Infinite groups: Sheet 3

November 20, 2019

Exercise 1. *Recall that, by Mal'cev's Theorem, given a nilpotent group G the following are equivalent (1) G is torsion-free; (2) its centre $Z(G)$ is torsion free; (3) every quotient $Z_{i+1}(G)/Z_i(G)$ from the upper central series is torsion-free. The example below shows that the same is not true for the lower central series.*

Given an integer $p \geqslant 2$, consider the following subgroup G of the integer Heisenberg group $H_3(\mathbb{Z})$:

$$G = \left\{ \begin{pmatrix} 1 & k & n \\ 0 & 1 & pm \\ 0 & 0 & 1 \end{pmatrix} \; ; \; k, m, n \in \mathbb{Z} \right\}.$$

(a) Prove that the commutator subgroup in G is:

$$C^2G = \left\{ \begin{pmatrix} 1 & 0 & pn \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; ; \; n \in \mathbb{Z} \right\}.$$

(b) Prove that the quotient G/C^2G is isomorphic to $\mathbb{Z}^2 \times \mathbb{Z}_p$.

Exercise 2. Show that for each finitely generated nilpotent group the Hirsch number equals the *Hirsch length* $h(G)$.

Exercise 3. Let $\mathcal{X}$ be a group–theoretic property (e.g. finite, abelian, nilpotent, free etc.)

A group H is *residually* $\mathcal{X}$ if for every $1 \neq h \in H$ there exists $N \lhd H$ with H/N satisfying $\mathcal{X}$ such that $h \notin N$.

Let $V = \mathbb{Z} \oplus \mathbb{Z}$ and let $G = V \rtimes \langle x \rangle$ be the semi-direct product where x is a matrix in $\mathrm{GL}_2(\mathbb{Z})$. Determine the terms of the lower central series $C^n(G)$ $(n \geq 1)$ in each of the following cases:

(i) $x = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. (ii) $x = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$. (iii) $x = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

In which cases is G (a) solvable, (b) nilpotent, (c) residually nilpotent?

Exercise 4. Prove that every infinite polycyclic group contains an infinite free abelian normal subgroup.

Exercise 5. 1. Let $\mathbb{F}_k$ denote the field with k elements. Use the 1-dimensional vector subspaces in $\mathbb{F}_k^2$ to construct a homomorphism $GL(2, \mathbb{F}_k) \to S_n$ for an appropriate n.

2. Prove that $GL(2, \mathbb{F}_2)$ and $GL(2, \mathbb{F}_3)$ are solvable.

Exercise 6. Recall that a finite sequence of vector subspaces

$$V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_k$$

in a vector space V is called a *flag* in V. If the number of the subspaces in such a sequence is maximal possible (equal $\dim(V) + 1$), the flag is called *full* or *complete*. In other words, $\dim(V_i) = i$ for all the subspaces of this sequence.

1. Prove that the subgroup $\mathcal{T}_n(\mathbb{K})$ of upper-triangular matrices in $GL(n, \mathbb{K})$, where $\mathbb{K}$ is a field, is solvable.

2. Use Part (1) to show that for a finite-dimensional vector space V, the subgroup G of $GL(V)$ consisting of elements g preserving a complete flag in V (i.e. $gV_i = V_i$, for every $g \in G$ and every i) is solvable.

3. Let V be a $\mathbb{K}$-vector space of dimension n, and let

$$V_0 = 0 \subset V_1 \subset \cdots \subset V_{k-1} \subset V_k = V$$

be a flag, not necessarily complete. Let G be a subgroup of $GL(V)$ preserving this flag. For every $i \in \{1, 2, \ldots, k-1\}$ let ρ_i be the projection $G \to GL(V_{i+1}/V_i)$. Prove that if every $\rho_i(G)$ is solvable, then G is also solvable.