C2.1a Lie algebras

Mathematical Institute, University of Oxford

Problem Sheet 4

Throughout this sheet we assume that all Lie algebras and all representations discussed are finite dimensional unless the contrary is explicitly stated, and we work over a field k *which is algebraically closed of characteristic zero.*

1. i) Show directly that if $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a surjective homomorphism of semisimple Lie algebras and $x = s + n$ is the Jordan decomposition of $x \in \mathfrak{g}_1$, then $\phi(x) = \phi(s) + \phi(n)$ is the Jordan decomposition of $\phi(x) \in \mathfrak{g}_2$.

ii) Show that homomorphisms between semisimple Lie algebras are compatible with the Jordan decomposition, that is, if $\mathfrak{g}_1, \mathfrak{g}_2$ are semisimple Lie algebras, and $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a homomorphism, then if $x = s + n$ is the Jordan decomposition of $x \in \mathfrak{g}_1$, $\phi(x) = \phi(s) + \phi(n)$ is the Jordan decomposition of $\phi(x)$ in $\mathfrak{g}_2$. (*For this part you may assume the fact, stated in lectures, that if* $x = s + n$ *is the Jordan decomposition of* x *and* $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ *is a representation, then* $\rho(s)$ *is semsimple and* $\rho(n)$ *is nilpotent.*)

2. i) Show that if V is a (finite dimensional) representation of $\mathfrak{sl}_2$ and $V = \bigoplus_{k\in\mathbb{Z}} V_k$ is the decomposition of V into (generalised) eigenspaces of h, then the number of irreducible constituents of V is equal to $\dim(V_0) + \dim(V_1)$.

ii) Show that if $\mathfrak{g} = \mathfrak{h} \oplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ is the Cartan decomposition of a semisimple Lie algebra $\mathfrak{g}$, and α, β and $\alpha + \beta$ are all in Φ, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. (*Hint: Use the representation theory of* $\mathfrak{sl}_2$. *You may assume all root spaces are* 1*-dimensional.*)

3. Use Weyl's theorem to give an alternative proof of the fact that any derivation of a semisimple Lie algebra $\mathfrak{g}$ is inner. (*Hint*: A derivation lets you construct a semi-direct product.)

4. Suppose that $\mathfrak{g}$ is a Lie algebra and (V, ρ) is a faithful finite-dimensional representation (so that we may think of $\mathfrak{g}$ as a subalgebra of $\mathfrak{gl}(V)$). Show that if V is irreducible and $\mathrm{tr}(\rho(x)) = 0$ for all $x \in \mathfrak{g}$, then $\mathfrak{g}$ is semisimple.

5. Let $\mathfrak{g} = \mathfrak{sp}_{2n}$ be the symplectic Lie algebra. Show that $\mathfrak{h}$, the space of matrices in $\mathfrak{g}$ which are diagonal, is a Cartan subalgebra, and thus find the roots of $\mathfrak{sp}_{2n}$. (*Optional: Do the same for the Lie algebras* $\mathfrak{so}_{2n}$.)

6. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. If $\Phi \subset \mathfrak{h}^*$ is the corresponding root system find an expression for the dimension of $\mathfrak{g}$ in terms of Φ. (In particular, the dimension of $\mathfrak{g}$ is determined by the root system).

7. lLet V be a $\mathbb{Q}$-vector space. A *lattice* in V is a discrete subgroup[1] $Q \subset V$ which spans V over $\mathbb{Q}$. Equivalently, a lattice is a subgroup Q of V of the form

$$\{\sum_i \lambda_i \beta_i : \lambda_i \in \mathbb{Z}\}$$

where $\{\beta_i\}_{i=1}^n$ is a basis of V. (You do not have to prove this).

Assume that V is equipped with an positive definite inner product $(-,-)$. A lattice $Q \subset V$ is called *integral* if $(\alpha, \beta) \in \mathbb{Z}$ for all $\alpha, \beta \in Q$. A lattice Q is called *even* if $(\alpha, \alpha) \in 2\mathbb{Z}$ for all $\alpha \in Q$.

i) Show that an even lattice is integral.

ii) Let $Q \subset V$ be an even lattice. Assume that the set $R_Q = \{\alpha \in Q : (\alpha, \alpha) = 2\}$ spans V. Show that R_Q is a root system in V.

iii) Let $V = \bigoplus_{i=1}^r \mathbb{Q}e_i$ equipped with the standard inner product $(e_i, e_j) = \delta_{ij}$. Let

$$\Gamma_r = \{\sum a_i e_i : \sum a_i \in 2\mathbb{Z} \text{ and either all } a_i \in \mathbb{Z} \text{ or all } a_i \in \mathbb{Z} + \frac{1}{2}.\}$$

Show that Γ_r is an even lattice if r is divisible by 8.

[1] That is, each element has an open ball around it which contains no other element of Q.

iv) (*Optional*) Consider $\Gamma = \Gamma_8 \subset \mathbb{Q}^8$. Show that V is spanned by the vectors $v \in \Gamma$ such that $(v, v) = 2$, and describe the roots in the resulting root system R_Γ.

v) (*Optional*) Consider the functional $t \in V^*$ given by

$$t = \sum_{i=1}^{7} (i-1)e_i^* + 23e_8^*,$$

where $\{e_i^*\}_{i=1}^8$ is the dual basis to $\{e_i\}_{i=1}^8$. Show that $0 \neq t(\alpha) \in \mathbb{Z}$ for all $\alpha \in R_\Gamma$. Calculate the set of roots $\alpha \in R_\Gamma$ with $t(\alpha) = 1$ and check it is a basis of V. Compute the matrix of the inner product with respect to this basis. (This step is similar to the proof that a root system has a base.)

vi) (*Optional*) Let H_7 denote the hyperplane V orthogonal to $e_7 + e_8$. Show that $R_\Gamma \cap H_7$ is a root system of and find a basis for H_7 contained in it (*hint: start with the basis in the previous part.*)

vii) (*Optional*) Let H_6 be the subspace orthogonal to $e_6 + e_7 + 2e_8$ and $e_7 + e_8$. Show that $R_\Gamma \cap H_6$ is a root system and calculate a basis for H_6 contained in it.