Delta functors

Wednesday, 28 October 2020 16:37

Let A, B be abelian categories

Def: A homological (cohomological) δ-functor between A and B is a collection of additive functors $T_n : A \to B$ ($T^n : A \to B$) for $n \geq 0$ with morphisms $\delta_n : T_n(C) \to T_{n-1}(A)$ ($\delta^n : T^n(C) \to T^{n+1}(A)$) defined for any s.e.s. $0 \to A \to B \to C \to 0$ in A. st.:

1) for any s.e.s $0 \to A \to B \to C \to 0$ we get a l.e.s.

$$\cdots \to T_{n+1}(C) \xrightarrow{\delta} T_n(A) \to T_n(B) \to T_n(C) \xrightarrow{\delta} T_{n-1}(A) \to \cdots$$

$$\left(\cdots \to T^{n-1}(C) \xrightarrow{\delta} T^n(A) \to T^n(B) \to T^n(C) \xrightarrow{\delta} T^{n+1}(A) \to \cdots \right)$$

Note: T_0 is right exact (T^0 is left exact)

2) for each morphism of s.e.s

$$\begin{array}{ccccccccc} 0 & \to & A' & \to & B' & \to & C' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \end{array}$$

we get comm. diag.

$$\begin{array}{ccc} T_n(C') & \xrightarrow{\delta} & T_{n-1}(A') \\ \downarrow & & \downarrow \\ T_n(C) & \to & T_{n-1}(A) \end{array}$$

$$\left(\begin{array}{ccc} T^n(C') & \xrightarrow{\delta} & T^{n+1}(A') \\ \downarrow & & \downarrow \\ T^n(C) & \xrightarrow{\delta} & T^{n+1}(A) \end{array} \right)$$

Ex: Homology is a homological δ-functor $H_* : Ch_{\geq 0}(A) \to A$

cohomology is a cohomological δ-functor $H^* : Ch^{\geq 0}(A) \to A$.

Ex: for any integer p define $T_0(A) = A/pA$

$T_1(A) = {}_pA = \{a \in A \mid pa = 0\}$

give a homological δ-functor (cohomological $T^0 = T$ $T^1 = T_0$) from Ab to Ab

This follows from the snake lemma applied to

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow p & & \downarrow p & & \downarrow p & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \end{array}$$

$$0 \to {}_pA \to {}_pB \to {}_pC \xrightarrow{\delta} A/pA \to B/pB \to C/pC \to 0$$

in a similar way we can define this for any $r \in R$.

This does not give a universal δ-functor. (Tor)

Def: A morphism of δ-functors $S = \{S_n\}$ $T = \{T_n\}$ is a collection of natural transformations $S_n \to T_n$ ($S^n \to T^n$) commutes with δ.

A homological δ-functor $T = \{T_n\}$ is universal if given any other δ-functor $S = \{S_n\}$ and a natural transformation $f_0 : S_0 \to T_0$ there exists a unique morphism $\{f_n : S_n \to T_n\}$ of δ-functors extending f_0.

A cohomological δ-functor $T = \{T^n\}$ is universal if given $S = \{S^n\}$ and $f^0 : T^0 \to S^0$ there exists a unique morphism $T \to S$ of δ-functors extending f^0.

Ex: if $F : A \to B$ is exact then $T_0 = F$ $T_n = 0$ for $n \neq 0$ is a universal δ-functor.

A way of constructing universal δ-functors (in categories with enough projectives / injectives) is by derived functors.