Why study Analytic Topology?

1. What exactly is Analytic Topology?

Topology is an enormously wide and varied area, and the topology courses on offer in section C reflect this.

If you're seriously considering this course, you'll have done some topology already. So you'll already have met the Hausdorff condition, compactness, and similar notions, and have seen some basic theorems about them.

This course will carry on along these lines. We'll prove that any product of compact spaces is compact (in a previous course you'll probably have seen this for a product of two compact spaces). We'll examine the properties of compact Hausdorff spaces, be impressed with how wonderful they are, and answer the question of when a topological space can be embedded inside one (thus inheriting some of the wonderfulness). And we'll look at generalisations of compactness, including one that is, rather usefully, possessed by all metric spaces.

I'll now look at some of these results a bit more closely.

2. Products of compact spaces

2.1. What do we mean by an infinite product of spaces?

The statement that any product of compact spaces is compact is called *Tychonoff's Theorem*; and the most revolutionay element of it is Tychonoff's notion of how to define an infinite product of topological spaces.

With hindsight, it's easy to see that Tychonoff's definition is the right one; it's even clearer if you know some category theory (I won't assume that you do). But at the time Tychonoff was writing, this wasn't clear at all. In the lectures, I'll try to make the definition look natural. But in doing so, I will be doing violence to history.

2.2. Proving Tychonoff's Theorem

And then we come to the proof. Tychonoff's Theorem is actually equivalent to the Axiom of Choice. So it'll come as no surprise if the proof uses some heavy technology. In the process of developing this technology, I'll answer the question of how to describe continuity and convergence, given that convergent sequences don't work.*

I'll need to use some set theory from time to time in this course. I won't assume you've done any set theory before, however; I'll explain it when it comes up.

3. Embedding a space in a compact Hausdorff space

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* If you've never encountered this idea before, see if you can contruct a topological space X, a subset A of X, and a point x in X, such that x belongs to the closure of A, but no sequence on A converges to x.

3.1. Why would you do such a thing?

The properties of compact Hausdorff spaces are so useful that it's a real nuisance when you come across a space that isn't.

The process of embedding a space as a (dense, but this comes for free) subspace of a compact Hausdorff space is known as *compactification.* It's often convenient to compactify a space X, prove a theorem about the compactification, and then draw consequences for X itself.

3.2. Compactifications large and small

In this course we'll examine two main methods of compactification, and find out when they're possible.

At the small end, there's the *Alexandroff one-point compactification.* $\mathbb{R}$ has one, for example (the circle); so does $\mathbb{C}$ (the Riemann sphere). One big advantage of the Alexandroff one-point compactification is that it's easy to understand and picture.

At the other end we have the *Stone-Čech compactification.* Where this one loses is in visualisability. I personally doubt if anyone can visualise the Stone-Čech compactification of, say, $\mathbb{R}$, or $\mathbb{N}$. But where it wins is in its properties. The unique property of the Stone-Čech compactification of a space X is that any continuous function f from X to a compact space K, can be extended to the Stone-Čech compactification of X. This puts us in a position to exploit the power of compactness, as it applies to f.

This power has been used to prove combinatorial theorems about the natural numbers by extending the binary function $+$ to the Stone-Čech compactification of $\mathbb{N}$ (and, for example, to find fixed points, that is, non-zero p such that $p + p = p$).

4. How does Analytic Topology connect with other courses?

There are a few connections with functional analysis. There is a bit of a flavour of set theory. C1.3 Analytic Topology doesn't, however, connect very closely with the other C1 courses, though I do end the course with a discussion of Stone duality, which has some bearing on mathematical logic.

5. Will I enjoy Analytic Topology?

If you liked your first topology course, you might well.

If your topological interests are more in the direction of the study of manifolds, or of geometry, then it may be less up your street.

If you really didn't like dealing with open covers, and closures, and neighbourhoods, and such things, then this course may not be for you.

R. W. Knight, May 2020