SHEET 3, EXERCISE 19

Please send comments/corrections to Jay Swar!

Exercise. Let k be a field, $R = k[x,y], M := R/(x,y)^2 \cong k \oplus kx \oplus ky$ as a k-module.
Consider the following Mod_R-SES's and compute the associated Tor-LES's:
$0 \to k \oplus k \to M \to k \to 0$:
i) LES from $M \otimes_R -$
$0 \to k \to \mathrm{Hom}_k(M,k) \to k \oplus k \to 0$:
ii) LES from $M \otimes_R -$
iii) LES from $k \otimes_R -$
$0 \to k^{\oplus 3} \to \frac{M \oplus M}{\langle (y,-x) \rangle} \to k \oplus k \to 0$:
iv) LES from $k \otimes_R -$.
For the SES $0 \to A \to B \to C \to 0$ above, I name morphisms $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$.

Solution. First, let's take a second to appreciate that the SES in iv $0 \to k^{\oplus 3} \to \frac{M \oplus M}{(y,-x)R} \to k^{\oplus 2} \to 0$ might appear, at first glance, to be ambiguous. If we think of the coproduct being taken in the category of k-modules, then by mapping to the first factor and quotienting, we have a SES of R-modules: $0 \to M \xrightarrow{q \circ i_1} \frac{M \oplus M}{(y,-x)R} \to \frac{R}{(x,y^2)} = \frac{k[y]}{y^2} \to 0$. We also have $0 \to \frac{(kx \oplus ky) \oplus (kx \oplus ky)}{(y,-x)R} \to \frac{M \oplus M}{(y,-x)R} \to k^{\oplus 2} \to 0$; this is a non-isomorphic SES of R-modules (e.g. since the first module has 0-action by x, y). Thus, the only reasonable way to interpret the $\oplus$'s in this question is as the coproducts in Mod_R (maybe this was obvious to the ever-alert; however when starting with parts i and ii, one might be distracted since it isn't difficult to see that the only possible Mod_R objects of the correct $\dim_k$ on the left and right have 0-action by x, y – hence why my Frankensteined answer awkwardly shows that). Note that $kx \oplus ky$ is allowed notation since kx in this usage refers to the R-submodule of M generated by x; in particular, this is Mod_R-isomorphic to k with 0-action by x, y, ie. $R/(x,y)$.

In preparation for what is to come, for $a \in R$, we write $a = a_0 + a_1 x + a_2 y + O((x,y)^2)$ (there'll be an obvious place where some n_i are still meant to be in R and I use n_{i0}, n_{i1}, n_{i2}!).

Let's note another important thing: there is not a unique injective R-module homomorphism $k^{\oplus 3} \to \frac{M \oplus M}{(y,-x)R}$, even though the image is unique (since for $a_i, b_i \in k$, we have that $x(a_0 + a_1 x + a_2 y, b_0 + b_1 x + b_2 y) = (a_0 x + b_0 y, 0)$ and $y(a_0 + a_1 x + a_2 y, b_0 + b_1 x + b_2 y) = (0, a_0 x + b_0 y)$ are both 0 iff $a_0 = b_0 = 0$). For example, mapping $\alpha : e_1 \mapsto (x,0), e_2 \mapsto (y,0), e_3 \mapsto (0,y)$ is not really preferred, yet is distinct, from the same map where instead $e_1 \mapsto (-x,0)$. However, the *image* of this map is unique $(\frac{(kx+ky) \oplus (kx+ky)}{(y,-x)R})$. This level of ambiguity arguably suggests that we should only "describe the maps in the induced LES" *up to unique isomorphism*. This is slightly different from earlier where we were in situations involving $\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{Z}}}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$ where there is an on-the-nose unique isomorphism. I'll do part iv) completely explicitly (i.e. I'll describe explicit morphisms by what they do to bases of our objects), and then I'll only describe the morphisms in parts i), ii), and ii) up to unique isomorphism.

Let us begin with the one I do completely explicitly, iv).

A $\mathrm{Mod}_{k[x,y]}$-projective resolution of $k = R/(x,y)$ is $\cdots \to 0 \to R \xrightarrow{d_2} R^2 \xrightarrow{d_1} R = P_0$ given by $d_1(f,g) = xf + yg$, $d_2(h) = (yh, -xh)$. The algorithmic approach to describing the morprhisms completely explicitly is: find resolutions for $k^{\oplus n}$, apply the horseshoe lemma to get a split SES of complexes, apply $k \otimes_R -$, and finally apply the snake lemma. The first three steps result in the following:

$$\begin{tikzcd}
& 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
& R^{\oplus 3} \arrow[r] \arrow[d, "\partial_2'"] & R^{\oplus 3} \oplus R^{\oplus 2} \arrow[r] \arrow[d, "\partial_2"] & R^{\oplus 2} \arrow[d, "\partial_2''"] \\
& (R^2)^{\oplus 3} \arrow[r] \arrow[d, "\partial_1'"] & (R^2)^{\oplus 3} \oplus (R^2)^{\oplus 2} \arrow[r] \arrow[d, "\partial_1"] & (R^2)^{\oplus 2} \arrow[d, "\partial_1''"] \\
& R^{\oplus 3} \arrow[r] \arrow[d, "\partial_0'"] & R^{\oplus 3} \oplus R^{\oplus 2} \arrow[r] \arrow[d, "\partial_0"] & R^{\oplus 2} \arrow[d, "\partial_0''"] \\
0 \arrow[r] & k^{\oplus 3} \arrow[r, "\alpha"'] & \frac{M \oplus M}{(y,-x)R} \arrow[r, "\beta"'] & k^{\oplus 2} \arrow[r] & 0
\end{tikzcd}
\overset{k \otimes_R -}{\rightsquigarrow}
\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
k^{\oplus 3} \arrow[r] \arrow[d, "\partial_2'"] & k^{\oplus 3} \oplus k^{\oplus 2} \arrow[r] \arrow[d, "\partial_2"] & k^{\oplus 2} \arrow[d, "\partial_2''"] \\
(k^2)^{\oplus 3} \arrow[r] \arrow[d, "\partial_1'"] & (k^2)^{\oplus 3} \oplus (k^2)^{\oplus 2} \arrow[r] \arrow[d, "\partial_1"] & (k^2)^{\oplus 2} \arrow[d, "\partial_1''"] \\
k^{\oplus 3} \arrow[r] \arrow[d, "\partial_0'"] & k^{\oplus 3} \oplus k^{\oplus 2} \arrow[r] \arrow[d, "\partial_0"] & k^{\oplus 2} \arrow[d, "\partial_0''"] \\
0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Where wlog we take $\alpha : e_1 \mapsto (x,0), e_2 \mapsto (y,0), e_3 \mapsto (0,y)$ and $\beta : (m,n) \mapsto (m_0,n_0)$ for $m,n \in R$ representing an element in $\frac{M\oplus M}{(y,-x)R}$. Let's narrate the horseshoe lemma: We lift ∂_0'' to get $R^{\oplus 2} \to \frac{M \oplus M}{(y,-x)R} : (m,n) \mapsto \overline{(m,n)}$. We subsequently won't bother with writing the overlines (which just indicates that the element is in a natural quotient of $R^{\oplus 2}$). Thus we have $\partial_0(r,s,t,a,b) = (r_0x + s_0y + a, t_0y + b)$. This is 0 iff $a_0 = b_0 = 0$ and $t_0 = -b_2, r_0 = -a_1, s_0 = -(a_2 + b_1)$. So $\ker \partial_0 = \{(-f + (x,y)R, -(g+m) + (x,y)R, -n + (x,y)R, xf + yg, xm + yn) : f,g,m,n \in R\}$ where by the $(x,y)R$ I mean any element in this ideal as opposed to a quotient – this naturally surjects onto $\ker \partial_0''$. We can then lift $\partial_1'' : (f,g,m,n) \mapsto (fx + gy, mx + ny)$ to get $\sigma_1 : (R^2)^{\oplus 2} \to R^{\oplus 3} \oplus R^{\oplus 2} : (f,g,m,n) \mapsto (-f, -g-m, -n, fx+gy, mx+ny)$. Thus we get $\partial_1(r,f,g,m,n) = (\partial_1'(r),0,0) + (-f,-g-m,-n,fx+gy,mx+ny)$ for $r \in (R^2)^{\oplus 3}$. Note $\ker \partial_1$ implies $x|g, y|f, x|n, y|m$, so g_0, f_0, m_0, n_0 are 0 for elements in this kernel. In fact, $f = yh, g = -xh, m = yp, n = -xp$. Set $r = (r,s,t,u,v,w) \in R^{\oplus 6}$. So we have $rx + sy = hy, tx + uy = py - xh, vx + wy = xp$. So we may write a lift of ∂_2'' to this kernel via $\sigma_2 : (h,p) \mapsto (h, 0, -h, p, p, 0, yh, -xh, yp, -xp)$. Now $\partial_2 = \partial_2' \oplus \sigma_2$, and our projective resolution terminates (since σ_2, ∂_2' are both injective).

Note that $\overline{\partial_1} = \overline{\partial_1' \oplus \sigma_1} : (r,s,t,u,v,w,f,g,m,n) \mapsto (-f_0, -g_0 - m_0, -n_0, 0, 0)$, and

$\overline{\partial_2} = \overline{\partial_2' \oplus \sigma_2} : (a,b,c,h,p) \mapsto (h_0, 0, -h_0, p_0, p_0, 0,0,0,0,0)$. The differentials in the left and right columns are all identically 0. So $\mathrm{Tor}_0^R(k, \frac{M\oplus M}{(y,-x)R}) = ke_4 \oplus ke_5$, $\mathrm{Tor}_1^R(k, \frac{M\oplus M}{(y,-x)R}) = \frac{\bigoplus_{i=1}^6 ke_i \bigoplus k(e_8 - e_9)}{k(e_1-e_3) \oplus k(e_4+e_5)} \cong k^{\oplus 5}$, $\mathrm{Tor}_2^R(k, \frac{M\oplus M}{(y,-x)R}) = \bigoplus_{i=1}^3 ke_i$.

So our LES looks like:

$$\begin{tikzcd}
& \cdots \arrow[r] & 0 \arrow[dll] \\
k^{\oplus 3} \arrow[r, "\alpha_2"] & \bigoplus_{i=1}^3 ke_i \arrow[r, "\beta_2"] & k^{\oplus 2} \arrow[dll, "\delta_2"'] \\
k^{\oplus 6} \arrow[r, "\alpha_1"] & \mathrm{Tor}_1^R(k, \frac{M\oplus M}{(y,-x)R}) \arrow[r, "\beta_1"] & k^{\oplus 4} \arrow[dll, "\delta_1"'] \\
k^{\oplus 3} \arrow[r, "\alpha_0"] & ke_4 \oplus ke_5 \arrow[r, "\beta_0"] & k^{\oplus 2} \arrow[dll, "\delta_0"'] \\
0 \arrow[r] & \cdots
\end{tikzcd}$$

and now the benefit of this method: we write the explicit maps:

- $\beta_0 : e_4 \mapsto e_1, e_5 \mapsto e_2$
- $\alpha_0 = 0$
- $\delta_1 : e_1 \mapsto -e_1, e_2 \mapsto -e_2, e_3 \mapsto -e_2, e_4 \mapsto e_3$ (we look at pre-images of $\overline{\partial_1}(e_i)$ for $i = 7,8,9,10$, resp.)
- $\beta_1 : e_8 - e_9 \mapsto e_2 - e_3$, all other $e_i \mapsto 0$

- $\alpha_1 : e_i \mapsto e_i$ for $i = 1, \dots, 6$ (note $e_1 = e_3, e_4 = -e_5$ in the target)
- $\delta_2 : e_1 \mapsto e_1 - e_3, e_2 \mapsto e_4 + e_5$ (we look at pre-images of $\overline{\partial_2}(e_i)$ for $i = 4, 5$, resp.)
- $\beta_2 = 0$
- $\alpha_2 : e_i \mapsto e_i$.

Note that we could have deduced many things (e.g. α_2 being an isomorphism and $\alpha_0 = 0$) just from knowing the groups in the long exact sequence, however we wouldn't have been able to write explicit non-zero maps. The algorithmic outline of this anwer can be mimicked for the other parts, but it's very painful to type up. For the remaining parts, I'll show how we might describe the maps up to canonical isomorphism (i.e. a choice of basis in any object would explicitly determine our maps if we went through the above pain!).

Caveat: I had decided on the $k^{\oplus n}$ notation later in my writing-timeline to emphasize that we are considering this object as a coproduct in Mod_R rather than some R-module which is Mod_k-isomorphic to k^n, apologies for the many subsequent inconsistencies.

i) Our outline here will be: find the groups in the LES, then see what exactness and our knowledge of the x, y-actions determine. The following is a $\mathrm{Mod}_{k[x,y]}$-projective resolution of M:

$\cdots \to 0 \to R^2 \xrightarrow{\partial_2} R^3 \xrightarrow{\partial_1} R \to 0$ with $\partial_1(f, g, h) = x^2 f + xyg + y^2 h$. To justify this, we check: $x^2 f + xyg + y^2 h = 0 \implies f = yf', h = xh' \implies x | g + yh', y | g + xf'$since $xf' + g + yh' = 0$.

If $x \nmid h'$, then $g = xg' - yh'$ and $f' = -g'$. Thus our triple looks like $(-yg', xg' - yh', xh')$.

If $y \nmid f'$, then $g = yg' - xf'$ and $h' = -g'$. Thus our triple looks like $(yf', yg' - xf', -xg')$.

Now suppose $h' = xh'', f' = yf''$. Thus $g = xyg''$ and we have $f'' + g'' + h'' = 0$. Our triple looks like $(y^2 f'', -xy(f'' + h''), x^2 h'')$.

We note that these are all in the image of $\partial_2(a, b) = (ya, -xa - yb, xb)$ which is also always in $\ker \partial_1$ and so we have our P_2. Further, $\ker \partial_2 = 0$ so our resolution terminates here.

So applying $- \otimes_R M$, we have $\mathrm{Tor}_i^R(M, M) = H_i(\cdots \to 0 \to M^2 \xrightarrow{\partial_2} M^3 \xrightarrow{\partial_1} M \to 0)$. $\mathrm{im}\partial_1 = 0, \ker \partial_1 = M^3, \mathrm{im}\partial_2 = \{(ya, -xa - yb, xb) : a, b \in M\} = \{(ya_0, -xa_0 - yb_0, xb_0) : a_0, b_0 \in k\}$. Note $\ker \partial_2 = \{(xa_1 + ya_2, xb_1 + yb_2) : a_i, b_i \in k\}$ has 0-action by x, y so $\mathrm{Tor}_2^R(M, M) = k^4$ with R-action given by $x\cdot = 0 = y\cdot$. $\mathrm{Tor}_1^R(M, M) \cong k^7$ as a k-mod. Note $\mathrm{Tor}_0^R(M, M) = M \otimes_R M = \frac{M}{(x,y)^2 M} = M$ as an R-module since $M = \frac{R}{(x,y)^2}$.

Applying $- \otimes_R k$, we have $\mathrm{Tor}_i^R(M, k) = H_i(\cdots \to 0 \to k^2 \xrightarrow{0} k^3 \xrightarrow{0} k \to 0) = \begin{cases} k & i = 0 \\ k^{\oplus 3} & i = 1 \\ k^{\oplus 2} & i = 2 \\ 0 & o/w \end{cases}$. Consequently,

$$\mathrm{Tor}_i^R(M, \bigoplus^n k) = \begin{cases} k^{\oplus n} & i = 0 \\ k^{\oplus 3n} & i = 1 \\ k^{\oplus 2n} & i = 2 \\ 0 & o/w \end{cases}.$$

So our LES looks like $\cdots \to 0 \to k^4 \xrightarrow{\alpha_2} k^4 \xrightarrow{\beta_2} k^2 \xrightarrow{\delta_2} k^6 \xrightarrow{\alpha_1} \mathrm{Tor}_1^R(M, M) \xrightarrow{\beta_1} k^3 \xrightarrow{\delta_1} k^2 \xrightarrow{\alpha_0} M \xrightarrow{\beta_0} k \to 0$ where all the k^n have 0-action by x, y as R-mods. We have $\beta_0 = \beta, \alpha_0 = \alpha$ from our original SES since $\beta_0(xm_1 + ym_2) = x\beta_0(m_1) + y\beta_0(m_2) = 0$, i.e. $\ker \beta_0 \supseteq kx \oplus ky$ (which must then be equality by exactness).[1] Since α_0 is injective, $\delta_1 = 0$ and β_1 is surjective. From the other side, α_2 is injective and so must be an isomorphism. Thus $\beta_2 = 0$ and so δ_2 is injective. Thus, α_1 is a projection onto k^4 with 0-action by x, y composed with an inclusion into $\mathrm{Tor}_1^R(M, M)$.

So we need to understand $0 \to k^4 \hookrightarrow \mathrm{Tor}_1^R(M, M) \to k^3 \to 0$. Note that $x \cdot \overline{(f_0 + xf_1 + yf_2, g_0 + xg_1 + yg_2, h_0 + xh_1 + yh_2)} = \overline{(xf_0, xg_0, xh_0)}$ and $y\cdot$ acts similarly. Thus, k^4 injects into $\frac{\overline{\{(xf_1 + yf_2, xg_1 + yg_2, xh_1 + yh_2) : f_i, g_i, h_i \in k\}}}{\mathrm{im}\partial_2 \cap \{\text{the above submodule}\}}$. But $\mathrm{im}\partial_2 \subseteq \{\text{the above submodule}\}$, and so we see $\frac{\overline{\{(xf_1 + yf_2, xg_1 + yg_2, xh_1 + yh_2) : f_i, g_i, h_i \in k\}}}{\mathrm{im}\partial_2 \cap \{\text{the above submodule}\}} \simeq k^4$ and we've completely described our LES (up to canonical isomorphisms).

ii) $\mathrm{Hom}_k(M, k) \cong ke_0^* + ke_1^* + ke_2^*$ where e_i^* represent the dual basis for $1, x, y$ resp. Note $x \cdot (f_0 e_0^* + f_1 e_1^* + f_2 e_2^*) = f_1 e_0^*$ and $y \cdot (f_0 e_0^* + f_1 e_1^* + f_2 e_2^*) = f_2 e_0^*$.[2] Now $k \hookrightarrow \mathrm{Hom}_k(M, k)$, so we note that x^2 acts by 0 on

[1] Of course, α_0 and β_0 can be seen as $\alpha \otimes k, \beta \otimes k$ from general theory.

[2] Probably there is a way to use the fact that if we apply $\mathrm{Hom}_k(-, k)$, then we are back to our first SES.

k. Thus x can't act on this k by a non-zero scalar, ie. x acts by 0. Similarly, y acts by 0 on the submodule k. The submodule of $\mathrm{Hom}_k(M,k)$ on which x and y act by 0 is ke_0^* which is one-dimensional$_k$ and so this injection (and thus the SES) is canonically determined and the k, k^2 have 0-action by x, y. (Of coure this deduction is unnecessary once one decides to exclusively read $\oplus$ to be the Mod_R-coproduct!)

We again have $\mathrm{Tor}_i^R(M,k^n) = \begin{cases} k^n & i=0 \\ k^{3n} & i=1 \\ k^{2n} & i=2 \\ 0 & o/w \end{cases}$ where each k^m has zero-action by x,y as an R-module.

Now applying $-\otimes_R \mathrm{Hom}_k(M,k)$ to our resolution of M, we consider: $\cdots \to 0 \to \mathrm{Hom}_k(M,k)^2 \xrightarrow{\partial_2} \mathrm{Hom}_k(M,k)^3 \xrightarrow{\partial_1} \mathrm{Hom}_k(M,k) \to 0$ where $\partial_1(f,g,h) = x^2f + xyg + y^2h, \partial_2(a,b) = (ya, -xa-yb, xb)$. We write $a = a_0e_0^* + a_1e_1^* + a_2e_2^* \in \mathrm{Hom}_k(M,k)$ for $a_i \in k$, etc.

So $\partial_1 = 0$. $\partial_2(a,b) = (a_2e_0^*, -a_1e_0^* - b_2e_0^*, b_1e_0^*)$, so $\mathrm{im}\partial_2 = (ke_0^*) \oplus (ke_0^*) \oplus (ke_0^*) \subseteq M^3$ and $\ker\partial_2 = \{(a_0e_0^* + a_1e_1^*, b_0e_0^* - a_1e_2^*) : a_1, a_0, b_0 \in k\}$. Thus, the non-trivial Tor_i are: $\mathrm{Tor}_0^R(M, \mathrm{Hom}_k(M,k)) = \mathrm{Hom}_k(M,k)$ (note Tor_0 can be seen easily anyways); $\mathrm{Tor}_1^R(M,\mathrm{Hom}_k(M,k)) \simeq k^6$ where x,y act by 0; and $\mathrm{Tor}_2(M,\mathrm{Hom}_k(M,k)) = \ker\partial_2 = k(e_0^*,0) \oplus k(e_1^*, -e_2^*) \oplus k(0,e_0^*)$ with the inherited x,y action from $\mathrm{Hom}_k(M,k)^2$.

So our LES looks like $\cdots \to 0 \to k^2 \xrightarrow{\alpha_2} \ker\partial_2 \xrightarrow{\beta_2} k^4 \xrightarrow{\delta_2} k^3 \xrightarrow{\alpha_1} k^6 \xrightarrow{\beta_1} k^6 \xrightarrow{\delta_1} k \xrightarrow{\alpha_0} \mathrm{Hom}_k(M,k) \xrightarrow{\beta_0} k^2 \to 0$ where the k^n all have 0-action given by x,y. $\mathrm{Hom}_k(M,k)$ is 3-dimensional$_{/k}$ so α_0 must be injective (since its source is k). The subspace of $\mathrm{Hom}_k(M,k)$ upon which x,y act by 0 is one-dimensional$_{/k}$ and so $\alpha_0 = \alpha, \beta_0 = \beta$. Thus $\delta_1 = 0$ and β_1 is surjective and thus an isomorphism. In turn, this implies $\alpha_1 = 0$ and δ_2 being a surjection, so β_2 is a surjection onto k followed by an inclusion into k^4. Now α_2 is a surjection; the subset of $\ker\partial_2$ on which x,y act by 0 is $k(e_0^*,0) \oplus k(0,e_0^*)$ which is 2-dimensional and so α_2 is an isomorphism onto this subset, and β_2 is the morphism $\ker\partial_2 \to \ker\partial_2/(k(e_0^*,0) \oplus k(0,e_0^*)) \simeq k \hookrightarrow k^4$.

iii) A $\mathrm{Mod}_{k[x,y]}$-projective resolution of k (where x,y act by 0) is $\cdots \to 0 \to R \xrightarrow{\partial_2} R^2 \xrightarrow{\partial_1} R = P_0$ given by $\partial_1(f,g) = xf + yg$, $\partial_2(h) = (yh, -xh)$.

Applying $-\otimes_R k^{\oplus n}$, we get $\mathrm{Tor}_i^R(k, k^{\oplus n}) = \begin{cases} k^{\oplus n} & i = 0,2 \\ k^{\oplus 2n} & i=1 \\ 0 & o/w \end{cases}$.

Applying $-\otimes_R \mathrm{Hom}_k(M,k)$, we have: $\cdots \to 0 \to \mathrm{Hom}_k(M,k) \xrightarrow{\partial_2} \mathrm{Hom}_k(M,k)^{\oplus 2} \xrightarrow{\partial_1} \mathrm{Hom}_k(M,k) \to 0$. Recall from $O(3)$ minutes ago: $\mathrm{Hom}_k(M,k) \cong ke_0^* + ke_1^* + ke_2^*$ where e_i^* represent the dual basis for $1,x,y$ resp. and $x \cdot (f_0e_0^* + f_1e_1^* + f_2e_2^*) = f_1e_0^*$, $y \cdot (f_0e_0^* + f_1e_1^* + f_2e_2^*) = f_2e_0^*$.

So $\partial_1(\sum f_ie_i^*, \sum g_ie_i^*) = (f_1+g_2)e_0^*$ and $\mathrm{Tor}_0^R(k,\mathrm{Hom}_k(M,k)) \simeq k^{\oplus 2}$. $\partial_2(f_0e_0^* + f_1e_1^* + f_2e_2^*) = (f_2e_0^*, -f_1e_0^*)$. Thus, $\mathrm{Tor}_2^R(k,\mathrm{Hom}_k(M,k)) \simeq R/(x,y) = k$.

So our LES looks like $\cdots \to 0 \to k \xrightarrow{\alpha_2} k \xrightarrow{\beta_2} k^2 \xrightarrow{\delta_2} k^2 \xrightarrow{\alpha_1} \mathrm{Tor}_1^R(k,\mathrm{Hom}_k(M,k)) \xrightarrow{\beta_1} k^4 \xrightarrow{\delta_1} k \xrightarrow{\alpha_0} k^2 \xrightarrow{\beta_0} k^2 \to 0$ where the k^n all have 0-action given by x,y. So β_0 is an isomorphism, $\alpha_0 = 0$, δ_1 is surjective, and β_1 maps onto $k^{\oplus 3}$. From the other side, α_2 is an isomorphism, $\beta_2 = 0$, δ_2 is thus an injection and so further must be an isomorphism. Thus $\alpha_1 = 0$. But thus, β_1 is an injection, and so we've shown $\mathrm{Tor}_1^R(k,\mathrm{Hom}_k(M,k)) \simeq k^{\oplus 3}$ (this can also be observed directly from $\ker\partial_1/\mathrm{im}\partial_2$) and all the maps are canonically determined.