Mapping cones and cylinders

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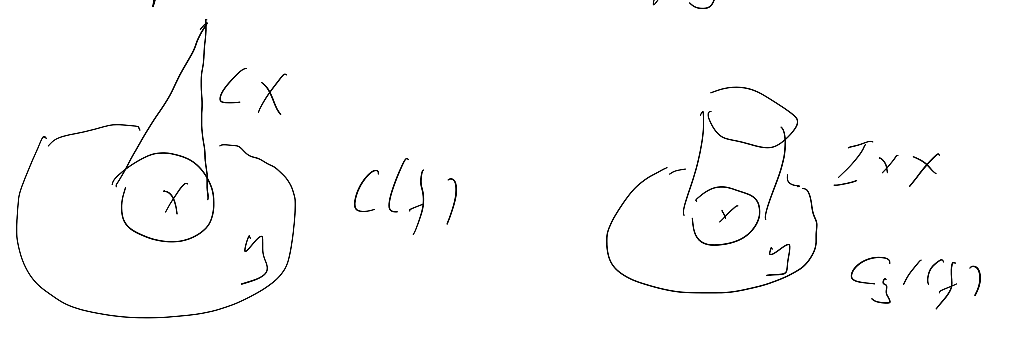
$f: X \to Y$ a cont. map of top. spaces.

mapping cone mapping cylinder



for complexes:

$f: B_\cdot \to C_\cdot$

$\text{cone}(f)_n = B_{n-1} \oplus C_n$

$d(b,c) = (-d(b),\ d(c) - f(b))$

$$\begin{pmatrix} -d_B & 0 \\ -f & d_C \end{pmatrix} : \begin{matrix} B_{n-1} \\ \oplus \\ C_n \end{matrix} \longrightarrow \begin{matrix} B_{n-2} \\ \oplus \\ C_{n-1} \end{matrix}$$

there is a s.e.s.

$$0 \to C \to \text{cone}(f) \xrightarrow{\delta} B[-1] \to 0$$

$c \mapsto (0,c)$

$(b,c) \mapsto -b$

the l.e.s.

$$\cdots \to H_{n+1}(\text{cone}(f)) \xrightarrow{\delta_*} H_n(B) \xrightarrow{\partial} H_n(C) \to H_n(\text{cone}(f)) \to \cdots$$

<u>lemma:</u> $\partial = f_*$

<u>proof:</u> $b \in B_n$ cycle we can lift it to $(-b, 0)$ in $\text{cone}(f)$ applying the diff. we get

$(db, fb) = (0, fb)$ so

$\partial[b] = [fb] = f_*[b]$

<u>cor:</u> $f: B_\cdot \to C_\cdot$ is a q.i. $\iff$ $\text{cone}(f)$ is exact.

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$\text{cyl}(f)_n = B_n \oplus B_{n-1} \oplus C_n$

$d(b, b', c) = (d(b) + b',\ -d(b'),\ d(c) - f(b'))$

$$\begin{pmatrix} d_B & id_B & 0 \\ 0 & -d_B & 0 \\ 0 & -f & d_C \end{pmatrix}$$

<u>check:</u> $d^2 = 0$

<u>Ex:</u> let $\text{cyl}(C)$ be the mapping cylinder of $id_C: C \to C$

$f, g: C \to D$ are chain homotopic $\iff$ $\exists s$ s.t. $(f, s, g): \text{Cyl}(C) \to D$

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let $\alpha: C \to \text{Cyl}(f)$ be the inclusion $c \mapsto (0,0,c)$

$\text{Cyl}(f) / \alpha(C) = \text{cone}(-id_B)$

which is null-homotopic (split exact) so if we look at the l.e.s of

$$0 \to C \xrightarrow{\alpha} \text{cyl}(f) \to \text{cone}(-id_B) \to 0$$

shows that α is a q.i.

we can also look at

$\beta: \text{cyl}(f) \to C$

$\beta(b, b', c) = f(b) + c$

see that $\beta\alpha = id_C$

<u>check:</u> $s(b, b', c) = (0, b, 0)$ is a chain homotopy from $id_{\text{cyl}(f)}$ to $\alpha\beta$.

so α is a chain homotopy eq. between C and $\text{Cyl}(f)$.

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let $0 \to B_\cdot \xrightarrow{f} C_\cdot \xrightarrow{g} D_\cdot \to 0$ be a s.e.s of complexes.

define $\varphi: \text{cone}(f) \to D$

$\varphi(b, c) = g(c)$

we get a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & 0 & \to & C & \to & \text{cone}(f) & \xrightarrow{\delta} & B[-1] \to 0 \\ & & & & \downarrow\alpha & & \| & & \\ 0 & \to & B & \to & \text{cyl}(f) & \to & \text{cone}(f) & \to & 0 \\ & & \| & & \downarrow\beta & & \downarrow\varphi & & \\ 0 & \to & B & \to & C & \to & D & \to & 0 \end{array}$$

since β is a q.i. we get that φ is a q.i. (5 lemma)

the l.e.s gives

$$\begin{array}{ccccccc} \xrightarrow{\partial} H_n(B) & \to & H_n(\text{cyl}(f)) & \to & H_n(\text{cone}(f)) & \xrightarrow{\partial} & H_{n-1}(B) \\ \| & & \downarrow\cong & & \downarrow\cong & & \| \\ \to H_n(B) & \to & H_n(C) & \to & H_n(D) & \xrightarrow{\partial} & H_{n-1}(B) \end{array}$$

<u>Ex:</u> $H_n(D) \cong H_n(\text{cone}(f)) \xrightarrow{-\delta} H_n(B[-1]) = H_{n-1}(B)$ is equal to the connecting morphism ∂ of the l.e.s of $0 \to B_\cdot \to C_\cdot \to D_\cdot \to 0$ so the l.e.s attached to f using $\text{cone}(f)$ is the same as the l.e.s. attached to $0 \to B \xrightarrow{f} C \xrightarrow{g} D \to 0$

do the same using g.