Projectives and Injectives

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Recall that an object P in $\mathcal{A}$ (abelian category) is called projective if $\mathrm{Hom}_{\mathcal{A}}(P,-)$ is exact.

I in $\mathcal{A}$ is called injective if $\mathrm{Hom}_{\mathcal{A}}(-,I)$ is exact.

$\mathrm{Hom}_{\mathcal{A}}$ is always left exact.

<u>Lemma:</u> 1) P is projective $\iff$

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists"'] \arrow[d] & \\ B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

2) I is injective $\iff$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[dl, dashed, "\exists"] \\ & I & \end{tikzcd}$$

An abelian category has enough projectives (injectives) if for any object M there is an epi. $P \to M \to 0$ where P is proj. (mono. $0 \to M \to I$ where I is injective).

<u>claim:</u> let R be a ring.

1) P is a proj. R-module $\iff$ P is a direct summand of a free module.

2) R-mod has enough projectives

<u>proof:</u> 1) Assume P is projective. Take a free module F surjecting onto P $\quad F \to P \to 0$

since P is projective this has a section

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists"'] \arrow[d, "id"] & \\ F \arrow[r] & P \arrow[r] & 0 \end{tikzcd}$$

so P is a direct summand of F.

check that free modules are projective ($\mathrm{Hom}_R(F,-)$ is exact) and direct summand of free is projective.

2) any R-module has a surjection from a free module.

<u>Baer's criterion:</u> A right R-module E is injective $\iff$ for every right ideal J of R and every $J \to E$ can be extended to a map $R \to E$.

<u>proof:</u> one direction is the definition applied to

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d] & R \arrow[dl, dashed, "\exists"] \\ & E & \end{tikzcd}$$

the other direction uses the axiom of choice (Zorn's lemma)

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & B \\ & E & \end{tikzcd}$$

look at the poset of extensions

$$\begin{tikzcd} A \arrow[r, phantom, "\subset"] \arrow[d, "\alpha"'] & A' \arrow[r, phantom, "\subset"] \arrow[dl, "\alpha'"] & B \\ E & & \end{tikzcd}$$

By Zorn's lemma there is a maximal extension $\alpha' : A' \to E$.

<u>claim:</u> $A' = B$.

suppose $A' \neq B$ let $b \in B \setminus A'$

$J = \{ r \in R \mid br \in A' \}$ is an ideal of R.

$J \xrightarrow{b} A' \to E$ extends to $f : R \to E$. let $A'' = A' + bR$ and define

$\alpha'' : A'' \to E$

$\alpha''(a + br) = \alpha'(a) + f(b)$.

this is well-defined and contradicts the maximality of α'. Hence $A' = B$.

<u>Cor:</u> if R is a PID ($\mathbb{Z}$ for example) then an R-module is injective iff it is divisible.

<u>Example:</u> in $Ab = \mathbb{Z}$-mod

$\mathbb{Q}$, $\mathbb{Z}_{p^\infty} = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}$ are injective.

Every inj. abelian group I is a direct sum $I = I_{tor} \oplus I_{free}$

I_{free} is a $\mathbb{Q}$-v.s. and I_{tor} is a direct sum of copies of $\mathbb{Z}_{p^\infty}$ for different primes p.

<u>claim:</u> Ab has enough injectives.

<u>proof:</u> for an abelian group A let

$$I(A) = \prod_{\mathrm{Hom}_{Ab}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$$

$I(A)$ is injective as a product of injectives.

there is a canonical map $e_A : A \to I(A)$ the φ-component of $e_A(a)$ ($\varphi \in \mathrm{Hom}_{Ab}(A, \mathbb{Q}/\mathbb{Z})$) is $\varphi(a)$. This map is injective as we can always find $f : A \to \mathbb{Q}/\mathbb{Z}$ with $f(a) \neq 0$ for any $a \in A$. uses the injectivity of $\mathbb{Q}/\mathbb{Z}$.

<u>Inj/proj and adjoints</u>

<u>Prop:</u> If an additive functor $R : \mathcal{B} \to \mathcal{A}$ is right adjoint to an exact functor and I is injective in $\mathcal{B}$ then $R(I)$ is inj. in $\mathcal{A}$.

If an additive functor $L : \mathcal{A} \to \mathcal{B}$ is left adjoint to an exact functor $R : \mathcal{B} \to \mathcal{A}$ and P is projective in $\mathcal{A}$ then $L(P)$ is projective in $\mathcal{B}$.

<u>proof:</u> we need to show that $\mathrm{Hom}_{\mathcal{A}}(-, R(I))$ is exact.

$\mathrm{Hom}_{\mathcal{A}}(-, R(I)) = \mathrm{Hom}_{\mathcal{B}}(L(-), I)$

this is a composition of two exact functors: L and $\mathrm{Hom}_{\mathcal{B}}(-, I)$, hence exact.

$\mathrm{Hom}_{Ab}(M, A) \cong \mathrm{Hom}_{mod\text{-}R}(M, \mathrm{Hom}_{Ab}(R, A))$

<u>Cor:</u> if I is an injective abelian group then $\mathrm{Hom}_{Ab}(R, I)$ is an injective R-module.

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let M be an R-module

$$I(M) = \prod_{\mathrm{Hom}_R(M, \mathrm{Hom}_{Ab}(R, \mathbb{Q}/\mathbb{Z}))} \mathrm{Hom}_{Ab}(R, \mathbb{Q}/\mathbb{Z})$$

this is injective as a product of injectives and there is a canonical mono

$e_M : M \to I(M)$

so R-mod has enough injectives