Tensor products

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Let k be a field and Vec_k the category of k-vector spaces.

recall: given $V, W, L \in \mathrm{Vec}_k$ denote by $\mathrm{Bilin}(V, W; L)$ the set of bilinear transformations $V \times W \to L$

Can we write this using category theory language?

Note that if $V, W \in \mathrm{Vec}_k$ then

$$\mathrm{Hom}_{\mathrm{Vec}_k}(V, W) \; (= \mathrm{Hom}_k(V, W))$$

is actually a k-vector space.

Let's denote this k-v.s. by

$$\underline{\mathrm{Hom}}(V, W) \in \mathrm{Vec}_k$$

Note: the category Vec_k is enriched over itself.

$$\mathrm{Bilin}(V, W; L) \simeq \mathrm{Hom}_{\mathrm{Vec}_k}(V, \underline{\mathrm{Hom}}(W, L))$$

$$\simeq \mathrm{Hom}_{\mathrm{Vec}_k}(W, \underline{\mathrm{Hom}}(V, L))$$

$\underline{\mathrm{Hom}}(-,-)$ inner Hom.

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Question: Is $\mathrm{Bilin}(V, W; -) : \mathrm{Vec}_k \to \mathrm{Set}$ representable? In other words, is there a v.s. $V \otimes_k W$ with a natural iso. of functors

$$\mathrm{Bilin}(V, W; -) \simeq \mathrm{Hom}_{\mathrm{Vec}_k}(V \otimes_k W, -)\,?$$

This is equivalent to asking for the universal bilinear transformation: a v.s. $V \otimes_k W$ with a bilinear map

$$V \times W \to V \otimes_k W \quad \text{s.t.}$$

for any other bilinear map

$$\begin{tikzcd} & V\otimes_k W \arrow[d, dashed, "\exists!"] \\ V\times W \arrow[ur] \arrow[r] & L \end{tikzcd}$$ commutes.

This is equivalent to: asking that the functor $\underline{\mathrm{Hom}}(W,-): \mathrm{Vec}_k \to \mathrm{Vec}_k$ has a left adjoint

$$\otimes_k W : \mathrm{Vec}_k \to \mathrm{Vec}_k$$

$$\mathrm{Hom}_{\mathrm{Vec}_k}(- \otimes_k W, -) = \mathrm{Hom}_{\mathrm{Vec}_k}(-, \underline{\mathrm{Hom}}(W,-))$$
$$\cong \mathrm{Bilin}(-, W; -)$$

Such a v.s. exists:

look at the v.s. spanned by $V\times W$ $\quad k[V\times W]$ and look at the subspace spanned by

$$(v+v', w) - (v,w) - (v',w)$$
$$(v, w+w') - (v,w) - (v,w')$$
$$(\alpha v, w) - \alpha(v,w) \qquad (v, \alpha w) - \alpha(v,w)$$

call it M

$$\begin{tikzcd} V\times W \arrow[r] \arrow[rr, bend right, "\text{bilinear}"'] & k[V\times W] \arrow[r] & k[V\times W]/M \end{tikzcd}$$

$$\mathrm{Hom}_{\mathrm{Vec}_k}\left(k[V\times W]/M, L\right) \simeq \mathrm{Bilin}(V,W;L)$$

so $V\otimes_k W$ exists.

Note: $V\otimes_k W$ is given up to a unique iso.

$$\begin{tikzcd} & V\otimes_k W \arrow[d, shift right, "\exists!"'] \\ V\times W \arrow[ur] \arrow[r] & (V\otimes_k W) \arrow[u, shift right, "\exists!"'] \end{tikzcd}$$

iso.

since $\mathrm{Bilin}(V,W;L) \simeq \mathrm{Bilin}(W,V;L)$

we get $V\otimes_k W \simeq W\otimes_k V$

$$k\otimes_k W \simeq W\otimes_k k \simeq W$$

$\mathrm{Trilin}(V_1,V_2,V_3;L) \simeq$

$\mathrm{Bilin}(V_1 \otimes_k V_2, V_3; L) \simeq$

$\mathrm{Bilin}(V_1, V_2 \otimes_k V_3; L)$

so there is a canonical iso.

$$(V_1 \otimes_k V_2) \otimes_k V_3 \simeq V_1 \otimes_k (V_2 \otimes_k V_3)$$

we get a closed symmetric monoidal category

$$(\mathrm{Vec}_k, - \otimes_k -, \underline{\mathrm{Hom}}(-,-))$$

Note: a k-algebra is a monoid object in $(\mathrm{Vec}_k, \otimes_k)$: an object $A \in \mathrm{Vec}_k$ with mult. $m: A \otimes_k A \to A$ and unit $i: k \to A$ s.t. the axioms of a monoid hold ass. unit.

$$\begin{tikzcd} A \otimes_k A \otimes_k A \arrow[r, "m \otimes id"] \arrow[d, "id \otimes m"'] & A \otimes A \arrow[d, "m"] \\ & \end{tikzcd}$$

commutes.

$$A \otimes A \xrightarrow{m} A$$

$$(ab)c = a(bc)$$

Since $- \otimes_k W$ is a left adjoint it commutes with arbitrary colimits.

$$\left(\bigoplus_{i \in I} V_i \right) \otimes_k W \simeq \bigoplus_{i \in I} (V_i \otimes_k W)$$

Note: $- \otimes_k W$ doesn't commute with infinite products in general.

Non-linear version: Set

the functor

$$\mathrm{Hom}_{Set}(Y, -) : Set \to Set$$

has a left adjoint $- \times Y$

$$\mathrm{Hom}_{Set}(X \times Y, Z) = \mathrm{Hom}_{Set}(X, \mathrm{Hom}_{Set}(Y, Z))$$

this is an example of a Cartesian closed category

In Vec_R $\overset{\text{binary}}{\text{product}} = \underbrace{\overset{\text{binary}}{\text{coproduct}}} = \text{biproduct}$

is $V \oplus W$. very different from

$V \otimes_R W$

$$\dim (V \oplus W) = \dim V + \dim W$$

$$\dim (V \otimes_R W) = \dim V \cdot \dim W.$$

Exactly the same works for any commutative ring R: the category R-mod is a closed symmetric monoidal category.

$$(R\text{-mod}, - \otimes_R -, \underline{\mathrm{Hom}}_R(-,-))$$

$$\mathrm{Hom}_{R\text{-mod}}(M \otimes_R N, L) \simeq \mathrm{Hom}_{R\text{-mod}}(M, \underline{\mathrm{Hom}}(N,L))$$

$- \otimes_R M$ is a left adjoint

so commutes with arbitrary (small) colimits

$$R^J \to R^I \to N \to 0$$

free presentation : generators and relations for the module N.

$$R^J \otimes_R M \to R^I \otimes_R M \to N \otimes_R M \to 0$$

$$R^I = \bigoplus_{i \in I} R$$

$$R^J = \bigoplus_{j \in J} R$$

$$R^I \otimes_R M = \Big(\bigoplus_I R\Big) \otimes_R M$$

$$= \bigoplus_I (R \otimes_R M)$$

$$= \bigoplus_I M = M^I$$

$$M^J \to M^I \to N \otimes_R M \to 0$$

Suppose $I \subset R$ is an ideal generated by $\{x_i\}_{i \in J}$

$$R^J \to R \to R/I \to 0$$

$$\begin{tikzcd}
R^J \otimes_R M \arrow[r] & R \otimes_R M \arrow[r] & R/I \otimes_R M \arrow[r] & 0 \\
M^J \arrow[u, phantom, sloped, "\cong"] & M \arrow[u, phantom, sloped, "\cong"] \arrow[ur] & &
\end{tikzcd}$$

$$R/I \otimes_R M \cong M/IM$$

if R is not commutative we have

$$- \otimes_R M : \text{mod-}R \to Ab$$

$M \in R\text{-mod}$

this has a right adjoint

$$Hom_{Ab}(M, -) : Ab \to \text{mod-}R$$

prop: if B is an R-S bimodule and C is a right S-module then

$Hom_S(B, C)$ is a right R module

$$(fr)(b) \overset{\text{def}}{=} f(rb)$$

$$Hom_S(B, -) : \text{mod-}S \to \text{mod-}R$$

is a right adjoint to

$$- \otimes_R B : \text{mod-}R \to \text{mod-}S$$

$$\text{Hom}_S(A \otimes_R B, C) \simeq \text{Hom}_R(A, \text{Hom}_S(B, C))$$

Remark: a way to prove this is by noticing / defining

$$M \otimes_R N \overset{\text{def}}{=} \text{Coeq}\,(M \otimes_{\mathbb{Z}} R \otimes_{\mathbb{Z}} N \rightrightarrows M \otimes_{\mathbb{Z}} N)$$

using properties of $- \otimes_{\mathbb{Z}} -$ and Coeq.