Category theory

Sunday, 4 October 2020 15:20

A category C consists of

ob C set/class

$X, y \in$ ob C $\quad \mathrm{Hom}_C(X, y)$ set

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & y \arrow[r, "g"] & Z
\end{tikzcd}$$

for all X $\quad id_X \in \mathrm{Hom}_C(X, X)$

$id_X \circ f = f$

$(f \circ g) \circ h = f \circ (g \circ h)$

Ex: A category with one object is a monoid

$\mathrm{Hom}_C(*, *)$ set with ass. mult. and identity = monoid.

Set category of sets

objects are sets and morphisms are functions.

Top   top. spaces   objects are
top. spaces and morphisms are
cont. maps.

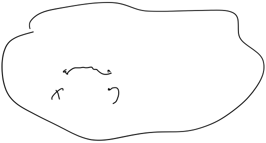
Ab   abelian groups

R ring R-mod   left R modules.

k field Vec_k vector spaces.

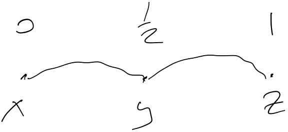
Grp , Ring , Ban_k

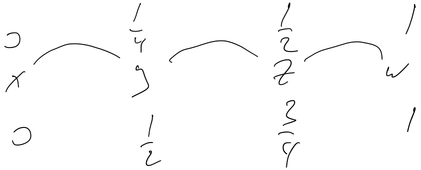
if X is a top. space



$\mathrm{ob}\ \Pi(X) = X$

$$\mathrm{Hom}_{\Pi(X)}(x,y) = \left\{ f : [0,1] \to X \;\middle|\; \begin{array}{l} f \text{ cont.} \\ f(0) = x \\ f(1) = y \end{array} \right\}$$





homotopy classes of paths

or we need ∞- category.

if C is a category,

define C^{op}:

$$\mathrm{ob}\, C^{op} = \mathrm{ob}\, C$$

$$\mathrm{Hom}_{C^{op}}(x, y) \overset{\mathrm{def}}{=} \mathrm{Hom}_C(y, x).$$

Vec_k^{op}

$$V \mapsto V^*$$

$$T \mapsto T^*$$

initial / final objects

C category. we call

I initial if for every

$X \in C$ there is a unique morphism

$I \xrightarrow{\exists!} X$.

$$\begin{tikzcd} I \arrow[r, shift left, "\exists!"] & I' \arrow[l, shift left, "\exists!"] \end{tikzcd}$$

isomorphism: $f: X \to Y$

$g: Y \to X$

$f \circ g = id \qquad g \circ f = id$

final object in C =

initial object in C^{op}.

F is final if for any

$X \in C \qquad X \xrightarrow{\exists!} F$

Set $\qquad \emptyset \qquad$ initial

$* \qquad$ final object

one-element set.

Zero object: if O is both initial and final it is called a zero object.

Ab, R-mod $\qquad 0$

$$\begin{tikzcd}
& 0 \arrow[dr] & \\
X \arrow[ur] \arrow[rr, squiggly, "0"] & & Y
\end{tikzcd}$$

monomorphism: $f: X \to Y$

for any $g_1, g_2 : Z \to X$

$$f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$$

or $\mathrm{Hom}_C(Z, X) \xrightarrow{f_*} \mathrm{Hom}_C(Z, Y)$

$$g \longmapsto f \circ g$$

f is monomorphism $\Leftrightarrow$ f_* is injective.

epimorphism: monomorphism in C^{op}

$f: X \to Y$

for all $g_1, g_2 : Y \to Z$

$$g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$$

$$\mathrm{Hom}(Y, Z) \xrightarrow{f^*} \mathrm{Hom}(X, Z)$$

$$g \longmapsto g \circ f$$

is injective.

Ex: in Set

mono = inj.

epi = surj.

Ring $\mathbb{Z} \to \mathbb{Q}$

mono

it is an epi of rings.

$k[x] \to k[x, x^{-1}]$ epi.

f mono + epi $\not\Rightarrow$ f iso.

Functors: $F: C \to D$

$F: \mathrm{ob}\, C \to \mathrm{ob}\, D$

$X \mapsto F(X)$

for all $X, Y \in C$

$\mathrm{Hom}_C(X, Y) \to \mathrm{Hom}_D(F(X), F(Y))$

$f \mapsto F(f)$

$$F(f \circ g) = F(f) \circ F(g)$$

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$$

Ex: Forgetful functors!

Forget: $Ab \to Set$

$R\text{-mod} \to Set$

$Top \to Set$

Free: $Set \to Ab$

$Set \to R\text{-mod}$

$Set \to Ring$

$* : Vect \to Vect^{op}$

$** : Vect \to Vect$

$$V \to V^{**}$$

$$Id \Rightarrow **$$

Natural transformations:

$F, G : C \to D$

$\alpha : F \Rightarrow G$

for all $x \in \mathrm{ob}\, C$ $\quad \alpha_x : F(x) \to G(x)$

$\alpha_x \in \mathrm{Hom}_D(F(x), G(x))$

for all $f : x \to y$

$$\begin{tikzcd}
F(x) \arrow[r, "\alpha_x"] \arrow[d, "F(f)"'] & G(x) \arrow[d, "G(f)"] \\
F(y) \arrow[r, "\alpha_y"'] & G(y)
\end{tikzcd}$$

commutes.

$Id : Vect \to Vect$

$** : Vect \to Vect$

$Id \Rightarrow **$

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & V^{**} \arrow[d] \\
W \arrow[r] & W^{**}
\end{tikzcd}$$

if C, D categories

$Fun(C, D)$ objects are functors

morphisms are natural transformations.

Cat category of (small) categories.

objects are small categories
morphisms are functors.

A, B, C are small categories

$$\mathrm{Hom}_{Cat}(A \times B, C) = \mathrm{Hom}_{Cat}(A, \mathrm{Fun}(B, C))$$

$A \times B$ is the cartesian product of A and B

$$\mathrm{ob}(A \times B) = \mathrm{ob}\, A \times \mathrm{ob}\, B$$

Cat is a cartesian closed category.

Note: the category of categories is a 2-category.

$$\begin{tikzcd} C \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & D \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$$

Equivalence of categories:

$$F : C \rightleftarrows D : G$$

$F \circ G = Id \qquad G \circ F = Id$

then C and D are

isomorphic cat.

$F \circ G \overset{\sim}{\Rightarrow} Id \qquad G \circ F \overset{\sim}{\Rightarrow} Id$

$F(G(X)) \cong X$

$G(F(y)) \cong y$

Adjoint Functors

$F : C \rightleftarrows D : G$

F and G are adjoint functors

F left adjoint, G right adjoint

a natural bijection

$\alpha_{x,y} : Hom_D(Fx, y) \xrightarrow{\sim} Hom_C(x, Gy)$

for all $x \in C$ $y \in D$

$g: x_2 \to x_1$ $h: y_1 \to y_2$

$$\begin{tikzcd}
\mathrm{Hom}_D(Fx_1, y_1) \arrow[r] \arrow[d] & \mathrm{Hom}_C(x_1, Gy_1) \arrow[d] \\
\mathrm{Hom}_D(Fx_2, y_2) \arrow[r] & \mathrm{Hom}_C(x_2, Gy_2)
\end{tikzcd}$$

commute.

Examples: Free is left adjoint to Forget.

Forget: $Ab \to Set$

Free : $Set \to Ab$

$\mathrm{Free}(X) = \mathbb{Z}[X]$ free abelian group on X.

$$\mathrm{Hom}_{Ab}(\mathbb{Z}[X], A) = \mathrm{Hom}_{Set}(X, \mathrm{forget}(A))$$

Forget : $\mathrm{Comm}_k \to \mathrm{Vect}_k$

Free : $\mathrm{Vect}_k \to \mathrm{Comm}_k$

$\mathrm{Free}(V) = S(V)$ symmetric alg. on V.

$$\mathrm{Hom}_{\mathrm{Comm}_k}(S(V), A) \simeq \mathrm{Hom}_{\mathrm{Vect}_k}(V, \mathrm{Forget}(A))$$

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limits / colimits

let $\underline{I}$ be a small category.

$F : \underline{I} \to C$

We call F a diagram in C.

A cone is an object $c \in C$ with morphisms $c \to F(i)$ for all $i \in \underline{I}$ s.t. for all $\alpha : i \to j$

$$\begin{tikzcd} & c \arrow[dl] \arrow[dr] & \\ F(i) \arrow[rr, "F(\alpha)"] & & F(j) \end{tikzcd}$$

Commutes.

there is a category of cones if a terminal object exists in the category of cones

we call it $\lim F$,

"universal cone".

dual: a co cone

$$\begin{tikzcd}
F(i) \arrow[r, "F(\alpha)"] \arrow[dr] & F(j) \arrow[d] & \cdots \\
 & c &
\end{tikzcd}$$

an initial object in the category of co cones is called $\operatorname{colim} F$.

Examples: if $\underline{I}$ is discrete category with only id. morphisms.

$F: \underline{I} \to C$ is a collection of objects in C

$\{X_i\}_{i \in \underline{I}}$ $X_i \in C$.

$\lim F = \underset{i \in \underline{I}}{\times} X_i$ products

$\operatorname{colim} F = \coprod_{i \in I} X_i$ coproduct.

Set: $\times_{i \in I} X_i$ Cartesian product

$\bigsqcup_{i \in I} X_i$ disjoint union

R-mod: $\{M_i\}_{i \in I}$ collection of R-modules

$\times_{i \in I} M_i$ product

$\bigoplus_{i \in I} M_i$ coproduct

Note: $M_1 \oplus M_2 \simeq M_1 \times M_2$

$$I = \{ \overline{i \rightrightarrows} j \}$$

$$\begin{tikzcd} \operatorname{Eq}(f,g) \arrow[r] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r] & \mathrm{Coeq}(f,g) \end{tikzcd}$$

if C has a zero object

recall we can define the

zero morphism

$$\begin{tikzcd} & 0 \arrow[dr] & \\ X \arrow[ur] \arrow[rr, "0"] & & Y \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Ker}(f) = \mathrm{Eq}(f,0) \arrow[r] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & Y \arrow[r] & \mathrm{Coeq}(f,0) = \mathrm{coker}(f) \end{tikzcd}$$

kernel

cokernel.

$f : M \to N$

R-mod: $\mathrm{Ker}(f) = f^{-1}(0)$

$= \{ m \in M \mid f(m) = 0 \}$

$\mathrm{coker}(f) = N / f(M)$

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let's assume that for

any $F : \underline{I} \to C$

$\lim F$, $\operatorname{colim} F$ exist.

$$\lim : \mathrm{Fun}(\underline{I}, C) \to C$$

$$F : \underline{I} \to C \mapsto \lim F$$

$$\operatorname{colim} : \mathrm{Fun}(\underline{I}, C) \to C$$

$$F : \underline{I} \to C \mapsto \operatorname{colim} F$$

$$\Delta : C \to \mathrm{Fun}(\underline{I}, C)$$

diagonal

$X \in C$

$\Delta X \in \mathrm{Fun}(\underline{I}, C)$

$(\Delta X)(i) = X$ all $i \in \underline{I}$.

$$\mathrm{Hom}_C(\operatorname{colim} F, X) = \mathrm{Hom}_{\mathrm{Fun}(\underline{I}, C)}(F, \Delta X)$$

$$\mathrm{Hom}_C(X, \lim F) = \mathrm{Hom}_{\mathrm{Fun}(\underline{I}, C)}(\Delta X, F)$$

colim is left adjoint to Δ

lim is right adjoint to Δ.

Suppose $L : C \rightleftarrows D : R$ is an adjunction.

$F : I \to C$ is a diagram in C.

$L \circ F : I \to D$ a diagram in D.

$$\mathrm{Hom}_D(\mathrm{colim}\, L \circ F, X) \simeq$$

$$\mathrm{Hom}_{\mathrm{Fun}(I, D)}(L \circ F, \Delta X) \simeq$$

$$\mathrm{Hom}_{\mathrm{Fun}(I, C)}(F, \Delta R(X)) \simeq$$

$$\mathrm{Hom}_C(\mathrm{colim}\, F, R(X)) \simeq$$

$$\mathrm{Hom}_D(L(\mathrm{colim}\, F), X)$$

natural bijection for all X.

Yoneda lemma $\Rightarrow$

$$\operatorname{colim} L \circ F \simeq L(\operatorname{colim} F)$$

left adjoint preserve colimits.

op: right adjoints preserve limits.

$$L(\operatorname{coker} f) \simeq \operatorname{coker} L(f)$$

$$R(\ker f) \simeq \ker R(f)$$

"left adjoints are right exact"

"right adjoints are left exact"