Derived functors

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Let $F: \mathcal{A} \to \mathcal{B}$ be a right exact functor. Assume that $\mathcal{A}$ has enough projectives. For each object A of $\mathcal{A}$ pick a projective resolution $P_\cdot \to A$.

<u>Def:</u> $L_i F(A) = H_i(F(P_\cdot))$

<u>Note:</u> Since $F(P_1) \to F(P_0) \to F(A) \to 0$ is exact so $L_0 F(A) = F(A)$.

<u>Lemma:</u> If $P_\cdot \to A$ and $Q_\cdot \to A$ are two projective resolutions then there is a canonical iso.

$$H_i(F(P_\cdot)) \xrightarrow{\sim} H_i(F(Q_\cdot))$$

<u>proof:</u> By the comparison thm there is a chain map $f: P_\cdot \to Q_\cdot$ lifting the identity giving

$$f_*: H_i(F(P_\cdot)) \to H_i(F(Q_\cdot))$$

any other lift $f': P_\cdot \to Q_\cdot$ is homotopic to f so $f_* = f'_*$. so f_* is canonical. We can also lift the identity to a map $g: Q_\cdot \to P_\cdot$. Note $g \circ f$ and id_P are both lifts of the identity so

$$g_* f_* = (gf)_* = (id_P)_*$$

and in the same way

$$f_* g_* = (id_Q)_*$$ so

f_* is an iso.

<u>Cor:</u> if A is proj. then $L_i F(A) = 0$ for $i \neq 0$.

<u>Lemma:</u> if $f: A' \to A$ is a morphism in $\mathcal{A}$ there is a natural map $L_i F(f): L_i F(A') \to L_i F(A)$

<u>proof:</u> let $P'_\cdot \to A'$ and $P_\cdot \to A$ be the chosen proj. resolutions. f lifts to a map $\tilde{f}: P'_\cdot \to P_\cdot$ giving a map

$$\tilde{f}_*: H_i F(P'_\cdot) \to H_i F(P_\cdot)$$

any other lift is homotopic to $\tilde{f}$ so the map $\tilde{f}_*$ is independent of the lift.

<u>prop:</u> $L_i F$ is an additive functor from $\mathcal{A}$ to $\mathcal{B}$.

<u>proof:</u> id_P lifts id_A so $L_i F(id_A)$ is the identity.
given $A' \xrightarrow{f} A \xrightarrow{g} A''$ and lifts $\tilde{f}, \tilde{g}$ of f, g then $\tilde{g}\tilde{f}$ lifts of gf so $g_* f_* = (gf)_*$ so $L_i F$ is a functor.
if $f_1, f_2: A' \to A$ then $\tilde{f}_1 + \tilde{f}_2$ lifts $f_1 + f_2$ so

$$f_{1*} + f_{2*} = (f_1 + f_2)_*$$

hence $L_i F$ is additive.

<u>Thm:</u> $\{L_i F\}$ form a homological δ-functor. If $\mathcal{A}$ has enough projectives then for any right exact functor $F: \mathcal{A} \to \mathcal{B}$ $\{L_i F\}$ forms a universal δ-functor.

<u>first step of the proof:</u>

Given a s.e.s.

$$0 \to A' \to A \to A'' \to 0$$

and proj. res. $P' \to A'$ and $P'' \to A''$ there is a proj. res. $P \to A$ s.t.

$$0 \to P' \to P \to P'' \to 0$$

is a s.e.s. of complexes each $0 \to P'_n \to P_n \to P''_n \to 0$ is split exact. Since F is additive

$$0 \to F(P'_n) \to F(P_n) \to F(P''_n) \to 0$$

is split exact in $\mathcal{B}$ and

$$0 \to F(P'_\cdot) \to F(P_\cdot) \to F(P''_\cdot) \to 0$$

is a s.e.s. of complexes. the resulting l.e.s.

$$\cdots \to L_i F(A'') \xrightarrow{\partial} L_{i-1} F(A') \to L_{i-1} F(A) \to \cdots$$

In a similar way if $F: \mathcal{A} \to \mathcal{B}$ is left exact we can define the right derived functors:

$$R^i F(A) = H^i F(I^\cdot)$$

where $A \to I^\cdot$ is a chosen injective resolution.

<u>Note:</u> $R^i F(A) = (L_i F^{op})^{op}(A)$

All the prop. above apply to right derived functors as well.

<u>Def:</u> for any object A of $\mathcal{A}$ $Hom_{\mathcal{A}}(A, -): \mathcal{A} \to Ab$ is left exact.

$$Ext^i_{\mathcal{A}}(A, B) \overset{def}{=} R^i Hom_{\mathcal{A}}(A, -)(B)$$

<u>Note:</u> We have the following eq.:
B is inj. $\Leftrightarrow$ $Hom_{\mathcal{A}}(-, B)$ is exact $\Leftrightarrow$ $Ext^i_{\mathcal{A}}(A, B) = 0$ for $i \neq 0$ and all A in $\mathcal{A}$ $\Leftrightarrow$ $Ext^1_{\mathcal{A}}(A, B) = 0$ for all A.

A is proj. $\Leftrightarrow$ $Hom_{\mathcal{A}}(A, -)$ is exact $\Leftrightarrow$ $Ext^i_{\mathcal{A}}(A, B) = 0$ $i \neq 0$ and all B $\Leftrightarrow$ $Ext^1_{\mathcal{A}}(A, B) = 0$ for all B.

<u>Note:</u> We will show that

$$R^* Hom(\ , B)(A) \simeq R^* Hom(A, -)(B) = Ext^*(A, B).$$

<u>Def:</u> let R be a ring and B a left R-module.

$- \otimes_R B$: mod-$R \to Ab$ is right exact

$$Tor_n^R(A, B) \overset{def}{=} L_n(- \otimes_R B)(A)$$