Analytic Topology: Problem sheet 0

1. (i) *Prove that every compact subset of a Hausdorff space is closed.*

Let X be a Hausdorff space, and let K be a compact subset of X.

We show that K is closed by showing that its complement is open.

Suppose that x is not an element of K.

For each $y \in K$, we use Hausdorffness of X to find disjoint open subsets U_y and V_y of X such that $x \in U_y$ and $y \in V_y$.

Now the family $\{V_y : y \in K\}$ is an open cover of K.

K is compact, so there exists a finite subcover $\{V_{y_i} : i < n\}$ (where n is some natural number).

Let $U = \bigcap_{i<n} U_{y_i}$.

Then for all $i < n$, $U \subseteq U_{y_i}$. So $U \cap V_{y_i} = \varnothing$. Also notice that U is a finite intersection of open sets, so it is open.

Let $V = \bigcup_{i<n} V_{y_i}$.

Then because $U \cap V_{y_i} = \varnothing$ for all i, $U \cap V = \varnothing$, and because $\{V_{y_i} : i < n\}$ is a cover of K, $K \subseteq V$. Hence $U \cap K = \varnothing$.

Hence for any $x \notin K$, we have an open set U such that $x \in U$ and $U \cap K = \varnothing$.

So the complement of K is open, and hence K itself is closed, as required.

(ii) *Give an example of a space X with a compact subset K which is not closed.*

There are many examples. The simplest is probably the two-point indiscrete space $X = \{0, 1\}$ (recall that a space X is indiscrete if and only if the only open sets are $\varnothing$ and the the whole space X), with $K = \{0\}$: K is neither the empty set nor the whole of X, so it is not closed, but it is finite, so it is compact.

2. (i) *Prove that every closed subset of a compact space is compact.*

Let X be a compact space, and let C be a closed subset.

Let $\mathscr{U}$ be an open cover of C.

Then because C is closed, $X \setminus C$ is open.

Thus $\mathscr{U} \cup \{X \setminus C\}$ is an open cover of X.

Now X is compact, so there exists $\mathscr{V}$ which is a finite subset of $\mathscr{U} \cup \{X \setminus C\}$ and is a cover for X.

The extra set $X \setminus C$ may or may not be a member of $\mathscr{V}$. If it is not, then $\mathscr{V}$ is already the finite subcover of $\mathscr{U}$ that we are seeking. If it is, we eliminate it, to obtain $\mathscr{V} \setminus \{X \setminus C\}$, which is a finite subset of $\mathscr{U}$, and is a cover of C.

Thus C is compact.

(ii) *Give an example of a space X with a closed subspace A which is not compact.*

There are many examples of this. One of the most familiar is $X = \mathbb{R}$ with the usual topology, and $A = [0, \infty)$.

3. *Prove that the image of any compact space under a continuous function is compact.*

Suppose that X is a compact space, and that $f : X \to Y$ is a continuous surjection.
Let $\mathscr{U}$ be an open cover of Y.
Then for each element U of $\mathscr{U}$, $f^{-1}(U)$ is an open set, because f is continuous.
Also the set $\{f^{-1}(U) : U \in \mathscr{U}\}$ is a cover of X.
Because X is compact, there is a finite subset $\{f^{-1}(U_i) : i < n\}$ which is a cover of X.
Then $\{U_i : i < n\}$ is a finite subset of $\mathscr{U}$ which is a cover for Y.
Hence Y is compact.

4. (i) *Prove that if X is a compact space, Y is a Hausdorff space, and $f : X \to Y$ is bijective and continuous, then it is a homeomorphism.*

It is sufficient to prove that $f^{-1} : Y \to X$ is continuous.

Recall that a function is continuous if and only if the inverse image of every closed set is closed.

So let C be a closed subset of X. We consider its inverse image under the function f^{-1}, namely $(f^{-1})^{-1}[C]$.

This is equal to the forward image $f[C]$ of C under f.
Now C is a closed subset of X, and X is compact.
Hence C is compact.
The image of any compact space under a continuous function is compact.
Hence $f[C]$ is compact.
Now Y is Hausdorff, so $f[C]$ is closed.

Thus the inverse image of any closed set under the map f^{-1} is closed, so f^{-1} is continuous.

Thus f is a homeomorphism.

(ii) *Give examples to show that the hypotheses that X is compact and that Y is Hausdorff cannot be omitted.*

We note that any function whose domain is discrete is continuous, and any function whose range is indiscrete is continuous.

Let X and Y be spaces of the same infinite size, such that X is discrete and Y is not, and let $f : X \to Y$ be a bijection. Then f is automatically continuous, but is not a homeomorphism.

Note that because X is infinite and discrete, it is not compact.

Now let X and Y be spaces of the same size, which must be at least two, such that Y is indiscrete and X is not, and let $f : X \to Y$ be a bijection. Then again f is automatically continuous, but is not a homeomorphism.

Note that because Y is indiscrete and has at least two distinct points, it is not Hausdorff.

5. *Let (X, d) be a metric space.*

(i) *Show that a subset A of X is closed if and only if every accumulation point a of a sequence $(a_n)_{n\in\mathbb{N}}$ of elements of A, is itself an element of A.*

First, suppose that A is closed. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of elements of A, and let a be an accumulation point of this sequence.

Recall that a belongs to the closure of A if and only if every open set containing a meets A.

But every open set containing a also contains a_n for infinitely many values of n, and thus contains at least one point of A, as required.

So a belongs to $\overline{A}$.

But A is closed, so $\overline{A} = A$, so a belongs to A.

Now suppose that whenever a is an accumulation point of a sequence $(a_n)_{n\in\mathbb{N}}$ on A, then a is an element of A.

We argue that A is closed, by showing that $\overline{A} \subseteq A$.

For suppose that a is an element of $\overline{A}$.

Then every open set containing a meets A.

Let a_n be a point of A contained in the ball of radius $1/n$ around a.

Then the sequence $(a_n)_{n\in\mathbb{N}}$ converges to a.

A fortiori, a is an accumulation point of the sequence.

By our hypothesis, a now belongs to A.

So we have shown that $\overline{A} \subseteq A$, and so A is closed.

(ii) *Show that if a is an accumulation point of a sequence $(a_n)_{n\in\mathbb{N}}$, then there is a subsequence of $(a_n)_{n\in\mathbb{N}}$ which converges to a.*

Since a is an accumulation point of the sequence $(a_n)_{n\in\mathbb{N}}$, every open set containing a contains a_n for infinitely many values of n.

For each natural number $k \geq 1$, let n_k be such that a_{n_k} is contained in the ball of radius $1/k$ about a, and such that for all k, $n_k < n_{k+1}$.

Then the sequence $(a_{n_k})_{k\in\mathbb{N}}$ converges to a, as required.

(iii) *Deduce that in a metric space, the topology can be completely described in terms of convergent sequences.*

This part just consists of restating what we've already proved.

A set A in a metric space is closed if and only if every accumulation point of any sequence on A is contained in A, if and only if the limit of every convergent sequence of elements of A is contained in A.

That is, we can tell whether A is closed purely by examining the convergent sequences of X.

Thus we can tell whether a set is *open* purely by looking at the convergent sequences of X; that is, the topology can be completely described in terms of convergent sequences.

6. *Let X be a Hausdorff space, let x be an element of X, and let C be a compact subset of X such that $x \notin C$. Prove that there exist disjoint open sets U and V such that $x \in U$ and $C \subseteq V$.*

Following the hint, we use Hausdorffness of X to show that for each $y \in C$ there exist disjoint open sets $U_y \ni x$ and $V_y \ni y$.

Now the family $\{V_y : y \in C\}$ is an open cover of C.

Let $\{V_{y_i} : i < n\}$ be a finite subcover.

Let $V = \bigcup_{i<n} V_{y_i}$ and let $U = \bigcap_{i<n} U_{y_i}$. U is a finite intersection of open sets, so it's open. (This is the point at which compactness is crucial.) Also, $U \cap V = \varnothing$; and $x \in U$ and $C \subseteq V$.

7. *Prove that a product of two compact spaces is compact.*

Let X and Y be compact spaces, and suppose that $\mathscr{U}$ is an open cover of $X \times Y$.

For each $x \in X$ and $y \in Y$, let $U_{x,y}$ be an open subset of X and $V_{x,y}$ be an open subset of Y such that for some element $W_{x,y}$ of the open cover $\mathscr{U}$,

$$(x, y) \in U_{x,y} \times V_{x,y} \subseteq W_{x,y}.$$

Fix x for a moment. Then for each $y \in Y$, $y \in V_{x,y}$, so $\{V_{x,y} : y \in Y\}$ is an open cover of the compact space Y. So let F_x be a finite subset of Y such that $\{V_{x,y} : y \in F_x\}$ covers Y.

Then the finite family

$$\{U_{x,y} \times V_{x,y} : y \in F_x\}$$

covers $\{x\} \times Y$.

Let $U_x = \bigcap_{y \in F_x} U_{x,y}$. Then U_x is a finite intersection of open sets, so it is open. Also $x \in U_x$, so

$$\{x\} \times Y \subseteq U_x \times Y = U_x \times \bigcup_{y \in F_x} V_{x,y} = \bigcup_{y \in F_x} U_x \times V_{x,y} \subseteq \bigcup_{y \in F_x} U_{x,y} \times V_{x,y}.$$

Now for each x, $x \in U_x$, so the open sets U_x cover the compact space X, so let G be a finite subset of X such that $\{U_x : x \in G\}$ covers X.

Then $X = \bigcup_{x \in G} U_x$.

Hence

$$X \times Y = \left(\bigcup_{x \in G} U_x \right) \times Y = \bigcup_{x \in G} (U_x \times Y) \subseteq \bigcup_{x \in G} \bigcup_{y \in F_x} U_{x,y} \times V_{x,y}.$$

It follows that the family $\{U_{x,y} \times V_{x,y} : x \in G,\, y \in F_x\}$ covers $X \times Y$.

Hence so does $\mathscr{U}' = \{W_{x,y} : x \in G,\, y \in F_x\}$.

Since G is finite, and all sets F_x are finite, so is $\mathscr{U}'$; so $\mathscr{U}'$ is a finite subcover.

The proof that $X \times Y$ is compact is now complete.

This argument can obviously be iterated to show that a product of three, four, five,... compact spaces is compact. However it provides no clue as to how to extend this to infinite products of compact spaces.