_{jk}(b)] = T_{ik}(ab)\,,\ [T_{ij}(a), T_{ki}(b)] = T_{kj}(-ab)\,,$$

   and that for all quadruples of distinct numbers i,j,k,ℓ,

$$[T_{ij}(a), T_{k\ell}(b)] = I\,.$$

4. Prove that $C^k\mathcal{U}_n(\mathbb{K}) \leqslant \mathcal{U}_{n,k+1}(\mathbb{K})$ for every $k \geqslant 0$. Deduce that $\mathcal{U}_n(\mathbb{K})$ is nilpotent.

Exercise 4. Which of the permutation groups S_n are nilpotent? Which of these groups are solvable?

Exercise 5. Let D_∞ be the infinite dihedral group. Recall that this group can be realized as the group of isometries of $\mathbb{R}$ generated by the symmetry $s : \mathbb{R} \to \mathbb{R}$, $s(x) = -x$ and the translation $t : \mathbb{R} \to \mathbb{R}$, $t(x) = x+1$, and as noted before $D_\infty = \langle t \rangle \rtimes \langle s \rangle$.

1. Give an example of two elements a, b of finite order in D_∞ such that their product ab is of infinite order.

2. Find $\mathrm{Tor}\, D_\infty$.

3. Is D_∞ a nilpotent group ? Is D_∞ polycyclic ?

4. Are any of the finite dihedral groups D_{2n} nilpotent?

Exercise 6. Let $\mathcal{T}_n(\mathbb{K})$ be the group of invertible upper-triangular $n \times n$ matrices with entries in a field $\mathbb{K}$.

1. Prove that $\mathcal{T}_n(\mathbb{K})$ is a semidirect product of its nilpotent subgroup $\mathcal{U}_n(\mathbb{K})$ introduced in Exercise Sheet 2, and the subgroup of diagonal matrices.

2. Prove that, if $\mathbb{K}$ has zero characteristic, the subgroup of $\mathcal{T}_n(\mathbb{K})$ generated by $I + E_{12}$ and by the diagonal matrix with $(-1, 1, \dots, 1)$ on the diagonal is isomorphic to the infinite dihedral group D_∞. Deduce that $\mathcal{T}_n(\mathbb{K})$ is not nilpotent.