Abelian categories

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Additive categories

We call a category C Ab-enriched category if $\mathrm{Hom}_C(x,y) \in Ab$ for all $x,y \in C$ and composition is bilinear

$$(f+g)\circ h = f\circ h + g\circ h$$

Remark: Enrichment

Top. enriched categories ...

Def: A functor $F: C \to D$ between Ab-enriched categories is called additive if

$$\mathrm{Hom}_C(x,y) \to \mathrm{Hom}_D(Fx,Fy)$$

is a group homomorphism for all $x,y \in C$.

Def: An Ab-enriched category C is called additive if it admits finite coproducts.

prop: In an Ab-enriched category a finite product is also a coproduct, this also includes the case of a terminal object being an initial object hence a zero object.

idea of proof: suppose $*$ is a terminal object. $id_*: * \to *$ is the unique morphism in $\mathrm{Hom}_C(*,*) \in Ab$. so $id_* = 0$

for any object A and a morphism $f: * \to A$ we have that $f = f\circ id_* = f\circ 0 = 0 \in \mathrm{Hom}_C(*,A)$. So there is a unique morphism $* \to A$ for any A and $*$ is an initial object.

$$\begin{tikzcd} X_1 \arrow[d, "id"'] \arrow[r, "\exists!"] \arrow[rrd, bend left, "0"] & X_1 \times X_2 \arrow[dl] \arrow[dr] & \\ X_1 & & X_2 \end{tikzcd} \qquad \begin{tikzcd} & X_1 \times X_2 \arrow[dl] \arrow[dr] & X_2 \arrow[l, "\exists!"'] \arrow[d, "id"] \arrow[lld, bend right, "0"'] \\ X_1 & & X_2 \end{tikzcd}$$

so you can check that these maps $X_1 \to X_1 \times X_2 \leftarrow X_2$ make $X_1 \times X_2$ into a coproduct $X_1 \sqcup X_2 \xrightarrow{\sim} X_1 \times X_2$, called biproducts $X_1 \oplus X_2$

Remark: doesn't extend to infinite products / coproducts.

$$Ab \qquad \coprod_{i\in I} A_i = \bigoplus A_i \subset \prod_{i\in I} A_i = \prod_{i\in I} A_i$$

Def: An additive category C is pre-abelian if any morphism has a kernel and a cokernel.

Note: this is equivalent to asking for finite limits and finite colimits.

$$Eq(f,g) = \ker(f-g)$$

$$Coeq(f,g) = \operatorname{coker}(f-g)$$

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if C is pre-abelian

$$\begin{tikzcd} \ker(f) \arrow[r] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r] & \operatorname{coker} f \\ & \operatorname{coker}(\ker f) \arrow[r, "\exists!"] \arrow[d, phantom, "\parallel"] & \ker(\operatorname{coker} f) \arrow[u] \arrow[d, phantom, "\parallel"] & \\ & \operatorname{coim} f & \operatorname{im} f & \end{tikzcd}$$

Def: call f strict if $\operatorname{coim} f \to \operatorname{im} f$ is an isomorphism.

Def: A pre-abelian category is abelian if all morphisms are strict.

Remark: This is equivalent to: every mono is a kernel and every epi is a coker

$\iff$ every mono is the kernel of its cokernel. every epi is the cokernel of its kernel.

Ex: $Ab = \mathbb{Z}$-mod

R-mod

if A is an abelian category so is A^{op}.

if A is an abelian category and C is a category $Fun(C,A)$ is abelian.

so $Fun(C^{op},A)$ is abelian

abelian presheaves on C.

abelian sheaves form an abelian category.

$Ban_{\mathbb{R}}$ Banach spaces over $\mathbb{R}$.

$V \overset{i}{\hookrightarrow} W$ open inclusion. not closed.

$$\operatorname{coker}(V \overset{i}{\hookrightarrow} W) = W/\bar{V}$$

$$\ker(W \to W/\bar{V}) = \bar{V} \neq V$$

so $V \overset{i}{\hookrightarrow} W$ is not strict.

TopAb, FiltAb are not abelian categories.

strict maps satisfy some nice conditions making them into quasi-abelian categories.

Quillen exact categories.

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in an abelian category every morphism has a factorisation

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \\ & \operatorname{im} f \arrow[ur] & \end{tikzcd}$$

$X \to \operatorname{im} f$ is epi

$\operatorname{im} f \to Y$ is mono

Def: A sequence of morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is called exact (at B) if $\ker g = \operatorname{im} f$ (as subobjects of B).

A short exact sequence is a sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

exact at all places $\Rightarrow$ f is mono, g is epi $\ker g = \operatorname{im} f$ so $f = \ker g$ $g = \operatorname{coker} f$.

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let C, D be abelian categories. An additive functor $F: C \to D$ is called right exact if for every s.e.s. $0 \to A \to B \to C \to 0$ the sequence $F(A) \to F(B) \to F(C) \to 0$ is exact.

F is called left exact if $0 \to F(A) \to F(B) \to F(C)$ is exact for every s.e.s $0 \to A \to B \to C \to 0$.

right exact $\Rightarrow$ preserves cokernels

left exact $\Rightarrow$ preserves kernels.

F is exact if it is both left and right exact.

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lemma: let A be an abelian category and let $M \in A$. Then the functor

$$\mathrm{Hom}_A(M,-): A \to Ab$$

is left exact, so if $0 \to A \to B \to C \to 0$ is a s.e.s in A then

$$0 \to \mathrm{Hom}_A(M,A) \to \mathrm{Hom}_A(M,B) \to \mathrm{Hom}_A(M,C)$$

is exact in Ab.

by applying op

lemma: $\mathrm{Hom}_A(-,M): A^{op} \to Ab$ is left exact.

Def: let A be an abelian category. An object P is called projective if $\mathrm{Hom}_A(P,-)$ is exact.

an object I is called injective if $\mathrm{Hom}_A(-,I)$ is exact.

Ex: in R-mod

$$\mathrm{Hom}_{R\text{-mod}}(R,M) = M$$

$$f \longmapsto f(1)$$

so $\mathrm{Hom}_{R\text{-mod}}(R,-)$ is the forgetful functor to Ab

so clearly $\mathrm{Hom}_{R\text{-mod}}(R,-)$ is exact and R is projective.

any free module $R^I = \coprod_I R$ is projective.

$$\mathrm{Hom}_{R\text{-mod}}(R, \oplus M_n) = \oplus M_n$$

$$\oplus M_n = \coprod M_n$$

Def: Let A be an additive category call $c \in A$ compact if

$$\coprod_{i\in I} \mathrm{Hom}_A(c,b_i) \to \mathrm{Hom}(c, \coprod_{i\in I} b_i)$$

is an isomorphism for any family $\{b_i\}_{i\in I}$ s.t. $\coprod_{i\in I} b_i$ exists.

Note: R is a compact object in R-mod.

Note also that for any R-module M we can find an epi $R^I \to M \to 0$

$\iff$ pick generators of M as an R-module.

Thm: let A be an abelian category with all (small) coproducts. Assume that P is a compact, projective generator. Then the functor

$$\mathrm{Hom}_A(P,-): A \to \mathrm{End}_A(P)^{op}\text{-mod}$$

is an equivalence of categories.

idea of proof: construct an adjoint functor

$$R = \mathrm{End}_A(P)^{op} \mapsto P$$

$$R^I \mapsto P^I$$

$$R^J \to R^I \to M \to 0$$

presentation of M by generators and relations

$$M \mapsto \operatorname{coker}(P^J \to P^I)$$

can write $P \otimes_R -$

Morita theory

$$R\text{-mod} \overset{?}{\simeq} S\text{-mod}$$

Freyd - Mitchell embedding theorem: if A is a small abelian category, there is a ring R and an exact fully faithful functor from A to R-mod.

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uses the Yoneda embedding

$$y: A \to Fun(A^{op}, Ab)$$