C8.6 Limit Theorems and Large Deviations in Probability

Sheet 4 HT 2021 (Sections 10, 11)

1. (i) Let $\delta > 0$ and $s \in E$, and X,Y two E-valued random variables on $(\Omega, \mathcal{F}, P)$. Show that

$$P\left[X \in B_s(2\delta)\right] + P\left[\rho(X,Y) > \delta\right] \geq P\left[Y \in B_s(\delta)\right],$$

where $B_s(\delta)$ denotes the open ball centered at s with radius δ. [*Hint.* If $a \in B_s(\delta)$ and $b \notin B_s(2\delta)$, then $\rho(a,b) > \delta$.]

(ii) Let $c > 0$ such that

$$\liminf_{\varepsilon\downarrow 0} \varepsilon \log P\left[\rho(X^\varepsilon, Y^\varepsilon) > \delta\right] \leq -c$$

where X^ε and Y^ε are E-valued random variables for every $\varepsilon \in (0,1)$. Show that

$$\liminf_{\varepsilon\downarrow 0} \varepsilon \log P\left[X^\varepsilon \in B_s(2\delta)\right] \vee (-c) \geq \liminf_{\varepsilon\downarrow 0} \varepsilon \log P\left[Y^\varepsilon \in B_s(\delta)\right].$$

In parts (iii) and (iv), suppose $\{X_n^\varepsilon : \varepsilon \in (0,1)\}$ is a family of random variables in a Polish space (E,ρ) on a probability space $(\Omega, \mathcal{F}, P)$, satisfying LDP with a good rate function I_n, where $n = 1, 2, \cdots$. Suppose $X_n^\varepsilon \to X^\varepsilon$ as $n \to \infty$ exponentially, i.e.

$$\lim_{n\to\infty} \limsup_{\varepsilon\downarrow 0} \varepsilon \log P\left[\rho(X_n^\varepsilon, X^\varepsilon) > \delta\right] = -\infty \tag{1}$$

for every $\delta > 0$.

(iii) Show that

$$\liminf_{\varepsilon\downarrow 0} \varepsilon \log P\left[X^\varepsilon \in B_s(2\delta)\right] \geq -\limsup_{n\to\infty} \inf_{B_s(\delta)} I_n(s)$$

for every $s \in E$ and $\delta > 0$.

(iv) Let $S \subset E$ be a closed subset, and $S^\delta = \{s \in E : \rho(s,S) < \delta\}$ for $\delta > 0$. Show that

$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log P\left[X^\varepsilon \in S\right] \leq -\lim_{\delta\downarrow 0} \liminf_{n\to\infty} \inf_{\overline{S^\delta}} I_n. \tag{2}$$

2. Let E, E' be two Polish spaces, $f_n : E \to E'$ be a sequence of continuous mappings, and $I : E \to [0,\infty]$ be a good rate function. Suppose f_n converges to f uniformly on $I_c = \{x : I(x) \leq c\}$ for every $c \geq 0$. Define

$$I'(s') = \{I(s) : s \in H \text{ such that } f(s) = s'\}$$

where $H = \{s \in E : I(s) < \infty\}$, and $I'(s') = \infty$ if $s' \in E' \setminus H$. Show that I' is a good rate function on E', that is, $I'_c = \{s' : I'(s') \leq c\}$ is compact for every $c \geq 0$.

3. Let $B = (B(t))_{t\geq 0}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, P)$, and $B^\varepsilon = \sqrt{\varepsilon} B$ for $\varepsilon \in (0,1)$. Suppose f_n be the mapping which sends a path w with time duration $[0,1]$ to its dyadic approximation

$$f_n(w)(t) = w(t_n^{k-1}) + 2^n(t - t_n^{k-1})(w(t_n^k) - w(t_n^{k-1}))$$

for $t \in \left[t_n^{k-1}, t_n^k\right]$, where $t_n^k = \frac{k}{2^n}$, $k = 0, \cdots, 2^n$, and $n = 1, 2, \cdots$.

(i) For every $T > 0$ and $\lambda > 0$ we have

$$P\left\{\sup_{s \le T} B(s) \ge \lambda T\right\} \le \exp\left(-\frac{\lambda^2}{2} T\right).$$

[*Hint.* You may use the fact that the running maximum $M_T = \sup_{s \le T} B(s)$ has a distribution with PDF

$$\frac{2}{\sqrt{2\pi T}} e^{-x^2/2T},$$

or consider the family of exponential martingales $\exp\left[\alpha B(t) - \frac{\alpha^2}{2} t\right]$ where $\alpha \in \mathbb{R}$.]

(ii) Then

$$P\left[\sup_{t \le 1} |B^\varepsilon(t) - f_n(B^\varepsilon)(t)| \ge \delta\right] \le 2^n \exp\left\{-\frac{2^n \delta^2}{8\varepsilon}\right\} \tag{3}$$

for every $\varepsilon \in (0,1)$ and $\delta > 0$. Hence deduce that $f_n(B^\varepsilon) \to B^\varepsilon$ exponentially as $n \to \infty$ as $C([0,1])$-valued random variables.

4. (*Wentzell-Freidlin's theory*) In this exercise we take the opportunity to develop a small part of Wentzell-Freidlin's small perturbation theory of dynamical systems. Let us concentrate on the one-dimensional case. Let $E = C_0([0,1], \mathbb{R})$ be the continuous path space in $\mathbb{R}$ starting at 0 with time duration $[0,1]$, equipped with the uniform norm. The mapping $f : w \to f(w)$ where $X = f(w)$ is defined by solving the integral equation:

$$X_t = w(t) + \int_0^t b(X_s) ds$$

where $b : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous.

(i) Show that $X : E \to E$ is well defined and is continuous.

(ii) Let P^ε (for every $\varepsilon \in (0,1)$) be the law of the solution $(X_t^\varepsilon)_{t \in [0,1]}$ to the stochastic integral equation

$$X_t^\varepsilon = \sqrt{\varepsilon} w(t) + \int_0^t b(X_s^\varepsilon) ds.$$

Show that (P^ε) satisfies a large deviation principle with some rate function I^b which you should specify.