C2.1a Lie algebras

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Problem Sheet 2

Assume throughout the problems that we work over a field k *which is algebraically closed of characteristic zero, unless the contrary is explicitly stated. This sheet covers material up to section* 10 *in the online notes, which corresponds in lectures to material up to the first half of week 4.*

1. Suppose that $\mathfrak{g}$ is a Lie algebra over the complex numbers. Show that $\mathfrak{g}$ is nilpotent if and only if any 2-dimensional subalgebra is abelian.

2. a) Let k be a field of characteristic 2 and let $\mathfrak{g} = \mathfrak{sl}_2(k)$. Show that $\mathfrak{g}$ is solvable (and even nilpotent) but that the natural two-dimensional representation of $\mathfrak{g}$ is irreducible. Conclude that Lie's theorem is not true in positive characteristic.

b) Let $\mathbb{C}[x]$ denote a polynomial ring in x, and consider the Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(\mathbb{C}[x])$ generated by the endomorphisms given by multiplication by x and $\frac{d}{dx}$. Show that $\mathfrak{g}$ is a three dimensional nilpotent Lie algebra, isomorphic to the Heisenberg algebra. Does $\mathfrak{g}$ fix a line in $\mathbb{C}[x]$? Why doesn't this contradict Lie's theorem?

3. Let V be a finite dimensional vector space, and let $\mathcal{F}$ be a flag $0 \subset V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V_n = V$ of subspaces where $\dim(V_i) = i$. If $\mathfrak{n}_\mathcal{F} = \{x \in \mathfrak{gl}(V) : x(V_i) \subseteq V_{i-1}\}$ and $\mathfrak{b}_\mathcal{F} = \{x \in \mathfrak{gl}(V) : x(V_i) \subseteq V_i\}$, then we have seen in lecture that $\mathfrak{n}_\mathcal{F}$ is an ideal in $\mathfrak{b}_\mathcal{F}$ and so we have an exact sequence

$$0 \longrightarrow \mathfrak{n}_\mathcal{F} \longrightarrow \mathfrak{b}_\mathcal{F} \longrightarrow \mathfrak{t} \longrightarrow 0$$

where $\mathfrak{t}$ is defined to be the quotient $\mathfrak{b}_\mathcal{F}/\mathfrak{n}_\mathcal{F}$. Show that this sequence is split, and that there are infinitely many splitting maps $s \colon \mathfrak{t} \to \mathfrak{b}_\mathcal{F}$.

4. Suppose that V is a finite dimensional vector space over an algebraically closed field k of characteristic zero, and $x, y \in \mathrm{End}_\mathsf{k}(V)$. Suppose that x and y commute with $z = [x, y] = xy - yx$. Show that z is nilpotent.

5. Recall an element x of a Lie algebra $\mathfrak{g}$ is said to be *regular* if it

$$\mathfrak{g}_{0,x} = \{y \in \mathfrak{g} : \exists n > 0, \mathrm{ad}(x)^n(y) = 0\}$$

has minimal possible dimension. Recall further that if V is a k-vector space and $x \in \mathfrak{gl}(V)$, then we may decompose V into the generalized eigenspaces of x, that is, $V = \bigoplus_\lambda V_\lambda$, where

$$V_\lambda = \{v \in V : \exists n \in \mathbb{N}, (x - \lambda)^n(v) = 0\}.$$

We define $x_s \in \mathfrak{gl}(V)$ to be the linear map given by $x_s(v) = \lambda.v$ for $v \in V_\lambda$. It is called the *semisimple* part of x. Clearly it is a diagonalisable linear map.

i) Let $x_n = x - x_s$. Check that x_n and x_s commute and that x_n is nilpotent.

ii) Show that $x \in \mathfrak{gl}(V)$ is regular if and only if x_s is regular.

iii) When is a semisimple (*i.e.* diagonalisable) element of $\mathfrak{gl}(V)$ regular?

iv) Exhibit a Cartan subalgebra of $\mathfrak{gl}(V)$, and describe the set of all regular elements of $\mathfrak{gl}(V)$.

[*Hint: For iii) pick a suitable basis of* V *to identify* $\mathfrak{gl}(V)$ *with* $\mathfrak{gl}_n$.]

6. Let $\mathfrak{g}$ be a nilpotent Lie algebra. Show that the Killing form on $\mathfrak{g}$ is identically zero.

7. (*Optional*: Let k be a field and let $\mathfrak{s}_\mathsf{k}$ be the 3-dimensional k-Lie algebra with basis $\{e_0, e_1, e_2\}$ and structure constants $[e_i, e_{i+1}] = e_{i+2}$ (where we read the indices modulo 3, so that we have for example $[e_2, e_0] = e_1$). Show that $\mathfrak{s}_\mathsf{k}$ is a simple Lie algebra. Show that $\mathfrak{s}_\mathbb{R}$ is not isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ but that $\mathfrak{s}_\mathbb{C} \cong \mathfrak{sl}_2(\mathbb{C})$. (*Hint: Consider characteristic polynomials.*)