Long exact sequences

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Thm: If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a s.e.s of complexes there is a natural map $\partial: H_n(C) \to H_{n-1}(A)$ the connecting homomorphism making

$$\cdots \to H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{f} H_n(B) \to H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

an exact seq.

(same with cochain complexes $\partial: H^n(C) \to H^{n+1}(A)$)

Snake lemma: given a comm. diagram

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "f"] & B' \arrow[r, "p"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C
\end{tikzcd}$$

with exact rows we get an exact seq.

$$\ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\partial} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h)$$

"$\partial(c') = i^{-1} g\, p^{-1}(c')$"

$c' \in \ker(h)$

(check that it doesn't depend on choices).

if $A' \to B'$ is monic so is $\ker(f) \to \ker(g)$ and if $B \to C$ is epic then so is $\operatorname{coker}(g) \to \operatorname{coker}(h)$

Remark: By Freyd-Mitchell embedding theorem we can assume we are working in R-mod for some R.

Apply snake lemma to

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r] \arrow[d, "d"] & B_n \arrow[r] \arrow[d, "d"] & C_n \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

we get that the rows of the commutative diagram

$$\begin{tikzcd}
& A_n/dA_{n+1} \arrow[r] \arrow[d, "d"] & B_n/dB_{n+1} \arrow[r] \arrow[d, "d"] & C_n/dC_{n+1} \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & Z_{n-1}(A) \arrow[r] & Z_{n-1}(B) \arrow[r] & Z_{n-1}(C)
\end{tikzcd}$$

are exact. Apply the snake lemma again to get that

$$H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C)$$

is exact. putting all these exact seq. together we get the long exact seq. on homology.

Ex: show that there is a functor

category of s.e.s of complexes in $\mathcal{A}$ $\longrightarrow$ long exact seq. in $\mathcal{A}$.