Complexes

Let A be an abelian category.

A chain complex is a family $\{C_n\}_{n\in\mathbb{Z}}$ of objects in A with morphisms $d_n : C_n \to C_{n-1}$ s.t. $d^2 = 0$ $d_{n+1} \circ d_n = 0$ $\forall n$.

d_n are called the differentials

$\ker d_n = Z_n$ n-cycles

$\operatorname{im} d_{n+1} = B_n$ n-boundaries

$$B_n \hookrightarrow Z_n \hookrightarrow C_n$$

$$H_n(C_\cdot) = Z_n / B_n = \operatorname{coker}(B_n \hookrightarrow Z_n)$$

n-th homology object of the complex.

We can form a category $Ch(A)$ with objects chain complexes and morphisms $u : C_\cdot \to D_\cdot$ a family $\{u_n : C_n \to D_n\}$ s.t.

$$\begin{array}{ccc} C_n & \xrightarrow{d_n} & C_{n-1} \\ {\scriptstyle u_n}\downarrow & & \downarrow{\scriptstyle u_{n-1}} \\ D_n & \xrightarrow{d_n} & D_{n-1} \end{array}$$

commutes for all n.

Ex: show that $u : C_\cdot \to D_\cdot$ induces a morphism $H_n(C_\cdot) \to H_n(D_\cdot)$ and

$H_*: Ch(A) \to A$ is a functor.

Def: $u : C_\cdot \to D_\cdot$ is called a quasi-isomorphism if the induced maps $H_n(C_\cdot) \to H_n(D_\cdot)$ are iso. for all n.

Ex: the following are equivalent:

① $C_\cdot$ is exact at every C_n

② $C_\cdot$ is acyclic $H_n(C_\cdot) \cong 0$ for all n.

③ $0 \to C_\cdot$ (here 0 is the complex with all objects $0 \in A$) is a quasi-isomorphism.

op

A cochain complex is given by $\{C^n\}_{n\in\mathbb{Z}}$ $d^n : C^n \to C^{n+1}$

$d^2 = 0$

$Z^n(C^\cdot) = \ker d^n$ n-cocycles

$B^n(C^\cdot) = \operatorname{im} d^{n-1}$ n-coboundaries.

$H^n(C^\cdot) = Z^n / B^n$ nth cohomology.

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A chain complex is bounded if $C_n = 0$ unless $a \le n \le b$

it has amplitude $[a, b]$.

bounded above $n > b \Rightarrow C_n = 0$

bounded below $n < a \Rightarrow C_n = 0$

we get full subcategories of $Ch(A)$: $Ch_b(A)$

$Ch_-(A)$ $Ch_+(A)$

$(Ch_{\ge 0}(A))$

Example: let X be a top. space and $S_k = S_k(X)$ the free R-module on the set of cont. maps $\Delta_k \to X$.

restriction to the ith face of Δ_k $0 \le i \le k$ gives a map $S_k \xrightarrow{\partial_i} S_{k-1}$

$d = \sum (-1)^i \partial_i$ gives a chain complex.

the singular chain complex of X. The nth homology of this chain complex is

$H_n(X; R)$ the nth singular homology.

Remark: the passage from a simplicial object to a chain complex is part of the Dold-Kan equivalence:

$$SA = \left\{\begin{array}{c}\text{simplicial}\\ \text{objects in}\\ A\end{array}\right\} \xrightarrow[\text{normalized chain complex}]{N} Ch_{\ge 0}(A)$$

$$\pi_*(X) \cong H_*(NX)$$

simplicial homotopy $\Longleftrightarrow$ chain homotopy

$$\begin{array}{ccccc} A & \xrightarrow{\varepsilon_0} & A \times \Delta[1] & \xleftarrow{\varepsilon_1} & A \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle h} & \swarrow{\scriptstyle g} & \\ & & B & & \end{array}$$

Def: a chain map $f : C_\cdot \to D_\cdot$ is null homotopic if there are maps $s_n : C_n \to D_{n+1}$ s.t. $f = ds + sd$

$$f_n = d_{n+1} s_n + s_{n-1} d_n$$

$$\begin{array}{ccccc} C_{n+1} & \xrightarrow{d} & C_n & \xrightarrow{d} & C_{n-1} \\ & \swarrow{\scriptstyle s_n} & \downarrow{\scriptstyle f_n} & \swarrow{\scriptstyle s_{n-1}} & \\ D_{n+1} & \xrightarrow{d} & D_n & \xrightarrow{d} & D_{n-1} \end{array}$$

$f, g : C_\cdot \to D_\cdot$ are called chain homotopic if $f - g$ is null homotopic

$\exists s$ $f - g = sd + ds$.

$f \sim g$.

Ex: if $f : C_\cdot \to D_\cdot$ is null homotopic then

$$f_* = 0 : H_*(C_\cdot) \to H_*(D_\cdot)$$

if $f \sim g$

$$f_* = g_* : H_*(C_\cdot) \to H_*(D_\cdot)$$

Ex: $Ch(A)$ is an abelian category. kernels and cokernels are computed componentwise

$0 \to A_\cdot \to B_\cdot \to C_\cdot \to 0$ is exact

$\Leftrightarrow$ $0 \to A_n \to B_n \to C_n \to 0$ is exact for all n.

Truncation

$$(\tau_{\ge n} C_\cdot)_i = \begin{cases} 0 & i < n \\ Z_n & i = n \\ C_i & i > n \end{cases}$$

$H_i(\tau_{\ge n} C_\cdot) = 0$ $i < n$

$H_i(\tau_{\ge n} C_\cdot) = H_i(C_\cdot)$ $i \ge n$.

$\tau_{<n} C = C / \tau_{\ge n} C$

$H_i(\tau_{<n} C) = H_i(C)$ $i < n$

$H_i(\tau_{<n} C) = 0$ $i \ge n$.

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we can also introduce the "stupid" truncations:

$$(\sigma_{<n} C)_i = \begin{cases} C_i & i < n \\ 0 & i \ge n \end{cases}$$

$H_n(\sigma_{<n} C) = C_n / B_n \ne H_n C$.

$\sigma_{\ge n} C = C / \sigma_{<n} C$.

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Translations

$(C_\cdot[p])_n = C_{n+p}$ $(C^\cdot[p])^n = C^{n-p}$

diff. $(-1)^p d$.

degree 0 of $C[p]$ is C_p.

$H_n(C_\cdot[p]) = H_{n+p}(C)$

$H^n(C^\cdot[p]) = H^{n-p}(C)$.

if $f : C_\cdot \to D_\cdot$ $f[p]_n = f_{n+p}$

$f : C^\cdot \to D^\cdot$ $f[p]^n = f^{n-p}$

$[p] : Ch(A) \to Ch(A)$.