Homological algebra

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Sheet 2

Exercise 1. Let k be a field, and let C be the abelian category of k-vector spaces. Let D be an arbitrary abelian category. Prove that every additive functor $C \to D$ is exact.

Exercise 2. Let R and S be rings, let $C := R$-Mod and $D := S$-Mod be the associated abelian categories of modules, and let $F : C \to D$ be an additive functor.

Assume that F sends short exact sequences to short exact sequences. Prove that it sends exact sequences (or any length) to exact sequences.

Exercise 3. Let R be a ring. Prove that an R-module P is projective iff every surjective map $A \to P$ admits a section.

Exercise 4. Let $R := \mathbb{Z}[\sqrt{-5}]$. Prove that the ideal generated by 2 and $1 + \sqrt{-5}$ is a projective R-module which is not free. *Hint:* show that the map $\left(\begin{smallmatrix} 2 & 1-\sqrt{-5} \\ 1+\sqrt{-5} & 2 \end{smallmatrix}\right) : R^{\oplus 2} \to M^{\oplus 2}$ is an isomorphism.

Exercise 5. Let R be a ring. Prove that for every sequence of R-modules $(M_i)_{n \in \mathbb{Z}}$, there exists a chain complex of free modules $C_\bullet$ such that $H_i(C_\bullet) \cong M_i$ for all $i \in \mathbb{Z}$.

Exercise 6. Let $\mathcal{A}$ be an arbitrary abelian category, and let $Ch(\mathcal{A})$ be the category of chain complexes of objects of $\mathcal{A}$. Given a morphism $f_\bullet : C_\bullet \to D_\bullet$ in $Ch(\mathcal{A})$, prove that the kernel of $f_\bullet$ is the chain complex $\big(\ldots \to \ker(f_n) \to \ker(f_{n-1}) \to \ldots\big)$.

The next exercise is a long and painful one which I don't expect you (or want you) to finish. But I do want you to start it. Write down what you think is approximately 50% of the proof, and then write "I give up" (or, if you don't want to give up, you may hand in a complete answer):

Exercise 7. Prove that a short exact sequence of chain complexes (of R-modules)

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_{n+1} & \to & B_{n+1} & \to & C_{n+1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_{n} & \to & B_{n} & \to & C_{n} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_{n-1} & \to & B_{n-1} & \to & C_{n-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_{n-2} & \to & B_{n-2} & \to & C_{n-2} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

induces a long exact sequence in homology

$$\ldots \to H_{n+1}(C_\bullet) \to H_n(A_\bullet) \to H_n(B_\bullet) \to H_n(C_\bullet) \to H_{n-1}(A_\bullet) \to \ldots$$

[For the definition of the so-called 'connecting homomorphism' $H_{n+1}(C_\bullet) \to H_n(A_\bullet)$, you may have a look at e.g. https://ncatlab.org/nlab/show/connecting+homomorphism*]*

(Hand in the Monday before the class, at 5pm)