TENSOR PRODUCTS AND CASIMIR OPERATORS

KEVIN MCGERTY

This note gives an account of the construction of Casimir elements following the approach of the lecture videos, which is somewhat more conceptual than the approach in the online lecture notes. It also gives details on tensor products, which are also discussed in the Part B representation theory course – see the section on Multilinear algebra in the notes for that course.

1. TENSOR PRODUCTS

Definition 1.1. Let V and W be vector spaces over a field k. The tensor product $V \otimes W$ is a k-vector space $V \otimes W$ equipped with a bilinear map $t\colon V \times W \to V \otimes W$ (where we write $v \otimes w$ for $t(v,w)$) which has the following universal property: If $B\colon V \times W \to U$ is a bilinear map taking values in a k-vector space U, then there exists a unique linear map $b\colon V \otimes W \to U$ such that $B = b \circ t$.

Remark 1.2. It is possible to construct $V \otimes W$ in various ways. If V and W are finite dimensional, we may pick a basis $\{e_1, \dots, e_n\}$ of V and a basis $\{f_1, \dots, f_m\}$ of W. Set $V \otimes W$ to be the vector space with basis $\{e_i \otimes f_j : 1 \leq i \leq n, 1 \leq j \leq m\}$, and if $v = \sum_{i=1}^n \lambda_i e_i$ and $w = \sum_{j=1}^m \mu_j f_j$, we set

$$t(v.w) = \sum_{i,j} (\lambda_i \mu_j) e_i \otimes f_j,$$

Given $B\colon V \times W \to U$ a bilinear map, we can define $b\colon V \otimes W \to U$ to be the unique linear map with $b(e_i \otimes f_j) = B(e_i, f_j)$. Provided you are happy with the axiom of choice, so that every vector space has a basis, the same construction gives the existence of the tensor product of arbitrary vector spaces.

Lemma 1.3. *Let V and W be vector spaces. There is a natural injective*[1] *map $\theta\colon V^* \otimes W \to Hom_{\mathsf{k}}(V,W)$ which is an isomorphism when V is finite-dimensional.*

Proof. (*c.f.* the proof that $\mathrm{ad}(x)$ is semisimple when x is). The map $(\alpha, w) \mapsto [v \mapsto \alpha(v).w]$ is bilinear[2], and so induces a linear map $\theta\colon V^* \otimes W \to \mathrm{Hom}_{\mathsf{k}}(V,W)$.

To see that it is injective, let $\{\delta_i : i \in I\}$ be a basis of V^*, and $\{f_k : k \in K\}$ be a basis of W. Then if $\gamma \in V^* \otimes W$, we by definition may write $\gamma = \sum_{(i,k)\in S} \lambda_{i,k} \delta_i \otimes f_k$, where the pairs (i,k) run over a finite subset S of $I \times K$. Now if we fix $k \in K$ we have

$$\sum_{i \in I:(i,k)\in S} \lambda_{i,k}\delta_i \otimes f_k = (\sum_{i \in I:(i,k)\in S} \lambda_{i,k}\delta_i) \otimes f_k,$$

thus setting $\phi_k = \sum_{i \in I:(i,k)\in S} \lambda_{i,k}\delta_i$ it follows $\gamma = \sum_{k \in S_K} \phi_k \otimes f_k$, where $S_K = \{k \in K : \exists i \in I, (i,k) \in S\}$. But then for any $v \in V$

$$0 = \theta(\gamma)(v) = \sum_{k \in S_K} \phi_k(v).f_k,$$

and so by the linear independence of the f_ks we must have $\phi_k(v) = 0$ for each k. Since this is true for all $v \in V$, it follows that $\phi_k = 0$, for each k, and hence $\gamma = 0$ as required.

Finally to see that θ is an isomorphism when V is finite dimensional, note that in that case we can assume our basis of V^* is dual to a basis $\{e_i : i \in I\}$ of V. But then if $\alpha \in \mathrm{Hom}_{\mathsf{k}}(V,W)$ it follows that $\alpha = \theta(\sum_{i\in I} \delta_i \otimes \alpha(e_i))$, as the two sides agree on the basis $\{e_i : i \in I\}$. □

Remark 1.4. Since we only use the cases where V and W are finite dimensional, the reader is welcome to ignore the generality the result is stated in and assume throughout that all vector spaces are finite dimensional. Here one can be a bit more concrete: if $\{e_1, \dots, e_n\}$ is a basis of V and $\{f_1, \dots, f_m\}$ is a basis of W, then taking the dual basis $\{\delta_1, \dots, \delta_n\}$ of V^* it is easy to see that the images of $\delta_i \otimes f_j$ under θ correspond to the elementary matrices E_{ij} under the identification of $\mathrm{Hom}_{\mathsf{k}}(V,W)$ given by the choice

[1] The proof of injectivity in the lecture video only works for finite dimensional V as it uses dual bases, the proof below works in general.

[2] There is a lot of linearity going on here! The map $(\alpha, w, v) \mapsto \alpha(v).w$ is linear in all of α, v and w. For fixed α, w, this shows that the map $v \mapsto \alpha(v).w$ is a linear map from V to W, while the linearity in α and w show the map which sends a pair (α, w) to the corresponding map from V to W is bilinear in α and w.

of bases for V and W, hence θ is an isomorphism. In general the image of θ is precisely the linear maps from V to W which have finite rank (as you can readily deduce from the proof of Lemma 1.3).

1.1. **Tensor products and $\mathfrak{g}$-representations.** Suppose that $\mathfrak{g}$ is a Lie algebra and (V,ρ) and (W,σ) are $\mathfrak{g}$-representations. We would like to understand if $V\otimes W$ has the structure of a $\mathfrak{g}$-representation. Now the map $\rho\oplus\sigma\colon \mathfrak{g}\to\mathfrak{gl}(V)\oplus\mathfrak{gl}(W)$ given by $(\rho\oplus\sigma)(x)=(\rho(x),\sigma(x))$ is a Lie algebra homomorphism, and the representations V and W are obtained from $\rho\oplus\sigma$ by composition with the obvious projection maps from $\mathfrak{gl}(V)\oplus\mathfrak{gl}(W)$ to $\mathfrak{gl}(V)$ and $\mathfrak{gl}(W)$ respectively, so it suffices to show that $V\otimes W$ carries the structure of a representation of $\mathfrak{gl}(V)\oplus\mathfrak{gl}(W)$ in some natural way.

We will analyze this in a couple of steps. Firstly, since $\mathfrak{gl}(V)\oplus\mathfrak{gl}(W)$ is a direct sum of Lie algebras, the following basic lemma shows that to give a representation of $\mathfrak{gl}(V)\oplus\mathfrak{gl}(W)$ on a vector space U is the same thing as giving an action $\alpha_1\colon\mathfrak{gl}(V)\to\mathfrak{gl}(U)$ of $\mathfrak{gl}(V)$ on U and an action $\alpha_2\colon\mathfrak{gl}(W)\to\mathfrak{gl}(U)$ of $\mathfrak{gl}(W)$ on U such that the images $\alpha_1(\mathfrak{gl}(V))$ and $\alpha_2(\mathfrak{gl}(W))$ in $\mathfrak{gl}(U)$ commute with each other[3]:

Lemma 1.5. *Let $\mathfrak{g}_1,\mathfrak{g}_2$ be Lie algebras and suppose that $\alpha_i\colon\mathfrak{g}_i\to\mathfrak{h}$ are Lie algebra homomorphisms. Then $\beta\colon\mathfrak{g}_1\oplus\mathfrak{g}_2\to\mathfrak{h}$ given by $\beta(x_1,x_2)=\alpha_1(x_1)+\alpha_2(x_2)$ is a Lie algebra homomorphism provided $[\alpha_1(\mathfrak{g}_1),\alpha_2(\mathfrak{g}_2)]=0$.*

Proof. This is a direct calculation. For $(x_1,x_2),(y_1,y_2)\in\mathfrak{g}_1\oplus\mathfrak{g}_2$ we have

$$\begin{aligned}[\beta(x_1,x_2),\beta(y_1,y_2)] &= [\alpha_1(x_1)+\alpha_2(x_2),\alpha_1(y_1)+\alpha_2(y_2)]\\ &= [\alpha_1(x_1),\alpha_1(y_1)]+[\alpha_1(x_1),\alpha_2(y_2)]+[\alpha_2(x_2),\alpha_1(y_1)]+[\alpha_2(x_2),\alpha_2(y_2)]\\ &= [\alpha_1(x_1),\alpha_1(y_1)]+[\alpha_2(x_2),\alpha_2(y_2)]\\ &= \alpha_1([x_1,y_1])+\alpha_2([x_2,y_2])\\ &= \beta(([x_1,y_1],[x_2,y_2])\\ &= \beta([(x_1,x_2),(y_1,y_2)])\end{aligned}$$

where in passing from the second to the third equality we use the assumption that $[\alpha_1(\mathfrak{g}_1),\alpha_2(\mathfrak{g}_2)]=[\alpha_2(\mathfrak{g}_2),\alpha_1(\mathfrak{g}_1)]=0$. $\square$

Now for any $x\in\mathfrak{gl}(V)$ and $y\in\mathfrak{gl}(W)$ it is straight-forward from the universal property of tensor products to obtain a linear map $x\otimes y\colon V\otimes W\to V\otimes W$: namely[4] $x\otimes y$ is characterized by the condition that

$$(x\otimes y)(v\otimes w)=x(v)\otimes y(w),\quad \forall v\in V, w\in W.$$

In particular, for any $x\in\mathfrak{gl}(V)$ we have $x\otimes 1_W\colon V\otimes W\to V\otimes W$. Now it is easy to check (using the universal property again) that the map $\iota_V\colon\mathfrak{gl}(V)\to\mathfrak{gl}(V\otimes W)$ given by $x\mapsto x\otimes 1$ is a map of associative algebras, *i.e.* for any $x_1,x_2\in\mathfrak{gl}(V)$ we have

$$\iota_V(x_1)\circ\iota_V(x_2)=(x_1\otimes 1_W)\circ(x_2\otimes 1_W)=(x_1x_2)\otimes 1_W=\iota_V(x_1x_2),$$

hence it is certainly a map of Lie algebras. Similarly we see that the map $\iota_W\colon\mathfrak{gl}(W)\to\mathfrak{gl}(V\otimes W)$ given by $y\mapsto 1_V\otimes y$ is a homomorphism of Lie algebras. But it is also clear that the images of ι_V and ι_W commute with each other:

$$\iota_V(x)\iota_W(y)(v\otimes w)=\iota_V(x)(v\otimes y(w))=x(v)\otimes y(w)=\iota_W(x(v)\otimes w)=\iota_W\iota_V(v\otimes w).$$

Hence by Lemma 1.5 we obtain a representation of $\mathfrak{gl}(V)\oplus\mathfrak{gl}(W)$ on $V\otimes W$ given by

$$(x,y)(v\otimes w)=(\iota_V\oplus\iota_W)(x,y)(v,w)=)x(v)\otimes w+v\otimes y(w).$$

Now returning to the general setting.

Definition 1.6. If (V,ρ) and (W,σ) are $\mathfrak{g}$-representations for an arbitrary Lie algebra $\mathfrak{g}$ then $V\otimes W$ becomes a $\mathfrak{g}$ representation via the composition

$$\mathfrak{g}\xrightarrow{\rho\oplus\sigma}\mathfrak{gl}(V)\oplus\mathfrak{gl}(W)\xrightarrow{\iota_V\oplus\iota_W}\mathfrak{gl}(V\otimes W)$$

More explicitly (and this is the only formula you really need to remember from this section!) $V\otimes W$ becomes a $\mathfrak{g}$-representation via the map $\rho\otimes\sigma\colon\mathfrak{g}\to\mathfrak{gl}(V\otimes W)$ where

$$(\rho\otimes\sigma)(x)(v\otimes w)=\rho(x)(v)\otimes w+v\otimes\sigma(x)(w),\quad\forall v\in V, w\in W. \tag{1.1}$$

[3] This is a Lie algebra analogue of the following statement for group representations: If G and H are groups both of which act linearly on a vector space V, then we get a representation of $G\times H$ on V provided the images of G and H in $\mathrm{GL}(V)$ commute with each other, where the action of $(g,h)\in G\times H$ on $v\in V$ is just given by $(g,h)(v)=g(h(v))=h(g(v))$.

[4] Using the universal property again, clearly $(v,w)\mapsto x(v)\otimes y(w)$ is a bilinear map from $V\times W$ since x and y are linear and t is bilinear, so it induces the linear map $x\otimes y$.

Remark 1.7. While the discussion in this section sought to explain how we might discover the action of a Lie algebra $\mathfrak{g}$ on a tensor product, if one simply guessed the action given by Equation (1.1), it is easy to check that it does indeed give a Lie algebra action, that is, a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{gl}(V \otimes W)$. It is a good exercise to do this computation for oneself.

Remark 1.8. It is easy to check from the definitions that the map θ given in Lemma 1.3 is a homomorphism of Lie algebra representations if V and W are.

1.2. **Bilinear forms.** Let $\mathrm{Bil}(V)$ be the space of bilinear forms on V, that is,

$$\mathrm{Bil}(V) = \{B \colon V \times V \to \mathsf{k} : B \text{ bilinear}\}.$$

From the definition of tensor products it follows that $\mathrm{Bil}(V)$ can be identified with $(V \otimes V)^*$. If V is a $\mathfrak{g}$-representation, this means $\mathrm{Bil}(V)$ also has the structure of $\mathfrak{g}$-representation: explicitly, if $B \in \mathrm{Bil}(V)$, then it yields a linear map $b \colon V \otimes V \to \mathsf{k}$ by the universal property of tensor products, and if $y \in \mathfrak{g}$, it acts on B as follows:

$$\begin{aligned} y(B)(v,w) &= y(b)(v \otimes w) \\ &= -b(y(v \otimes w)) \\ &= -b(y(v) \otimes w + v \otimes y(w)) \\ &= -B(y(v), w) - B(v, y(w)). \end{aligned}$$

If we apply this to $(V, \rho) = (\mathfrak{g}, \mathrm{ad})$, then the condition that $B \in \mathrm{Bil}(\mathfrak{g})^{\mathfrak{g}}$, becomes that for all $x, y, z \in \mathfrak{g}$,

$$\begin{aligned} 0 &= y(B)(x,z) \\ &= -B(\mathrm{ad}(y)(x), z) - B(x, \mathrm{ad}(y)(z)), \\ &= -B([y,x], z) - B(x, [y,z]) \\ &= B([x,y], z) - B(x, [y,z]), \end{aligned}$$

that is, $B([x,y],z) = B(x,[y,z])$. Thus the condition that a symmetric bilinear form is invariant is exactly that B is an invariant vector in $\mathrm{Bil}(V)$.

Remark 1.9. There is a natural map $V^* \otimes V^* \to (V \otimes V)^*$ given by $(\delta_1 \otimes \delta_2)(v_1 \otimes v_2) = \delta(v_1)\delta(v_2)$. When V is finite dimensional, it is easy to check that this is an isomorphism (in general it is only injective).

Now using the above, we have isomorphisms of $\mathfrak{g}$-representations

$$\mathrm{Bil}(\mathfrak{g}) = (\mathfrak{g} \otimes \mathfrak{g})^* \cong \mathfrak{g}^* \otimes \mathfrak{g}^* \cong \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}^*)$$

If β is an invariant symmetric blinear form on $\mathfrak{g}$ then it yields a homomorphism $\tau \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}^*)^{\mathfrak{g}}$, that is, a $\mathfrak{g}$-homomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$. If β is non-degenerate, the τ is an isomorphism.

2. Casimir elements

In this section we construct, for representations V of a Lie algebra $\mathfrak{g}$ whose trace form is nondegenerate, a non-zero $\mathfrak{g}$-homomorphism from V to itself. We then show that if $\mathfrak{g}$ is semisimple, this construction can be applied whenever $\ker(\rho) \neq \mathfrak{g}$. We being with a basic fact which we essentially saw in our examples of Lie algebras at the start of the course.

Lemma 2.1. *If V is a vector space then the map $ad\colon \mathfrak{gl}(V) \to \mathfrak{gl}(\mathfrak{gl}(V))$ given by $ad(a)(x) = ax - xa$ has image in $Der_{\mathsf{k}}(End_{\mathsf{k}}(V))$, that is $ad(a)(x.y) = ad(a).x + x.ad(a)(y)$. It follows that the composition map $\mathfrak{gl}(V) \times \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ induces a map of $\mathfrak{gl}(V)$-representations*

$$m \colon \mathfrak{gl}(V) \otimes \mathfrak{gl}(V) \to \mathfrak{gl}(V), \quad x \otimes y \mapsto x \circ y$$

Proof. The first statement is an easy calculation:

$$\begin{aligned} \mathrm{ad}(a)(x).y + x.\mathrm{ad}(a)(y) &= (ax - xa).y + x(ay - ya) \\ &= a.xy - xay + xay - xy.a \\ &= \mathrm{ad}(a)(x.y). \end{aligned}$$

For the second part of the Lemma, one simply notes that for $a, x, y \in \mathfrak{gl}(V)$ we have

$$\begin{aligned} m(a.(x \otimes y)) &= m(\mathrm{ad}(a)(x) \otimes y + x \otimes \mathrm{ad}(a)(y)) \\ &= \mathrm{ad}(a)(x).y + x.\mathrm{ad}(a)(y) \\ &= \mathrm{ad}(a)(x.y) = \mathrm{ad}(a)(m(x \otimes y)). \end{aligned}$$

so that m is a homomorphism of $\mathfrak{gl}(V)$-representations. $\square$

Definition 2.2. Suppose that (V,ρ) is a $\mathfrak{g}$-representation and that its trace form $\beta = t_V$ is non-degenerate. Then β induces an isomorphism $\tau\colon \mathfrak{g} \to \mathfrak{g}^*$, and hence we have a sequence of homomorphisms of $\mathfrak{g}$-representations

$$\mathrm{Hom}_{\mathsf{k}}(\mathfrak{g},\mathfrak{g}) \xrightarrow{\theta^{-1}} \mathfrak{g}^* \otimes \mathfrak{g} \xrightarrow{\tau^{-1}\otimes 1} \mathfrak{g}\otimes\mathfrak{g} \xrightarrow{\rho\otimes\rho} \mathfrak{gl}(V)\otimes\mathfrak{gl}(V) \xrightarrow{m} \mathfrak{gl}(V).$$

where the first map is the inverse of the map θ from Lemma 1.3, and the map $m\colon \mathfrak{gl}(V)\otimes\mathfrak{gl}(V)\to\mathfrak{gl}(V)$ is given by the composition of the map of the previous Lemma. (Note that the first two maps are isomorphisms.)

As the identity element $\mathrm{id}_{\mathfrak{g}}$ is clearly an invariant vector in $\mathrm{Hom}_{\mathsf{k}}(\mathfrak{g},\mathfrak{g})$, applying to it the above sequence of $\mathfrak{g}$-homomorphisms yields an invariant vector C_V in $\mathrm{Hom}_{\mathsf{k}}(V,V)$, that is, a $\mathfrak{g}$-homomorphism from V to itself. We call this the *Casimir operator*.

We can make the Casimir operator explicit as follows: Pick a basis $\{x_1),\dots,x_n)\}$ of $\mathfrak{g}$. The nondegeneracy of the form t_V on $\mathfrak{g}$ implies that there is a unique basis $\{y_1,\dots,y_n\}$ which is dual to the basis $\{x_1,\dots,x_n\}$ in the sense that $t_V(x_i),y_j)) = 1$ if $i=j$ and is 0 otherwise. If $\tau\colon\mathfrak{g}\to\mathfrak{g}^*$ the it is clear that $y_) = \tau^{-1}(\delta_i)$, where $\{\delta_i : 1\leq i\leq n\}$ denotes the basis of $\mathfrak{g}^*$ dual to the basis $\{x_1,\dots,x_n\}$.

Now the identity element of $\mathrm{Hom}_{\mathsf{k}}(\mathfrak{g}),\mathfrak{g})$ corresponds to the element $\sum_{i=1}^n \delta_i\otimes x_i$, where $\{\delta_i : 1\leq i\leq n\}$ is the dual basis to $\{x_1,\dots,x_n\}$, as can be checked by computing on the basis $\{x_1,\dots,x_n\}$. Then it follows immediately from the definitions that

$$C_V = \sum_{i=1}^n \rho(y_i)\rho(x_i),$$

Lemma 2.3. *Let (V,ρ) be a representation of a Lie algebra $\mathfrak{g}$ such that t_V is non-degenerate on $\mathfrak{g}$. Then the Casimir operator C_V satisfies $tr(C_V) = \dim(\mathfrak{g})$. In particular it is non-zero.*

Proof. This is immediate from the above formula for C_V.

$$\mathrm{tr}(C_V) = \mathrm{tr}(\sum_{i=1}^n \rho(y_i),\rho(x_i)) = \sum_{i=1}^n \mathrm{tr}(\rho(y_i)\rho(x_i)) = \sum_{i=1}^n t_V(y_i,x_i) = \sum_{i=1}^n 1 = \dim(\mathfrak{g}).$$

□

Now suppose that $\mathfrak{g}$ is semisimple, and (V,ρ) an arbitrary representation of $\mathfrak{g}$. While it is not the case that the trace form t_V must be non-degenerate, the following Lemma identifies its radical:

Lemma 2.4. *Suppose that $\mathfrak{g}$ is semisimple and (V,ρ) is a representation of $\mathfrak{g}$. Then the radical of t_V is precisely the kernel of ρ. In particular if (V,ρ) is faithful then t_V is nondegenerate.*

Proof. The image $\rho(\mathfrak{g})\subseteq\mathfrak{gl}(V)$ of $\mathfrak{g}$ is a semisimple Lie algebra (since $\mathfrak{g}$ is) and the statement of the Lemma is exactly that t_V is nondegenerate on $\rho(\mathfrak{g})$. But the radical $\mathfrak{r} = \mathrm{rad}(t_V)$ is an ideal of $\rho(\mathfrak{g})$. Now Proposition 11.2 shows that if we let $(D^k\mathfrak{r})_{k\geq 0}$ be the derived series of $\mathfrak{r}$, we must have $D^{k+1}\mathfrak{r}\subsetneq D^k\mathfrak{r}$ whenever $D^k\mathfrak{r}\neq\{0\}$, thus $\mathfrak{r}$ must be solvable. Since $\rho(\mathfrak{g})$ is semisimple, this forces the radical to be zero as required. □

Definition 2.5. If $\mathfrak{g}$ is a semisimple Lie algebra and (V,ρ) is a $\mathfrak{g}$-representation on which $\mathfrak{g}$ acts non-trivially (so that $\rho(\mathfrak{g})\neq\{0\}$) then, by Lemma 2.4, the above construction can be applied to V as a representation[5] of the semisimple Lie algebra $\rho(\mathfrak{g})$, yielding a $\mathfrak{g}$-endomorphism[6] of V, with trace $\mathrm{tr}(C_V) = \dim(\rho(\mathfrak{g}))$. Thus for any representation (V,ρ) of a semisimple Lie algebra $\mathfrak{g}$ with $\rho(\mathfrak{g})\neq\{0\}$ we obtain a non-zero $\mathfrak{g}$-endomorphism C_V of V which we will call the Casimir of V.

Remark 2.6. If $\mathfrak{g}$ is simple, rather than just semisimple, then by Schur's Lemma $\mathrm{Hom}_{\mathsf{k}}(\mathfrak{g},\mathfrak{g})^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g},\mathfrak{g})$ is one-dimensional (the scalar multiples of the identity). Since $\mathrm{Hom}_{\mathsf{k}}(\mathfrak{g},\mathfrak{g})\cong\mathfrak{g}\otimes\mathfrak{g}$ as $\mathfrak{g}$-representations, the invariants $(\mathfrak{g}\otimes\mathfrak{g})^{\mathfrak{g}}$ in $\mathfrak{g}\otimes\mathfrak{g}$ must also be one-dimensional (the image of the scalar multiples of the identity under any isomorphism). If we pick a non-zero element $C\in(\mathfrak{g}\otimes\mathfrak{g})^{\mathfrak{g}}$, then, for any representation on which $\mathfrak{g}$ acts non-trivially, there is a non-zero scalar λ_V such that

$$C_V = \lambda_V.m\circ(\rho\otimes\rho)(C)$$

Thus the Casimir operators C_V, up to scaling, all come from the same element of $\mathfrak{g}\otimes\mathfrak{g}$.

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[5]with action map the inclusion map from $\rho(\mathfrak{g})$ into $\mathfrak{gl}(V)$.

[6]Since by construction C_V commutes with every element of $\rho(\mathfrak{g})$.

Example 2.7. Let us take $\mathfrak{g} = \mathfrak{sl}_2$. Then the trace form $t(x, y) = \mathrm{tr}(x.y)$ is non-degenerate and invariant, with

$$t(e, f) = t(e, h) = 1, \quad t(h, h) = 2, \quad t(e, e) = t(f, f) = t(e, h) = t(f, h) = 0$$

Thus the corresponding isomorphism $\tau\colon \mathfrak{sl}_2 \to \mathfrak{sl}_2^*$ gives

$$(\tau^{-1} \otimes 1)(\theta^{-1}(\mathrm{id}) = f \otimes e + \frac{1}{2} h \otimes h + e \otimes f.$$

For any $\mathfrak{sl}_2$-representation (V, ρ) we thus get a $\mathfrak{g}$-endomorphism of V by applying $m \circ (\rho \otimes \rho)$ to this element, namely $\rho(e)\rho(f) + \frac{1}{2}\rho(h)^2 + \rho(f)\rho(e)$. This is exactly the operator used in Sheet 3 of the problem set.