C2.1a Lie algebras

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Problem Sheet 3

Assume throughout the problems that we work over a field k which is algebraically closed of characteristic zero, and all Lie algebras and representations are finite dimensional over k, unless the contrary is explicitly stated.

1. Let κ denote the Killing form on $\mathfrak{gl}_n(\mathbb{C})$ and let $\mathfrak{h}, \mathfrak{n}_+, \mathfrak{n}_-$ denote the subspaces of diagonal, strictly upper triangular and strictly lower triangular matrices respectively.

i) Show that $\mathfrak{h}$ is orthogonal to $\mathfrak{n}_+ \oplus \mathfrak{n}_-$ and that the restriction of κ to $\mathfrak{n}_+ \oplus \mathfrak{n}_-$ is nondegenerate. (*Hint*: It is probably useful to calculate the values of the Killing form on matrix coefficients).

ii) Calculate $\mathfrak{n}_+^\perp$.

iii) Describe the radical of the restriction of κ to $\mathfrak{h}$ and conclude that the restriction of κ to $\mathfrak{sl}_n(\mathbb{C})$ is nondegenerate.

2. Show that the Killing form for $\mathfrak{sl}_n$ is given by

$$\kappa(x, y) = 2n.\mathrm{tr}(x.y),$$

(where $\mathrm{tr}(x.y)$ denotes the ordinary trace, *i.e.* the trace form for the vector representation).

The next few questions of this exercise sheet classify all the irreducible finite dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$.

Recall that if we let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

then e, f and h give a basis of $\mathfrak{sl}_2$ with relations

$$[h, e] = 2e, [h, f] = -2f \quad \text{and} \quad [e, f] = h.$$

Hence, a representation of $\mathfrak{sl}_2(\mathbb{C})$ consists of a vector space V over $\mathbb{C}$ together with three endomorphisms E, F and H satisfying

$$HE - EH = 2E, HF - FH = -2F \quad \text{and} \quad EF - FE = H.$$

(We recover the representation $\phi : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$ by setting $\phi(e) = E$, $\phi(f) = F$ and $\phi(h) = H$.)

We will also need a partial ordering on k: since k has characteristic zero it contains a copy of $\mathbb{Q}$, and we will say that $a < b$ if $b - a \in \mathbb{Q}_{>0}$. If $I \subset \mathsf{k}$ is a finite subset of k we say $\lambda \in I$ is *maximal* if $\lambda < \mu$ implies $\mu \notin I$.

In the rest of this problem set we always assume that V is *finite dimensional.*

3. a) Show that the endomorphisms E and H satisfy the relation

$$(H - (\lambda + 2))^k E = E(H - \lambda)^k.$$

(Here $\lambda \in \mathbb{C}$ and we write λ instead of $\lambda \cdot \mathrm{id}_V$.) Deduce that if $v \in V$ belongs to the generalised λ-eigenspace of H, then Ev belongs to the generalised $(\lambda + 2)$-eigenspace.

b) Deduce a similar statement for the action of F on the generalised eigenspaces of H.

c) Let λ be an eigenvalue for H which is a maximal element of the set of eigenvalues of H in the sense described above. Use a) to show that $EV_\lambda = 0$.

d) Use b) to deduce that for large enough n we have $F^n(v) = 0$.

4. a) Show the relation (for $n \geq 1$)

$$HF^n = F^n H - 2nF^n.$$

b) Show ($n \geq 1$ as before)

$$EF^n = F^n E + nF^{n-1}H - n(n-1)F^{n-1}.$$

c) Deduce that, if $v \in V$ is a vector such that $Ev = 0$ then

$$E^n F^n v = nE^{n-1}F^{n-1}(H-(n-1))v = n!\prod_{i=1}^{n}(H-(i-1))v.$$

d) Let λ be a maximal eigenvalue of H (in the above sense) and let V_λ denote the generalised λ-eigenspace. Use 4(d) and (c) to deduce that H acts diagonalisably on V_λ and that λ is a non-negative integer.

5. a) Let λ be a maximal eigenvalue of H as in the previous question, and choose a non-zero vector $v \in V_\lambda$. We know by Questions 2 and 3 that $Ev = 0$ and that λ is an non-negative integer. Show the relations:

$$HF^k v = (\lambda - 2k)F^k v,$$
$$EF^k v = k(\lambda - (k-1))F^{k-1}v.$$

Deduce that $F^{\lambda+1}v = 0$ and that the $F^i v$ for $0 \leq i \leq \lambda$ are linearly independent and span a simple submodule of V.

b) Check that the above relations define an $\mathfrak{sl}_2(\mathbb{C})$-module for any non-negative integer λ. Deduce that there is (up to isomorphism) a unique simple module $V(\lambda)$ of dimension $\lambda + 1$ for all non-negative integers λ.

6. Let $\mathfrak{g}$ be a semisimple Lie algebra, and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ be its Cartan decomposition. To each $\alpha \in \Phi$ one may attach an element $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)} \in \mathfrak{h}$ (where $\lambda \mapsto t_\lambda$ is the isomorphism $\mathfrak{h}^* \to \mathfrak{h}$ induced by the Killing form restricted to $\mathfrak{h}$). Show that if V is a finite-dimensional representation of $\mathfrak{g}$ then the eigenvalues of h_α on V are all integers. (Here the elements $h_\alpha \in \mathfrak{h}$ are as given in Proposition 17.5 of the online lecture notes – you may just assume that Proposition when doing this question.)

7. *(Optional harder exercise)* Let V be an arbitrary finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$.

a) Let $\lambda \in \mathbb{Z}$ be maximal amongst the eigenvalues of H, and let $V_\lambda \subset V$ denote the λ-eigenspace. Suppose that V has the property that $Ev = 0$ implies that $v \in V_\lambda$. Show that V is completely reducible.

b) Consider the endomorphism $c = EF + FE + \frac{1}{2}H^2$. Show that c commutes with E, F and H. (c is called the *Casimir* element.) Deduce from Schur's lemma that c acts as a scalar on any irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. Compute the scalar with which c acts on $V(m)$.

c) Show that V is completely reducible.

(*This result also follows from the more general complete reducibility result due to Weyl, but it is nice to see a more explicit proof for an algebra as small as* $\mathfrak{sl}_2$*.*)