## Homological algebra

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## Sheet 1

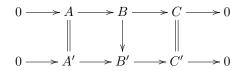
**Exercise 1.** Let A, B, C be R-modules. Show that there exists canonical R-module isomorphisms  $\operatorname{Hom}(A \oplus B, C) \cong \operatorname{Hom}(A, C) \oplus \operatorname{Hom}(B, C)$  and  $\operatorname{Hom}(A, B \oplus C) \cong \operatorname{Hom}(A, B) \oplus \operatorname{Hom}(A, C)$ .

More generally, in a category which admits sums and products, prove that  $\operatorname{Hom}(\bigoplus_{i \in \mathcal{I}} M_i, N) = \prod_{i \in \mathcal{I}} \operatorname{Hom}(M_i, N)$  and  $\operatorname{Hom}(M, \prod_{i \in \mathcal{I}} N_i) = \prod_{i \in \mathcal{I}} \operatorname{Hom}(M, N_i)$ .

**Exercise 2.** A monomorphism is a morphism f that satisfies  $(f \circ g_1 = f \circ g_2) \Rightarrow (g_1 = g_2)$ . Equivalently, it is a morphism  $f: X \to Y$  with the property that whenever two morphisms  $g_1, g_2: Z \to X$  are distinct, they remain distinct after composing them with f. Dually, an *epimorphism* is a map f that satisfies  $(g_1 \circ f = g_2 \circ f) \Rightarrow (g_1 = g_2)$ .

Let  $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$  be a short exact sequence. Prove, using the language of category theory, that f is always a monomorphism, and that g is always an epimorphism.

**Exercise 3.** Consider a commutative diagram of the following form, where the rows are exact:



Prove that the map  $B \to B'$  is an isomorphism.

**Exercise 4.** Let R := k[x, y] where k is a field. Let  $M_1 := R^2/\langle (x, 0), (y^2, -x), (0, y) \rangle$  and  $M_2 := R/\langle x^2, xy, y^3 \rangle$ . (Here, the symbol  $\langle \ldots \rangle$  means 'submodule generated by'.) Provide examples of non-split short exact sequences of R-modules

$$0 \to M_1 \to ?? \to M_2 \to 0.$$

**Exercise 5.** Prove that every short exact sequence of abelian groups  $0 \to A \to B \to \mathbb{Z} \to 0$  splits. Prove that every short exact sequence of abelian groups  $0 \to \mathbb{Q} \to B \to C \to 0$  splits.

**Exercise 6.** Prove that  $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p:\text{prime}} \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ .

**Exercise 7.** Prove that, in general, there is no isomorphism  $\operatorname{Hom}(M, \bigoplus_{i \in \mathcal{I}} N_i) = \bigoplus_{i \in \mathcal{I}} \operatorname{Hom}(M, N_i)$ .

**Exercise 8** (super-hard). Prove that, surprisingly, the natural inclusion  $\bigoplus_{i \in \mathbb{N}} \mathbb{Z} \to \text{Hom}(\prod_{i \in \mathbb{N}} \mathbb{Z}, \mathbb{Z})$  is an isomorphism.