Double complexes

Monday, 12 October 2020 15:30

$Ch(A)$ is an abelian category. So we can form $Ch(Ch(A))$. We want a functor

$Ch(Ch(A)) \to Ch(A)$ which is a sort of a right adjoint to the constant functor.

In order to get nicer formulas we will define an isomorphic category to $Ch(Ch(A))$.

Def: A double complex (bicomplex) in A is a family $\{C_{p,q}\}$ of objects in A with maps

$d^h: C_{p,q} \to C_{p-1,q}$ $\quad d^v: C_{p,q} \to C_{p,q-1}$

$(d^h)^2 = (d^v)^2 = 0$

$d^v d^h + d^h d^v = 0$

$$\begin{array}{ccc} C_{p-1,q} & \leftarrow & C_{p,q} \\ \downarrow & & \downarrow \\ C_{p-1,q-1} & \leftarrow & C_{p,q-1} \end{array}$$

anti-commutes!

Note: if we replace $d^v_{p,q}$ by $(-1)^p d^v_{p,q}$ we get that the squares commute so we have an object in $Ch(Ch(A))$ and this gives an iso. of categories.

Total complexes

given a bicomplex $C = \{C_{p,q}\}$ we can define complexes

$$Tot^{\Pi}(C)_n = \prod_{p+q=n} C_{p,q}$$

$$Tot^{\oplus}(C)_n = \bigoplus_{p+q=n} C_{p,q}$$

$d = d^h + d^v$ **Note:** $d^2 = 0$

Note: for this to be always well-defined we need that A has countable products/coproducts.

Acyclic assembly lemma:

Let $C = \{C_{p,q}\}$ be a double complex. If ① C is an upper half-plane complex with exact columns or a ② right half plane complex with exact rows then $Tot^{\Pi}(C)$ is acyclic.

If C is an ③ upper half-plane complex with exact rows or a ④ right half plane complex with exact columns then $Tot^{\oplus}(C)$ is acyclic.

Note: We will prove ①. Check that the rest follow from ① or use the same proof idea. Need less: for ① and ③ need diagonals are bounded in lower right, for ② and ④ diagonals are bounded in upper left.

proof of ①: let C be an upper half-plane bicomplex with exact columns.

let's show that $H_0(Tot^{\Pi}(C)) = 0$; by translation we will get that $Tot^{\Pi}(C)$ is acyclic.

let $c = (\dots, c_{-2,2}, c_{-1,1}, c_{0,0}) \in \prod C_{-p,p} = Tot^{\Pi}(C)_0$ be a 0-cycle. We will use induction to find elements $b_{-p,p+1}$ s.t.

(*) $d^v(b_{-p,p+1}) + d^h(b_{-p+1,p}) = c_{-p,p}$

so we will get an element $b \in \prod C_{-p,p+1}$ s.t. $d(b) = c$ showing $H_0(Tot^{\Pi}(C)) = 0$.

let $b_{1,0} = 0$ for $p = -1$

$c_{0,-1} = 0$ so $d^v(c_{0,0}) = 0$

since the 0^{th} column is exact there is a $b_{0,1} \in C_{0,1}$ s.t. $d^v(b_{0,1}) = c_{0,0}$. By induction

$d^v(c_{-p,p} - d^h(b_{-p+1,p})) =$

$d^v(c_{-p,p}) + d^h d^v(b_{-p+1,p}) =$

$d^v(c_{-p,p}) + d^h(c_{-p+1,p-1}) -$

$d^h d^h(b_{-p+2,p-1}) = 0$

since the p^{th} column is exact there is $b_{-p,p+1}$ s.t.

$d^v(b_{-p,p+1}) = c_{-p,p} - d^h(b_{-p+1,p})$

This lemma will be used to show that the derived functors of $\otimes$ and Hom don't depend on which variable you derive: balancing Tor and Ext.

Spectral sequences