Projective resolutions

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Let $\mathcal{A}$ be an abelian category.

Def: Let M be an object of $\mathcal{A}$. A left resolution of M is a complex $P_\cdot$ with $P_i = 0$ $i<0$ and a map $\varepsilon: P_0 \to M$ s.t.

$$\cdots \to P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\varepsilon} M \to 0$$

is exact.

If each P_i is projective we call it a projective resolution.

Note: this is equivalent to

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "0"] & P_1 \arrow[r] \arrow[d, "0"] & P_0 \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

is a quasi-iso.

Remark: By applying op we get the notion of a right resolution

$$0 \to M \to I^0 \to I^1 \to \cdots$$

and injective resolutions.

Lemma: If $\mathcal{A}$ has enough projectives then every object has a projective resolution.

proof: Let M be an object. Let $\varepsilon_0: P_0 \to M$ be an epi. where P_0 is proj. Let $M_0 = \ker(\varepsilon_0)$. By induction given M_{n-1} let $\varepsilon_n: P_n \to M_{n-1}$ be an epi. where P_n is proj. and $M_n = \ker(\varepsilon_n)$. Let d_n be the composition

$$P_n \to M_{n-1} \to P_{n-1}.$$

$$d_n(P_n) = M_{n-1} = \ker(d_{n-1})$$

so this is a projective resolution.

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & & & \\
 & & & M_1 \arrow[ur] \arrow[dr] & & & & & \\
\cdots \arrow[rr] & & P_2 \arrow[ur] \arrow[rr] & & P_1 \arrow[rr] \arrow[dr] & & P_0 \arrow[r] & M \arrow[r] & 0 \\
 & M_2 \arrow[ur] & & & & M_0 \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Comparison thm: Let $P_\cdot \xrightarrow{\varepsilon} M$ be a proj. resolution of M (enough to assume that $\cdots \to P_1 \to P_0 \to M \to 0$ is a complex with P_i projective) and $f': M \to N$. Then for every resolution $Q_\cdot \xrightarrow{\eta} N$ there is a chain map $f: P_\cdot \to Q_\cdot$ lifting f' ($\eta \cdot f_0 = f' \cdot \varepsilon$). This lifting is unique up to a chain homotopy equivalence.

proof: good exercise in the definitions.

Horseshoe lemma: suppose

$$0 \to A' \to A \to A'' \to 0$$

is a s.e.s. and $P' \xrightarrow{\varepsilon'} A'$ and $P'' \xrightarrow{\varepsilon''} A''$ are projective resolutions. Then there is a proj. resolution $P_\cdot \xrightarrow{\varepsilon} A$ s.t. $P_n = P_n' \oplus P_n''$ s.t.

$$0 \to P' \xrightarrow{i} P \xrightarrow{\pi} P'' \to 0$$

is a s.e.s of complexes where i is the inclusion $i_n: P_n' \to P_n' \oplus P_n''$ and π is the projection $\pi_n: P_n' \oplus P_n'' \to P_n''$.

proof: exercise.

Ex: write the dual propositions for injective resolutions.