C8.5: Introduction to Schramm-Loewner Evolution

Dmitry Belyaev

1 Introduction

There are several important lattice models that have a scaling limit as mesh of the lattice goes to zero. Moreover, these limits are invariant under conformal transformations. This is known in some cases and conjectured in others.

1.1 Brownian motion

One of motivating examples is the statement that the Random Walk converges to the Brownian Motion. This example has several important features:

- Discrete model converges to a continuous model when mesh of the lattice goes to zero.
- The limit is independent of the lattice. This is what is known as *universality*. In the case of the random walk this is true under rather mild assumptions on the regularity of the lattice.
- The limit is more symmetric than the discrete model. For example, the random walk on the square lattice is only symmetric with respect to vertical and horizontal reflections and rotations by $\pi/2$. The two-dimensional Brownian motion is invariant under all rotation. In fact, it is (up to change of time) is *conformally invariant*. This means that a conformal image of BM has the same law as BM. The precise statement is given below.

To explain the last result we first have to introduce some notations.

Definition 1.1 *A complex martingale is a process M_t in $\mathbb{C}$ or $\mathbb{C}^d$ that is a vector martingale when $\mathbb{C}$ or $\mathbb{C}^d$ is regarded as a real vector space.*

However there are changes brought on by the complex structure. It is clear that if M is a complex martingale then so is its complex conjugate $\bar{M}_t$. The bracket process is a bit more tricky to define in this setting - because there is a choice! Should it be bilinear or sesquilinear

$$\begin{aligned} \langle \alpha M, \beta N \rangle &= \\ & ?\alpha\beta \langle M, N \rangle \\ & ?\alpha\bar{\beta} \langle M, N \rangle . \end{aligned}$$

There is no canonical correct answer and we adopt the first convention.

Definition 1.2 *Let M and N be two continuous square integrable complex martingales then $\langle M, N\rangle_t$ is the unique bounded variation, adapted, and continuous $\mathbb{C}$-valued process with initial value zero so that*

$$M_t N_t - \langle M, N\rangle_t$$

is a martingale.

It is easy to see that if $M_t = R_t + iS_t$, $\tilde{M}_t = \tilde{R}_t + i\tilde{S}_t$ where R_t, S_t etc. are real valued martingales then

$$\langle M, M\rangle_t = \left\langle R, \tilde{R}\right\rangle_t - \left\langle S, \tilde{S}\right\rangle_t + i\left(\left\langle R, \tilde{S}\right\rangle_t + \left\langle \tilde{R}, S\right\rangle_t\right).$$

Definition 1.3 *A continuous complex (local) martingale M_t is a* conformal (local) martingale *if $\langle M, M\rangle_t \equiv 0$.*

Definition 1.4 *Complex Brownian motion is $Z_t = X_t + iY_t$ where the real and imaginary parts X_t and Y_t are independent real BM.*

It is very easy to check that this is a conformal martingale.

Next, we need some standard results adapted to the complex case.

Lemma 1.5 (Levy characterization of BM) *Let Z_t be a complex valued continuous adapted process. Then Z_t is a complex Brownian motion if and only if*

1. *Z_t is a conformal local martingale.*
2. *$\langle Z, \bar{Z}\rangle_t = 2t$.*

Lemma 1.6 (Dubins-Schwarz change of time) *If Z_t is a conformal martingale with continuous sample paths then $\left\langle Z, \bar{Z}\right\rangle_t$ is positive, increasing and continuous. Moreover, if $\left\langle Z, \bar{Z}\right\rangle_t$ is strictly increasing*[1] *and*

$$\tau(s) := \inf\left\{t \mid \left\langle Z, \bar{Z}\right\rangle_t > 2s\right\}$$

is finite for all s then $Z_{\tau(s)}$ is a complex Brownian motion on $\left(\Omega, \mathcal{F}_{\tau(s)}, \mathbb{P}\right)$.

Theorem 1.7 (Ito's Lemma - Complex Variable case) [2] *Suppose that Z is a continuous complex martingale and that f is a C^2 function then*

$$\begin{aligned} f(Z_t) - f(Z_0) &= \int_0^t \partial f(Z_s)\ dZ_s + \int_0^t \bar{\partial} f(Z_s)\ d\bar{Z}_s \\ &\quad + \left(\int_0^t \partial\partial f(Z_s)\ d\langle Z, Z\rangle_s + \int_0^t \bar{\partial}\bar{\partial} f(Z_s)\ d\left\langle \bar{Z}, \bar{Z}\right\rangle_s\right) \\ &\quad + 2\int_0^t \partial\bar{\partial} f(Z_s)\ d\left\langle Z, \bar{Z}\right\rangle_s \end{aligned}$$

The proof is a matter of checking algebra.

[1] This extra condition is required in the complex case. Without it $\langle X, X\rangle$ might stay constant while $\langle Y, Y\rangle$ increases, and thus X would be constant while Y varies; we would not be able to time change this to 2-dimensional Brownian motion.

[2] Here we use standard notations $\bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ and $\partial = \partial_z = \frac{1}{2}(\partial_x - i\partial_y)$.

Exercise 1.8 *Let Z_t be a conformal martingale and f be an analytic function. Show that $f(Z)$ is a martingale and*

$$f(Z_t) - f(Z_0) = \int_0^t f'(Z_s)\ dZ_s.$$

Verify that

$$\left\langle f(Z), \overline{f(Z)} \right\rangle_t = \int_0^t f'(Z_s) \overline{f'(Z_s)\, d\langle Z, Z\rangle_s}$$

We are now in a position to prove conformal invariance of complex Brownian motion.

Theorem 1.9 (Lévy) *If f is an entire function and Z is a conformal local martingale with $\left\langle Z, \bar{Z}\right\rangle_t$ strictly increasing, then $f(Z_t)$ is a conformal local martingale. It's bracket $\left\langle f(Z), \overline{f(Z)}\right\rangle_t = \int_0^t |f'(Z_s)|^2\ d\left\langle Z, \bar{Z}\right\rangle_s$ is positive, strictly increasing and continuous and if $\tau(s) := \inf\left\{t \mid \left\langle f(Z), \overline{f(Z)}\right\rangle_t > 2s\right\}$ then $f(Z_\tau)$ is a complex Brownian motion.*

This theorem follows immediately from the Levy characterization and Dubins-Schwarz change of time. The only non-trivial part is to show that if Z is a Brownian Motion then the new time runs to infinity. The precise statement is in the following exercise

Exercise 1.10 *Let Z_t be a complex Brownian Motion and f be an entire non-constant function. Show that $\int_0^\infty |f(Z_t)|^2 d(2t) = \infty$. [*Hint: the Brownian Motion is recurrent.*]*

1.2 Other lattice models

There is a whole range of critical lattice models that appear in statistical physics that are conjectured to exhibit similar behaviour. They have scaling limits, these scaling limits are independent of the lattice and they are conformally invariant.

The goal of this course is to introduce a one-parameter family of continuous random curves that are the only possible conformally invariant scaling limits of these models. These curves are called Schramm-Loewner Evolution (SLE). They were introduced by Oded Schramm in 1998 and led to a revolution in our understanding of critical lattice models.

1.3 Prerequisites

Theory of SLE requires tools from different areas of mathematics. The main ingredients come from stochastic analysis. It is assumed that you should be familiar with foundations of stochastic analysis, in particular, you should be comfortable with results like Ito formula and optional stopping theorem. The other main ingredient is complex analysis. It is assumed that you are familiar with foundations of complex analysis. We will need several results about conformal transformations but they will be briefly introduced in the next section. You will benefit from attending C4.8 Complex Analysis: Conformal Maps and Geometry, nut it is not strictly necessary.

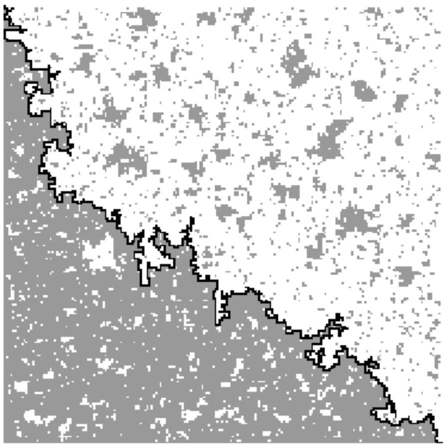


Figure 1: An interface between two clusters in the critical Ising model on the square lattice.

2 Complex analysis

In this section we will discuss the basic complex analysis results that will be needed later. In this section all functions will be *conformal transformations* on their respective domains. This means that they are complex analytic one-to-one maps. Alternative term is *univalent functions*. All domains will be open and in most cases simply-connected. Proof of the Riemann uniformization theorem could be found in many advanced complex analysis textbooks. One of the the standard references is the book by L. Ahlfors [1]. All necessary information can be found in my book [3]. You can also consult Lawler's book [6] which introduces necessary complex analysis using more probabilistic methods.

2.1 Riemann Uniformization Theorem

I will assume that you know some basic facts about Möbius transformations:

1. For every choice of distinct z_1, z_2, z_3 and w_1, w_2, w_3 there is unique Möbius transformation μ such that $\mu(z_i) = w_i$.

2. All Möbius transformations that map the unit disc $\mathbb{D}$ onto itself are of the form
$$\mu(z) = e^{i\theta}\frac{z - z_0}{1 - z\bar{z}_0},$$
moreover one can show that all *conformal* automorphisms of $\mathbb{D}$ are of this form.

3. All Möbius transformations that preserve the upper half-plane $\mathbb{H}$ are of the form
$$\mu(z) = \frac{az + b}{ac + d}$$
where a, b, c, d are real and $ad - bc > 0$. Moreover all *conformal* automorphisms of $\mathbb{H}$ are of this form.

4. All Möbius transformations of $\mathbb{C}$ are of the form $az + b$

One of the main results is the Riemann uniformization theorem which classifies all simply connected domains.

Theorem 2.1 (Riemann Uniformization) *Let Ω be a simply connected open domain in the complex sphere $\hat{C}$ and let z_0 be a point in Ω. There are three cases: $\hat{C}\setminus$ is empty, contains one point or contains at least three points. In these three cases there are conformal maps ϕ which map Ω onto $\hat{C}$, $\mathbb{C}$, and $\mathbb{D}$ respectively. Moreover in all the cases there is a map ϕ such that $\phi(z_0) = 0$ and the argument of $\phi'(z_0)$ is zero (we will write this as $\phi'(z_0) > 0$). In the last case this map is unique.*

In this statement we normalize the map by its value at one interior point and by the argument of the derivative at this point. There are other standard normalizations. If the boundary of domain is sufficiently nice, for example it is a Jordan curve, then the Riemann map is continuous up to the boundary and we can normalize so that the the domain is mapped onto the upper half-plane $\mathbb{H}$ in such a way that two fixed boundary points are mapped to 0 and ∞.

2.2 Basic properties

In this section we formulate several standard results about univalent functions in the unit disc. All these results can be found in C4.8 lecture notes or in many books on function theory. For example [4, 8, 3].

By S we denote the class of functions that are univalent in the unit disc and have expansion

$$f(z) = z + a_2 z^2 + a_3 z^2 + \dots .$$

An important example of a function from the class S is the Koebe function

$$K(z) = z + 2z^2 + 3z^3 + \dots .$$

Theorem 2.2 (Schwarz lemma) *Let f be an analytic function in $\mathbb{D}$ such that $f(0) = 0$ and $|f(z)| \le 1$ for all $z \in \mathbb{D}$. Then*

$$|f(z)| \le |z|, \qquad |f'(0)| \le 1.$$

Moreover, if $|f'(0)| = 1$ or $|f(z)| = |z|$ for some $z \ne 0$, then $f(z) = e^{i\theta} z$ all z for some $\theta \in \mathbb{R}$.

Theorem 2.3 (Growth Theorem) *Let $f \in S$, then for all $|z| = r$*

$$\frac{r}{(1+r)^2} \le |f(z)| \le \frac{r}{(1-r)^2}.$$

Moreover, if the equality happens in one of these inequalities for some $z \ne 0$, then f is a rotation of the Koebe function, namely, there is $\theta \in \mathbb{R}$ such that $f(z) = e^{-i\theta} K(e^{i\theta} z)$.

Theorem 2.4 *Let $f \in S$, then for all $|z| = r$*

$$\frac{1-r}{(1+r)^3} \le |f'(z)| \le \frac{1+r}{(1-r)^3}.$$

Moreover, if the equality happens in one of these inequalities for some $z \ne 0$, then f is a rotation of the Koebe function.

Theorem 2.5 (Koebe distortion) *Let f be a univalent map on a simply connected domain Ω and denote $\omega' = f(\Omega)$. Then for every $z \in \Omega$*

$$\frac{1}{4} \le |f'(z)| \frac{\operatorname{dist}(z, \partial\Omega)}{\operatorname{dist}(f(z), \partial\Omega')} \le 4.$$

Theorem 2.6 (Beurling distortion) *Let Ω be a simply connected domain and $K_2 \subset K_2$ be compacts such that $\Omega \setminus K_i$ is simply connected. We assume that $B(z_0, r_0) \subset \Omega \setminus K_2$ and $\operatorname{diam}(K_2 \setminus K_1) < \epsilon < r_0/10$. Let $g : \Omega \setminus K_1 \to \mathbb{D}$ be a conformal transformation such that $g(z_0) = 0$. Denote $A = g(K_2 \setminus K_1$. Then there is a constant which depends on r_0 only such that $\operatorname{diam}(A) \le \sqrt{\epsilon}$.*

This theorem gives us a uniform bound on the distortion near the boundary.

Definition 2.7 *The* kernel *of a family of simply-connected domains (Ω_n) with respect to w_0 is the largest simply connected domain Ω such that $w_0 \in \Omega$ and every closed subset of Ω belongs to all Ω_n for sufficiently large n. If there are no such domains, then we define the kernel to be $\{w_0\}$.*

We say that Ω_n converges to the kernel Ω if Ω is the kernel for every subsequence of (Ω_n). This convergence is called Carathéodory or kernel convergence*.*

Theorem 2.8 (Carathéodory) *Let (Ω_n) be a sequence of simply connected domains containing w_0. Let f_n be conformal maps from $\mathbb{D}$ onto Ω_n normalized by $f_n(0) = w_0$ and $f'(0) > 0$. Then the following conditions hold:*

1. *If $f_n to f$ uniformly on compact sets, then $\Omega_n \to \Omega = f(\mathbb{D})$.*
2. *If $\Omega_n \to \Omega$ in the Carathéodory sense and $\Omega \neq \mathbb{C}$, then $f_n \to f$ uniformly on compact sets, where f is the univalent map onto Ω normalized by $f(0) = w_0$ and $f'(0) > 0$.*
3. *In both cases described above, if the limit of domains is not a singleton or, equivalently, the limiting map f is not a constant, the inverse functions f_n^{-1} converge to f^{-1} locally uniformly.*

Theorem 2.9 (Poisson formulas) *Let u be harmonic in $\mathbb{D}$ and continuous in the closed unit disc, then*

$$u(z) = \frac{1}{2\pi} \int_{\partial\mathbb{D}} u(\zeta) P_{\mathbb{D}}(\zeta, z) |\mathrm{d}\,\zeta|,$$

where

$$P_{\mathbb{D}}(\zeta, z) = \frac{1 - |z|^2}{|\zeta - z|^2} = \operatorname{Re}\left(\frac{\zeta + z}{\zeta - z}\right).$$

The function P is called the Poisson kernel *in $\mathbb{D}$. It is the real part of the function which is called* the Schwarz kernel*.*

Let u be a harmonic function in $\mathbb{H}$ which is continuous up to the boundary and decay at infinity sufficiently fast, then

$$u(z) = \int_{\mathbb{R}} u(t) \frac{y}{(x-t)^2 + y^2} \mathrm{d}\,t = \int u(t) P_{\mathbb{H}}(t, z) \mathrm{d}\,t,$$

where the Poisson kernel in $\mathbb{H}$ is equal to $y/(\pi((x-t)^2 + y^2))$ is the imaginary part of the Schwarz kernel $1/(\pi(t - z))$.

Cauchy formula allows to represent an analytic function through its boundary values. These Poisson kernel formulas allow to represent an analytic function using only real or imaginary values on the boundary.

Theorem 2.10 (Schwarz formulas) *Let f be an analytic function in $\mathbb{D}$ which is continuous up to the boundary, then*

$$\begin{aligned} f(z) =& \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \mathrm{Re}\, f(e^{i\theta}) \mathrm{d}\,\theta + i \mathrm{Im}\, f(0) \\ =& \frac{1}{2\pi}\int_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z} \mathrm{Re}\, f(\zeta) |\mathrm{d}\,\zeta| + i \mathrm{Im}\, f(0). \end{aligned}$$

Similarly, if f is analytic in $\mathbb{H}$, continuous up to the boundary and decays at infinity at least as $|z|^{-\alpha}$ for some $\alpha > 0$, then

$$f(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Im}\, f(t)}{t-z} \mathrm{d}\, t.$$

2.3 Half-plane capacity

Definition 2.11 *The subset K of the upper half-plane $\mathbb{H}$ is called a compact $\mathbb{H}$-hull (or a hull for simplicity) if K is bounded, $K = \mathbb{H} \cap \bar{K}$ and $\mathbb{H} \setminus K$ is simply connected.*

Lemma 2.12 *Let K be a compact $\mathbb{H}$-hull, then there is a unique conformal transformation $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ such that*

$$\lim_{z\to\infty} g_K(z) - z = 0.$$

This normalization is called the *thermodynamic normalization.* In the future g_K will always denote the conformal transformation with thermodynamic normalization.

Definition 2.13 *For a hull K we define its half-plane capacity by*

$$\mathrm{hcap}(K) = \lim_{z\to\infty} z(g_K(z) - z)$$

From the definition of the capacity one can easily derive that the capacity is translationaly invariant and satisfies a simple scaling relation

Lemma 2.14 *Let K be a hull, then* $\mathrm{hcap}(K + x) = \mathrm{hcap}(K)$ *and* $\mathrm{hcap}(\lambda K) = \lambda^2 \mathrm{hcap}(K)$ *where $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}^+$.*

Example 2.15 *Let $K = \{z = x + iy : y > 0, x^2 + y^2 < 1\}$ the the upper half-disc, then* $\mathrm{hcap}(K) = 1$.

Joukowsky map $z + 1/z$ maps the unit disc and its complement onto the complex plane with the slit from -2 to 2. In particular it maps the complement of K onto $\mathbb{H}$.

Example 2.16 *Let K be the interval from 0 to it then* $\mathrm{hcap}(K) = t^2/2$

Example 2.17 *Suppose* $0 < \alpha < 1$ *and let* K *be the interval from the origin to* $\alpha^\alpha(1-\alpha)^{1-\alpha}e^{i\alpha\pi}$. *Then it is possible to show that*[3]

$$g_K^{-1}(z) = (z + (1-\alpha))^\alpha (z-\alpha)^{1-\alpha} = z - \frac{\alpha(1-\alpha)}{2z} + \dots$$

hence $\operatorname{hcap}(K) = \alpha(1-\alpha)/2$.

Lemma 2.18 *Let* K *be a hull,* B_t *be a complex Brownian motion started from* $z = x + iy \in \mathbb{H} \setminus K$ *and let* $\tau = \inf\{t > 0 : B_t \notin \mathbb{H} \setminus K\}$ *be the first exit time. Then*

1. $\operatorname{Im}(z) = \operatorname{Im} g_K(z) + \mathbb{E}^z(\operatorname{Im} B_\tau)$
2. $\operatorname{hcap}(K) = \lim_{y\to\infty} y\mathbb{E}^{iy}(\operatorname{Im} B_\tau)$
3. *If the set* K *is inside the unit disc, then*

$$\operatorname{hcap}(K) = \frac{2}{\pi}\int_0^\pi \mathbb{E}^{e^{i\theta}}(\operatorname{Im} B_\tau)\sin\theta d\theta$$

Proof is left as an exercise.

Lemma 2.19 *Let* $\Omega = \mathbb{H} \setminus K$, $f_K : \mathbb{H} \to \Omega$, *then*

$$f_K(z) = z - \frac{\operatorname{hcap}(K)}{z} + \dots$$

and

$$\operatorname{hcap}(K) = \frac{1}{\pi}\int_{-\infty}^{\infty} \operatorname{Im} f_K(x) \mathrm{d}\, x.$$

Proof. The first part follows immediately from the thermodynamical normalization of $g_K = f_K^{-1}$.

To prove the second part we consider $f_K(z) - z$. This function is analytic in $\mathbb{H}$ and decays as $O(z^{-1})$ at infinity. By Schwarz formula

$$f(z) - z = \frac{1}{\pi}\int \frac{\operatorname{Im}(f_K(t) - t)}{t - z}\mathrm{d}\, t = \frac{1}{\pi}\int \frac{\operatorname{Im} f_K(t)}{t-z}\mathrm{d}\, t.$$

Multiplying by z and passing to the limit as $z \to \infty$ we have

$$z(f_K(z) - z) = -\operatorname{hcap}(K) + O(z^{-1}).$$

On the other hand, since K is compact, so $\operatorname{Im} f_K(t) = 0$ outside of some compact set, we have

$$\frac{1}{\pi}\int \left(-1 + \frac{t}{t-z}\right)\operatorname{Im} f_K(t)\mathrm{d}\, t \to \frac{1}{\pi}\int \operatorname{Im} f_K(t)\mathrm{d}\, t.$$

■

The integral formula for capacity immediately implies the following corollaries:

[3] This is a Schwarz–Christoffel map

Corollary 2.20 *The capacity is positive.*

Corollary 2.21 *Capacity is additive in the following sense: Let A_1 and A_2 be two disjoint hulls. Denote by A their union and by $\tilde{A}_2$ the image of A_2 under g_{A_1}. Then* $\mathrm{hcap}(A) = \mathrm{hcap}(A_1) + \mathrm{hcap}(\tilde{A}_2)$.

Corollary 2.22 *Capacity is an increasing function of domain. Namely, let $A_1 \subset A_2$ be two hulls, then* $\mathrm{hcap}(A_1) \leq \mathrm{hcap}(A_2)$

2.4 Mapping-out functions

In this section we study properties of mapping-out functions g_K. This class of functions is closely related to the class Σ but there are some important distinctions.

Lemma 2.23 *Let K be a half-plane hull and g_K be a corresponding mapping. Suppose that $x \in \mathbb{R}$ is to the right of K, that is $[x, \infty)$ does not intersect the closure of K, then*

$$g_K(x) > x. \tag{1}$$

For points that are to the left of K the inequality is reversed.

This lemma has a simple geometric interpretation: the mapping out function pushes the hull down and it pushes the real line away towards infinity. The proof is a straightforward application of Schwarz integral formula applied to f_K and is left as an exercise.

Exercise 2.24 *Use the Schwarz formula for $f_K(z) - z$ to prove Lemma 2.23.*

Lemma 2.23 could be used to obtain two similar results: one states that $g_K(x)$ is monotone with K and the other gives an upper bound on $g_K(x) - x$ given a bound on K.

Lemma 2.25 *Suppose that $K_1 \subset K_2$ are two hulls and let x be a point to the right of K_2 then*

$$g_{K_1}(x) \leq g_{K_2}(x) \tag{2}$$

where equality happens for some x is and only if $K_1 = K_2$.

Proof. Let us define $K = g_{K_1}(K_2 \backslash K_1)$ which is a compact hull unless $K_1 = K_2$. Assuming that this is the case we consider the function g_K. By uniqueness of the Riemann map with thermodynamic normalization $g_{K_2}(z) = g_K(g_{K_1}(z))$. Applying (1) to g_K on the right-hand side we get

$$g_{K_2}(x) > g_{K_1}(x).$$

■

Corollary 2.26 *Let us assume that a non-trivial compact hull K is inside the unit disc $\mathbb{D}$ and that $x > 1$, then*

$$x < g_K(x) < x + \frac{1}{x}. \tag{3}$$

For $x < -1$ we have $x > g_K(x) > x + 1/x$.

Proof. This result is a combination of the previous lemma applied to $K_1 = K$ and $K_2 = \mathbb{D} \cap \mathbb{H}$ and a simple observation that $g_{\mathbb{D}}(z) = z + 1/z$. ∎

Next we show that a similar result holds in the upper half plane as well, namely that there is a *uniform* estimate for $|g_K(z) - z|$ given a bound on K.

Lemma 2.27 *Let K be a half-plane hull, $H = \mathbb{H} \setminus K$ and $g_K : H \to \mathbb{H}$ the corresponding conformal map, then*

$$|g_K(z) - z| \le 3\mathrm{rad}(K), \tag{4}$$

for all $z \in H$.

Proof. We start by assuming that $K \subset \mathbb{D}$ and consider $(x^+, \infty) = g_K((1,\infty))$. By Corollary 2.26 $x^+ \in [1,2]$. For all $x > 1$ we have $0 \le g_K(x) - x \le 1/x \le 1$, or, equivalently, $|f_K(x) - x| \le 1$ for all $x \ge x^+$. In a similar way we define x^- by $(-\infty, x^-) = g_K((-\infty,-1))$ and by the same argument $|f_K(x) - x| \le 1$ for $x < x^-$. We already know that $(x^-, x^+) \subset (-2,2)$. Since $K \subset \mathbb{D}$ the image $f_K((x^-,x^+))$ is inside $\mathbb{D}$, this proves that $|f_K(x) - x| \le 3$ on (x^-, x^+). Combining all these estimates we can see that $|f_K(x)-x| \le 3$ and since $f_K(z) - z \to 0$ at infinity, by maximum modulus principle, $|f_K(z) - z| \le 3$ everywhere in $\mathbb{H}$. This is equivalent to $|g_K(z) - z| \le 3$ in the complement of K.

Finally, to prove the general case we assume that $K \subset \{z \in \mathbb{H} : |z - \zeta| \le r\}$ for some $r > 0$ and $\zeta \in \mathbb{R}$. Let $\tilde{K} = (K - \zeta)/r$ be the rescaled and shifted hull. By basic properties of mapping-out functions

$$g_K(z) = r g_{\tilde{K}}\left((z-\zeta)/r\right) + \zeta.$$

Combining this formula with an estimate $|g_{\tilde{K}}(z) - z| \le 3$ we prove (4).

∎

With a bit more work we can get the uniform bound for the next term as well.

Lemma 2.28 *Let K be a half-plane hull which is inside $|z| \le R$ and $z \in \mathbb{H}$ with $|z| \ge 10R$. Then*

$$\left| g_K(z) - z - \frac{\mathrm{hcap}(K)}{z} \right| \le \frac{10R\,\mathrm{hcap}(K)}{|z|^2}. \tag{5}$$

Constant 10 appearing in this lemma is not sharp, it could be improved, but the particular value is not really important. Also note that the condition that K is inside the disc of radius R is very similar to $\mathrm{rad}(K) \le R$, but we assume that the centre of the disc is at the origin. Without this assumption there is no universal estimate in terms of $\mathrm{rad}(K)$. Indeed, let us consider K to be the half-disc of radius 1 centred at x_0. Then $g_K(z) = z + (z - x_0)^{-1}$ and

$$|g(z) - z - 1/z| = \frac{|x_0|}{|z - x_0||z|}$$

which is not uniformly bounded.

Proof. The Schwarz formula for $f_K(z) - z$ and the integral formula for the capacity give

$$g_K(z) - z - \frac{\mathrm{hcap}(K)}{z} = \frac{1}{\pi}\int \mathrm{Im}\, f_K(x) \left(\frac{1}{g_K(z) - x} - \frac{\mathrm{hcap}(K)}{z} \right) \mathrm{d}\, x.$$

Since $|z| > 10R$ and $g_K(z) - z| \le 3R$ by (4), for $|x| \le 2R$ we have

$$\left|\frac{1}{g_K(z) - x} - \frac{\mathrm{hcap}(K)}{z}\right| \le \frac{5R}{|z||z - 5R|} \le \frac{10R}{|z|^2}.$$

By comparing K with the half-disc of radius R we see that $\operatorname{Im} f_K(x) = 0$ if $|x| > 2R$. Combining this with previous estimates we obtain

$$\left|g_K(z) - z - \frac{\mathrm{hcap}(K)}{z}\right| \le \frac{10R}{|z|^2}\frac{1}{\pi}\int \operatorname{Im} f_K(x) \mathrm{d}\, x = \frac{10R\mathrm{hcap}(K)}{|z|^2}.$$

■

3 Loewner Evolution

Let $\gamma(t)$ be a simple curve in the upper half-plane. For simplicity we assume that it starts at the origin, but it could start at any point on the real line. We assume $\gamma(t) \to \infty$ as $t \to \infty$. Let us fix some time $t > 0$ and consider the domain $\mathbb{H}_t = \mathbb{H} \setminus \gamma[0, t]$. By $b(t)$ we denote the capacity of $\gamma[0, t]$ and by g_t the map $g_{\gamma[0,2]}$ – the map from $\mathbb{H}_t$ onto $\mathbb{H}$ with thermodynamic normalization. By f_t we denote g_t^{-1} and for $s < t$ we define $\phi_{s,t} = g_s(f_t)$. Finally, by $u_t = u(t)$ we denote $g_t(\gamma(t))$. We assume that $\mathrm{hcap}(\gamma([0, t]))$ is a differentiable function of t. This assumption is not really restrictive, by change of time we can make it any given monotone function of t, in many cases we will assume that $\mathrm{hcap}(\gamma([0, t])) = 2t$. When $\mathrm{hcap}(\gamma([0, t])) = 2t$ we say that γ is *parametrized by capacity*.

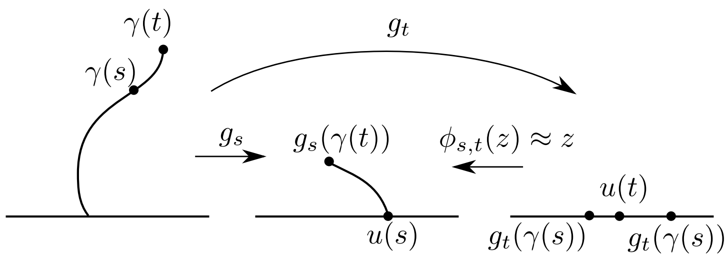


Figure 2: Chordal Loewner Evolution

Our goal is to derive differential equations satisfied by f_t and g_t.

First, we claim that u_t is continuous. The idea is very simple. By continuity, for any ϵ if $|s - t|$ is sufficiently small, then $\mathrm{diam}\gamma([s, t]) \le \epsilon$. By Beurling inequality

$$\mathrm{diam}(y_{s,t}) = \mathrm{diam}(g_s(\gamma([s, t]))) \le c\sqrt{\epsilon}.$$

By Lemma 2.27

$$|\phi_{s,t}(z) - z| \le c\sqrt{\epsilon}.$$

Applying this to $z = u_t$ and using the previous estimate we have

$$|u_t - u_s| \le c\sqrt{\epsilon}.$$

Next we claim that g_t satisfies a certain differential equation. To derive this differential equation we apply the half-plane version of Schwarz formula to $\phi_{s,t}(z) - z$ to write

$$\phi_{s,t}(z) - z = \frac{1}{\pi} \int \frac{1}{x - z} \operatorname{Im} \phi_{s,t}(x) \mathrm{d}\, x.$$

By construction $g_s = \phi_{s,t}(g_t)$, so by plugging g_t instead of z in the formula above we obtain

$$g_s(z) - g_t(z) = \frac{1}{\pi} \int \frac{1}{x - g_t(z)} \operatorname{Im} \phi_{s,t}(x) \mathrm{d}\, x.$$

Since $\operatorname{Im} \phi_{s,t}(x) = 0$ outside of a shrinking neighbourhood of $u(t)$, dividing by $s - t$ and passing to the limit as $s \nearrow t$ we obtain

$$\dot{g}_t(z) = \frac{1}{u(t) - g_t(z)} \lim \frac{1}{s - t} \frac{1}{\pi} \int \operatorname{Im} \phi_{s,t}(x) \mathrm{d}\, x.$$

By Lemma 2.19 and additive property of capacity the last integral is equal to $\operatorname{hcap}(\gamma([0,t])) - \operatorname{hcap}(\gamma([0,s]))$. This implies that the right hand side is equal to

$$\frac{\partial_t \operatorname{hcap}(\gamma([0,t]))}{g_t(z) - u(t)}.$$

Now we can formulate the main result:

Theorem 3.1 *Let γ be a curve in $\mathbb{H}$ satisfying all assumptions above and let f_t and g_t be the corresponding conformal maps and $u(t) = g_t(\gamma(t))$, then g_t satisfies an ordinary differential equation*

$$\dot{g}_t(z) = \frac{\partial_t \operatorname{hcap}(\gamma([0,t]))}{g_t(z) - u(t)}, \qquad g_0(z) = z \tag{6}$$

and f_t satisfies

$$\dot{f}_t(z) = -f_t'(z) \frac{\partial_t \operatorname{hcap}(\gamma([0,t]))}{z - u(t)}, \qquad f_0(z) = z. \tag{7}$$

We have already proved the first part of the theorem. To prove (7) we differentiate the identity $f_t(g_t(z)) = z$ with respect to t and use (6).

The equations (6) and (7) are known as (chordal) Loewner Evolutions or (chordal) Loewner Differential Equations.

Sometimes is is useful to consider the so called *radial* version of this theory. In this case γ is a simple curve inside $\mathbb{D}$ which grows from a boundary points (the default choice is 1) towards the origin. By g_t we denote the conformal map $g_t : \mathbb{D} \setminus \gamma([0,t]) \to \mathbb{D}$ normalized by $g_t(0) = 0$ and $g_t'(0) > 0$. We parametrize γ in such a way that $g_t'(0) = e^t$. We denote $\lambda_t = g_t(\gamma_t)$. Then, using a very similar argument, one can show that

$$\dot{g}_t(z) = g_t(z) \frac{\lambda_t + g_t(z)}{\lambda_t - g_t(z)} \tag{8}$$

and

$$\dot{f}_t(z) = -z f_t'(z) \frac{\lambda_t + z}{\lambda_t - z}. \tag{9}$$

3.1 Solving Loewner Evolution

In the previous section we have shown that given a nice curve γ we can define u_t and g_t and show that the family of maps g_t satisfies a differential equation which depends on u_t. Surprisingly this process can be reversed: given some function u_t, one can solve ODE and the solution will be given by a family of conformal maps.

Theorem 3.2 (Loewner) *Let μ_t be a family of non-negative Borel measures on $\mathbb{R}$ such that $t \mapsto \mu_t$ is continuous (in weak topology) and the measures are uniformly bounded. Namely, for every t there is M_t such that for all $0 \le s \le t$ we have $\mu_s(\mathbb{R}) \le M_t$ and $\operatorname{supp} \mu_s \subset [-M_t, M_t]$. Let $g_t(z)$ be the solution of*

$$\partial_t g_t(z) = \dot{g}_t(z) = \int_{\mathbb{R}} \frac{\mu_t(du)}{g_t(z) - u}, \qquad g_0(z) = 0. \tag{10}$$

For each $z \in \mathbb{H}$ we define

$$T_z = \inf\{t > 0 : g_t(z) \text{ is defined and } g_t(z) \in \mathbb{H}\}.$$

Define

$$\mathbb{H}_t = \{z \in \mathbb{H} : T_z > t\}.$$

Then $g_t : \mathbb{H}_t \to \mathbb{H}$ is the Riemann map with thermodynamic normalization

$$g_t(z) = z + \frac{b(t)}{z} + O(z^{-2}),$$

where

$$b(t) = \int_0^t \mu_s(\mathbb{R}) ds.$$

In the future we will be mostly interested in the case $\mu_t = 2\delta_{u_t}$. In this case we say that u_t is the driving function of the Loewner Evolution which now takes the form

$$\dot{g}_t(z) = \frac{2}{g_t(z) - u_t}, \qquad g_0(z) = z. \tag{11}$$

Proof. First of all we observe

$$\dot{g}_t = \int \frac{\operatorname{Re}(g_t - u) - \operatorname{Im}(g_t - u)}{|g_t - u|^2} \mu_t(du)$$

This implies that $\operatorname{Im} g_t(z)$ is strictly decreasing with time. In other words, all points move towards $\mathbb{R}$.

Let $z \neq w$ and define $\Delta_t(z, w) = g_t(z) - g_t(w)$, the equation (10) implies

$$\begin{aligned}
\dot{\Delta}_t(z, w) &= \int \left[\frac{1}{g_t(z) - u} - \frac{1}{g_t(w)} - u \right] \mu_t(du) \\
&= -\Delta_t(z, w) \int \frac{\mu_t(du)}{(g_t(z) - u)(g_t(w) - u)}
\end{aligned}$$

with the initial condition

$$\Delta_0(z, w) = z - w.$$

This equation could be integrated:

$$\Delta_t(z,w) = (z-w)\exp\left(-\int_0^t \int_{\mathbb{R}} \left((g_s(z)-u)^{-1}(g_s(w)-u)^{-1}\right) \mu_s(du)ds\right)$$

In particular this implies that if $g_t(z)$ and $g_t(w)$ are bounded away from $\mathbb{R}$ then $\Delta(z,w)/(z-w)$ is uniformly bounded.

$$\begin{aligned}
g_t'(z) &= \lim w \to w \frac{\Delta_t(z,w)}{z-w} \\
&= \lim_{w\to z} \exp\left(-\int_0^t \int_{\mathbb{R}} \left((g_s(z)-u)^{-1}(g_s(w)-u)^{-1}\right) \mu_s(du)ds\right) \\
&= \exp\left(-\int_0^t \int_{\mathbb{R}} (g_s(z)-u)^{-2}\mu_s(du)ds\right)
\end{aligned}$$

This proves that $g_t(z)$ is analytic in z. Moreover, the integral formula for Δ_t implies that $\Delta(z,w) \neq 0$ if $z \neq w$, hence g_t is one-to-one on $\mathbb{H}_t$. Therefore g_t is a conformal transformation on $\mathbb{H}_t$ and $g_t(\mathbb{H}_t) \subset \mathbb{H}$.

By dominated convergence theorem we can differentiate (10) with respect to z and obtain a useful formula for evolution of g_t'

$$\dot{g}_t'(z) = -g_t'(z)\int_{\mathbb{R}} (g_t(z)-u)^{-2}\mu_t(du), \qquad g_0'(z) = 1.$$

Next we have to show that g_t is onto, namely that for every $w \in \mathbb{H}$ there is a point in $\mathbb{H}_t$ such that $g_t(z) = w$. This is done by considering the "backward flow". We fix w and t and consider the initial value problem for $0 < s < t$

$$\dot{h}_s(w) = -\int \frac{\mu_{t-s}(du)}{g_s(w)-u}, \qquad h_0(w) = w.$$

Since the imaginary part of h_s is increasing, the solution exists for all $0 \leq s \leq t$. However, if h_s is a solution with $h_0(w) = w$, then $g_s := h_{t-s}$ is the solution of (10) with $g_t = w$. In other words $w = g_t(h_t(w))$.

Finally for large z

$$\dot{g}_t(z) = \frac{\mu_t(\mathbb{R})}{g_t(z)} + \cdots = \frac{\mu_t(\mathbb{R})}{z} + \ldots$$

hence

$$g_t(z) = z + \frac{\int_0^t \mu_s(\mathbb{R})ds}{z} + \ldots.$$

■

Sometimes it is convenient to consider the sets $K_t = \mathbb{H}\backslash\mathbb{H}_t$. It is not difficult to see that the sets K_t are increasing $\mathbb{H}$-hulls.

Example 3.3 [4] *If $u_t = 0$, then the Loewner Evolution* (11) *could be solved explicitly and the corresponding hulls K_t are vertical intervals from the origin.*

[4] See problem sheet 2

Example 3.4 [5] *If $u_t = c\sqrt{t}$ then the hulls K_t are of the form $\gamma[0,t]$ where*

$$\gamma(t) = 2\sqrt{t}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha-1/2} e^{i\pi(1-\alpha)}$$

and α is the only solution to

$$c = \frac{2\alpha(2\alpha-1)}{\sqrt{\alpha(1-\alpha)}}.$$

In the radial case the result is very similar.

Theorem 3.5 (Radial Loewner Evolution) *As before we start with a continuous uniformly bounded family of measures μ_t, but now they are on the unit circle $\mathbb{T}$. For each z we denote by $g_t(z)$ the solution of initial value problem*

$$\dot{g}_t(z) = g_t(z)\int_0^{2\pi} \frac{e^{i\theta}+g_t(z)}{e^{i\theta}-g_t(z)}\mu_t(d\theta), \qquad g_0(z) = z. \tag{12}$$

We define T_z as the supremum of all t such that the solution is defined up to time t and $g_t(z) \in \mathbb{D}$. Let $\mathbb{D}_t = \{z : T_z > t\}$. Then g_t is the unique conformal transformation of $\mathbb{D}_t$ onto $\mathbb{D}$ such that $g_t(0) = 0$ and $g_t'(0) > 0$. Moreover,

$$\ln g_t'(0) = \int_0^t \mu_s(\mathbb{T})ds.$$

The proof of this theorem is exactly the same as before, so we will not repeat it here.

Note: the radial version when the riving measures are δ-measures is the original equation that was introduced by Loewner in 1923.

3.2 When K_t is a curve

As we can see, when K_t is a simple curve $\gamma([0,t])$ then the corresponding measures are δ-measures, so we have a Loewner evolution driven by a function, moreover, this function is continuous. Under some additional assumptions, (almost) the same is true is γ is not necessarily simple. This raises a natural question: if a Loewner evolution is driven by a continuous function, is it true that K_t is given by a curve?

This is not quite true, but there is a simple geometric condition equivalent to the continuity of the driving function. This result was obtained by Pommerenke [7] in the radial case. Here we formulate a chordal version of the same result.

Theorem 3.6 *Let $f_t(z)$ be a Loewner evolution and K_t be the corresponding family of growing compacts. Let us assume that $0 \le t \le T \le \infty$. By $\mathbb{H}_t$ we denote $\mathbb{H} \setminus K_t$. Then the following two conditions are equivalent.*

1. *There is a continuous function u_t such that f_t and $g_t = f_t^{-1}$ satisfy the Loewner differential equations driven by u_t.*

[5] See problem sheet 3.

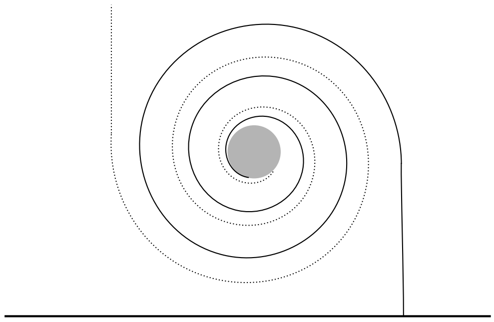


Figure 3: Spiral domain has no trace

2. *For every $\epsilon > 0$ there is δ such that for all $0 \le s < t \le T$ with $(t-s) \le \delta$ there is a cross-cut of $\mathbb{H}_s$ with diameter at most ϵ which separates $\mathbb{H}_s \setminus \mathbb{H}_t$ from infinity.*

Definition 3.7 *Let K_t be a growing family of half-plane hulls (i.e. compact sets in $\mathbb{H}$ such that $\mathbb{H} \setminus K_t$ is simply connected). We say that a curve γ is the* trace *of K_t if $\mathbb{H}_t = \mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma([0,t])$.*

The geometric condition in the theorem above is very close to the condition that there is a trace. Unfortunately, this is not quite true. Let us consider the following family: K_t is a solid spiral curve for $t < t_0$ (see Figure 3), K_{t_0} is the solid spiral with the limiting disc, for $t > t_0$ we add parts of the dotted spiral. It is not hard to show that this family of domains satisfy the second condition of Theorem 3.6, so the corresponding Loewner evolution is driven by a continuous curve. On the other hand, at time t_0 we add the entire disc to K_t, so there is no trace.

The following theorem gives a sufficient condition for the existence of the trace.

Theorem 3.8 *Let g_t be a chordal or radial Loewner evolution driven by a continuous function $u(t)$ and let K_t be the corresponding growing sets. Let us assume that K_t have locally connected boundary for every t (which implies that the conformal maps are continuous up to the boundary).*

Let us define

$$\gamma(t) = \lim_{y \to 0} g_t^{-1}(u(t) + iy). \tag{13}$$

Then $\gamma(t)$ is right continuous and the Loewner chain is generated by γ. Namely, $\mathbb{H}_t = \mathbb{H} \setminus K_t$ is the unbounded component of the complement of the closure of $\gamma([0,t])$.

We would like to mention that the curve γ does not have to be continuous. This could be seen from a simple counterexample shown in Figure 4. In this example we start with a curve $\gamma(t)$ which we assume to be parametrized by capacity. For $t < 1/2$ this curve behave like this: it starts as an arc of a semi-circle, then it makes a long excursion inside the unit disc, then it continues along

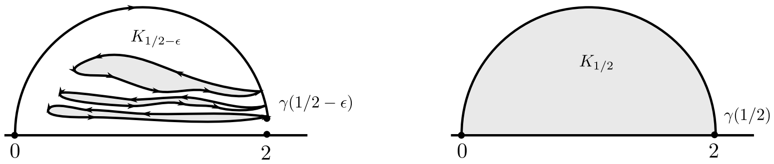


Figure 4: An example of a curve which is not locally connected, only right continuous, but the corresponding Loewner chain is driven by a continuous driving function and the hulls are locally connected.

the semi-circle, makes an excursion and so on. We can assume that the points from which the curve makes an excursion accumulate towards 2, for simplicity me can assume that nth excursion starts from a point which is 2^{-n} away from 2. We also assume that all excursions are of diameter at least 1/2.

By comparing $\gamma([0,t])$ with the half-disc and a circular slit, we can see that $\mathrm{hcap}(\gamma([0,t])) \nearrow 1$, and $\gamma([0,t])$ converges to the half-disc in the Carathéodory sense as $t \to 1/2$. This implies that it is indeed possible to parametrise this curve by capacity for $t \in [0,1/2)$. As $t \to 1/2$, this curve is obviously not continuous. We define $\gamma(1/2) = 2$.

For all $s < t < 1/2$, the increment $\gamma([s,t])$ could be separated from infinity by a short cross cut (the remaining part of the semi-circle). This shows that this curve satisfies the condition (2) of Theorem 3.6. This implies that the corresponding Loewner chain is generated by a continuous function.

Finally, for all $t < 1/2$, K_t is generated by a very nice continuous curve, hence it is locally connected. At $t = 1/2$, the hull is the half-disc and also simply connected. This shows that the hulls K_t for $t \in [0,1/2]$ satisfy all assumptions of Theorem 3.8, but the 'trace' curve γ is only right continuous, but not left continuous. Moreover, the trace itself $\gamma([0,1/2])$ is not locally connected, but the hulls K_t are locally connected.

Finally, we would like to mention the result of Lind which gives a condition implying that K_t is generated by a simple continuous curve.

Theorem 3.9 *Let g_t be the chordal Loewner evolution driven by $u(t)$ such that*

$$\sup_{s\neq t} \frac{|u(t)-u(s)|}{|t-s|^{1/2}} = \|u\|_{1/2} < 4,$$

then K_t is generated by a simple continuous curve γ.

3.3 Basic properties of Loewner Evolution

Markov or semi-group property Let $0 \le s \le t$ and define $g_{s,t}$ by

$$g_t(z) = g_{s,t}(g_s(z)).$$

Let us fix s and define $g_t^* = g_{s,s+t}$, then

$$\partial_t g_t^*(z) = \dot{g}_t(z) = \int_{\mathbb{R}} \frac{\mu_{s+t}(du)}{g_t^*(z)-u}, \qquad g_0(z) = 0.$$

This property is a direct corollary of the semi-group property of ODE. Running evolution for the time $t+s$ is the same as running it for the time s, taking the result as the initial condition and running it for time s.

Scaling property Let g_t be LE driven by a function u_t and let K_t be the corresponding hulls.

$$\dot{g}_t = \frac{2}{g_t - u_t}.$$

Let us consider the family of scaled hulls λK_t. We know that

$$g_{\lambda K}(z) = \lambda g_K(z/\lambda) = z + \frac{\lambda^2 \mathrm{hcap}(K)}{z} + \dots$$

We use the standard normalization $\mathrm{hcap}(K_t) = 2t$, and hence $\mathrm{hcap}(\lambda K_t) = 2\lambda^2 t$. Tohave the standard parametrization for the scaled domains we use the following change of time: $\tilde{K}_t = \lambda K_{t/\lambda^2}$, then $\mathrm{hcap}(\tilde{K}_t) = 2t$. Define

$$\tilde{g}_t(z) = \lambda g_{t/\lambda^2}(z/\lambda). \tag{14}$$

By construction $\tilde{g}_t$ map $\mathbb{H} \setminus \tilde{K}_t$ onto $\mathbb{H}$ and have thermodynamic normalization. Simple differentiation shows that

$$\partial_t \tilde{g}_t(z) = \frac{2}{\tilde{g}_t(z) - \lambda u_{t/\lambda^2}}.$$

Hence $\tilde{g}_t$ is LE drive by $\tilde{u}_t = \lambda u_{t/\lambda^2}$.

Note: If K_t is a straight interval from the origin (parametrized by capacity), then $\tilde{K}_t = K_t$ and hence $\tilde{u}_t = u_t$. In other words the driving function should be invariant with respect the Brownian scaling. The only *deterministic* functions with this property are the multiples of $\sqrt{t}$. Essentially the same argument gives that the hulls corresponding to $u_t = c\sqrt{t}$ must be scaling-invariant, and hence they must be straight intervals.

Lemma 3.10 *Let K_t be generated by the chordal Loewner evolution driven by u_t. Suppose that $u_t \in [a,b]$ for all t, then $K_t \subset [a,b] \times [0, 2\sqrt{t}]$.*

Proof. The main idea is quite simple: under the Loewner evolution all points go down until they hit the real line. The fastest decay happens when the driving measure is concentrated right below the point. Which means that the fastest decay correspond to the vertical slit.

Let $y_t = \operatorname{Im} z_t = \operatorname{Im} g_t(z)$. By considering the imaginary part of the Loewner evolution we have

$$\dot{y}_t = -\frac{2y_t}{|z_t - u_t|^2} \geq -\frac{2}{y_t}.$$

This differential inequality is easy to integrate

$$y_t^2 \geq y_0^2 - 4t.$$

This means that z_t can't hit the real line before time $y_0^2/4$. Equivalently, all points in K_t can't have the imaginary part larger that $2\sqrt{t}$. So $K_t \subset \mathbb{R} \times [0, 2\sqrt{t}]$.

By considering the real part we have

$$\dot{x}_t = \frac{2(x_t - u_t)}{|z_t - u_t|^2}.$$

If $x_0 > b$, then $\dot{x}_t > 0$ and $x_t > b$. This proves that z_t is never equal to u_t, so the points with the real part larger than b can't be in K_t. The same argument works for $x < a$. This proves that $K_t \subset [a, b] \times \mathbb{R}$. ■

4 Schramm-Loewner Evolution

4.1 Schramm's Principle

It is a common belief in statistical physics, that many lattice models have conformally invariant scaling limits. We have seen so far one example: random walk converges to Brownian motion which is conformally invariant. For most of the other models that are of interest, this result is very hard to prove. Moreover, it is not easy to state it properly, it is not clear in what sense they might converge and what is the limiting object.

Fortunately, for many models one can set up the boundary conditions in such a way that the model produces a curve connecting two marked points on the boundary. We will be working with this curve. The general setup for convergence of random curves is relatively simple. For a given lattice mesh δ this curve (sometimes it is called interface) is a piecewise linear continuous curve which connects two points a, b on the boundary of a domain Ω. This curve is random, which means that its law is a probability measure on the space of all continuous curves connecting a and b inside Ω. Let us denote this measure by $\mu_\delta(\Omega, a, b)$. Convergence means that $\mu_\delta(\Omega, a, b) \to \mu(\Omega, a, a)$ in the space of all probability measures on continuous curves (where we do not distinguish between curves that differ by change of parametrization).

Let us define these measures for all simply connected domains Ω and all pairs of boundary points a, b. We will assume that the measures $\mu(\Omega, a, b)$ have two properties: Domain Markov property and Conformal Invariance. These assumptions are motivated by physics predictions and computer simulations.

Definition 4.1 *We say that measures $\mu(\Omega, a, b)$ satisfy the* Domain Markov property *if for curve $\gamma : \mathbb{R} \to \Omega$ that has no self-crossings and such that $\gamma(0) = a$ we have*

$$\mu(\Omega, a, b)|\gamma[0, 1] = \mu(\Omega \setminus \gamma[0, 1], \gamma(1), b).$$

In other words if we condition measure on the event that the curve starts as γ that its continuation will have the same distribution as our random curve in the domain Ω with γ removed.

We say that $\mu(\Omega, a, b)$ is conformally invariant *if for every conformal map $\phi : \Omega \to \Omega'$ with $\phi(a) = a'$ and $\phi(b) = b'$ we have*

$$\phi(\mu(\Omega, a, b)) = \mu(\Omega', a', b').$$

Theorem 4.2 (Schramm's Principle) *Let us consider a family of random measures on paths $\mu(\Omega, a, b)$ that satisfy domain Markov and conformal invariance properties. Let γ be the random curve in $\mathbb{H}$ given by $\mu(\mathbb{H}, 0, \infty)$. Let us* *parametrize γ by capacity, so that* $\text{hcap}(\gamma[0,t]) = 2t$. *We can describe γ by LE with driving function u_t (this function is also random). Then u_t must be equal to cB_t where B_t is standard one-dimensional Brownian motion and c is a constant.*

The complete proof is rather technical, but its main idea is simple. If we reformulate domain Markov property and conformal invariance in terms of the driving functions, the we get that u_t mast be continuous with stationary independent increments, hence it is a Brownian motion with constant speed.

4.2 Definition and Basic Properties

The Schramm's principles motivates the definition of SLE which stands for Schramm-Loewner Evolution or Stochastic-Loewner Evolution. According to Schramm's principle SLE curves are the only possible conformally invariant scaling limits of lattice models interfaces.[6]

Definition 4.3 $\text{SLE}_\kappa = \text{SLE}(\kappa)$ *where parameter $\kappa \in [0, \infty)$ is the solution of the Loewner Evolution driven by $u_t = \sqrt{\kappa} B_t$, where B_t is Brownian motion*

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \qquad g_0(z) = z.$$

Note: occasionally by SLE we mean not only g_t but also $K_t = \mathbb{H} \setminus \mathbb{H}_t$.

One of the very important properties of SLE is that it is generated by a curve. This is a rather difficult and technical theorem, so we state it without proof.

Theorem 4.4 (Rohde-Schramm, Lawler-Schramm-Werner) *Let g_t be $SLE(\kappa)$ and $\mathbb{H}_t$ be the corresponding family of shrinking domains, then with probability one there is a curve $\gamma(t)$ such that $\mathbb{H}_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$. Alternatively, SLE has a trace with probability one.*

Proposition 4.5 *SLE curves are scale invariant and invariant with respect to the transformation $(x, y) \mapsto (-x, y)$*

Proof. As we proved for LE, the scaling of the domain corresponds to the following transformation of the driving function:

$$u_t \mapsto \tilde{u}_t = \lambda u_{t/\lambda^2}.$$

We know that the Brownian motion is invariant under this scaling, hence the distribution of the scaled SLE is the same as the distribution of the SLE. (In fact this is just a partial case of conformal invariance of SLE.)

The second property is also simple: one can easily show that this transformation of the domain corresponds to the change of sign in the driving function. On the other hand the distribution of B_t is the same as the distribution of $-B_t$. (Note that this transformation is not analytic, so we can not use conformal invariance of SLE.) ∎

Another property which follows immediately from the properties of LE is a kind of Markov property.

[6] Domain Markov property is satisfied by all relevant lattice model interfaces, it corresponds to the fact that the models have only local interactions.

Proposition 4.6 *Let g_t be* SLE(κ)*, let us fix some time t and consider $\tilde{g}_s = g_{t,t+s}$ which is defined by $g_{t+s} = \tilde{g}_s \circ g_t$. Then $\tilde{g}_s$ is also* SLE(κ)*.*

Proof. By semi-group property $\tilde{g}$ is also a solution of LE driven by $\tilde{u}_s = u_{t+s}$. By Markov property of Brownian motion it has the same distribution as Brownian motion started from u_t and hence $\tilde{g}_s$ is SLE started from u_t. ∎

4.3 Phase-transitions

SLE curves (traces) have very different behaviour depending on the parameter κ. For $\kappa \in [0, 4]$, they are simple curves that go to infinity, for $\kappa \in (4, 8)$, they form 'loops' or 'bubbles' that cover the entire plane, for $\kappa \geq 8$, they are space-filling curves.

Theorem 4.7 (SLE phase transition) *Let γ be* SLE(κ) *trace, then*

1. *If $\kappa \in [0, 4]$ then γ is a simple curve a.s.*
2. *If $\kappa \in (4, 8)$, then for all $z \in \mathbb{H}$, $z \notin \gamma$ a.s. but $\cup_{t>0} K_t = \mathbb{H}$ a.s.*
3. *If $\kappa \geq 8$, then $\gamma([0, \infty)) = \bar{\mathbb{H}}$ a.s.*

We will not give the complete proof of this theorem, instead we present the main computation showing the origin of this phase transition and prove some simple cases.

We start by investigating the boundary behaviour of SLE.

Lemma 4.8 (Boundary phase transition) *Let γ be* SLE(κ) *trace, then*

1. *If $\kappa \in [0, 4]$ then $\gamma(0, \infty) \cap \mathbb{R} = \emptyset$ a.s.*
2. *If $\kappa \in (4, 8)$, then for all $x, y > 0$, γ intersects $[x, \infty)$ a.s;*

   $$\mathbb{P}(\gamma \text{ hits } [x, x+y)) = \phi\left(\frac{y}{x+y}\right)$$

   where ϕ is an explicit explicit functions defined by (18)*.*
3. *If $\kappa \geq 8$, then $\mathbb{R} \subset \gamma([0, \infty))$ a.s.*

In order to prove this lemma we first need a technical statement about Bessel processes.

Lemma 4.9 *Let x_t be a process defined by*

$$\mathrm{d}\, x_t = \mathrm{d}\, g_t(x) - \sqrt{\kappa} \mathrm{d}\, B_t = \frac{2 \mathrm{d}\, t}{x_t} - \sqrt{\kappa} \mathrm{d}\, B_t, \quad x_0 = x.$$

For $0 < r < x < y < R < \infty$ we denote $T_x = \inf\{s : x_t = 0\}$, $\tau = \inf\{s : x_s \notin [r, R]\}$ and $T(R) = \inf\{s : x_x = R\}$. Then

1. *For $\kappa \leq 4$ we have $\mathbb{P}(T_x = \infty) = 1$.*
2. *For $\kappa > 4$ we have $\mathbb{P}(T_x = \infty) = 0$.*

3. For $4 < \kappa < 8$

$$\mathbb{P}(T_x = T_y) = \frac{\int_0^{x/y}(1-s)^{8/\kappa-2}s^{-4/\kappa}\mathrm{d}\,s}{\int_0^1(1-s)^{8/\kappa-2}s^{-4/\kappa}\mathrm{d}\,s},$$

4. $\mathbb{P}(T_x = T_y) = 0$ *for* $\kappa \geq 8$.

Proof.
Define $F(x) = \mathbb{P}^x(x_\tau = r)$ where $\mathbb{P}^x$ denotes the law of the process with $x_0 = x$. By Markov property $F(x_{t\wedge\tau})$ is a martingale (it is easy to see that $\mathbb{P}^x(x_\tau = r|\mathcal{F}_t) = F(x_{t\wedge\tau})$).

Applying Ito formula to $F(x_t)$ we get

$$\mathrm{d}\,F(x_t) = \left(\frac{2}{x_t}F'(x_t) + \frac{\kappa}{2}F''(x_t)\right)\mathrm{d}\,t - F'(x_t)\mathrm{d}\,B_t.$$

Note: a priori we don't know that F is twice differentiable. We assume it for now and will justify later.

Since we know that $F(x_t)$ is a martingale, its drift must be identically equal to 0. Together with obvious boundary conditions we have

$$\frac{2}{x}F'(x) + \frac{\kappa}{2}F''(x) = 0, \quad F(r) = 1, \ F(R) = 0.$$

This equation can be integrated and we find

$$F(x) = \frac{R^{1-4/\kappa} - x^{1-4/\kappa}}{R^{1-4/\kappa} - r^{1-4/\kappa}}, \quad \kappa \neq 4, \tag{15}$$

and for $\kappa = 4$

$$F(x) = \frac{\ln(R) - \ln(x)}{\ln(R) - \ln(r)}. \tag{16}$$

Next we are going to justify the use of Ito formula. Let us consider $M_t = F(x_{t\wedge\tau})$ where F is given by (15) or (16). The computation above implies that M_t is a local martingale, and since it is bounded, it must be a martingale. By optional stopping theorem

$$F(x) = F(x_0) = \mathbb{E}(M_\infty|\mathcal{F}_0) = \mathbb{P}(x_\tau = r).$$

This proves that $M_t = F_t$ and that $F(x)$ is twice differentiable.
Case $\kappa \leq 4$. From explicit formulas we can see that

$$\mathbb{P}(x_\tau = r) \to 0, \quad r \to 0.$$

This implies that for all $R > 0$

$$\mathbb{P}(T(R) < T_x) = 0,$$

where $T(R) = \inf\{s : x_x = R\}$ is the first time the process hits R. Since this is true for all R

$$\mathbb{P}(T_x = \infty) = 1.$$

Note that for $\kappa = 4$ if we fix r, then $F(x) \to 1$ as $R \to \infty$. This shows that although the process does not hit 0 it will come arbitrary close to it.

Case $\kappa > 4$.

Again, consider the limit $r \to 0$. We get

$$\mathbb{P}(T(R) < T_x) = 1 - \lim_{r \to 0} F(x) = \left(\frac{x}{R}\right)^{1-4/\kappa} \to 0, \ R \to \infty.$$

This implies that $\mathbb{P}(T_x = \infty) = 0$.

Finally, we consider $\phi(x, y) = \mathbb{P}(T_x = T_y)$. By monotonicity and the above results we have $T_x \le T_y < \infty$ with probability one. By the same argument as before $\phi(x_t, y_t)$ is a martingale. Assuming that ϕ is in C^2 and applying Ito formula we have

$$\mathrm{d}\,\phi(x_t, y_t) = (\frac{2}{x_t}\mathrm{d}\,t + \sqrt{\kappa}\mathrm{d}\,B_t)\phi_x + (\frac{2}{y_t}\mathrm{d}\,t + \sqrt{\kappa}\mathrm{d}\,B_t)\phi_y + \frac{\kappa}{2}(\phi_{xx} + 2\phi_{xy} + \phi_{yy})\mathrm{d}\,t.$$

Drift must be zero and hence ϕ is a solution of the following PDE

$$\frac{2}{x}\phi_x + \frac{2}{y}\phi_y + \frac{\kappa}{2}\phi_{xx} + \frac{\kappa}{2}\phi_{yy} + \kappa\phi_{xy}.$$

By scale invariance of SLE, the function ϕ depends on x/y only, namely $\phi(x, y) = h(x/y)$. Changing variable to $t = x/y \in [0, 1]$ we get

$$h''(t)\kappa t(1-t) = h'(t)2(t(\kappa - 2) - 2).$$

This equation can be rewritten as

$$(\ln h')' = \frac{2}{\kappa}\left(-\frac{2}{t} - \frac{4-\kappa}{1-t}\right).$$

In this form the equation can be integrated and the general solution is

$$h(t) = c_1 \int_0^t (1-s)^{8/\kappa - 2} s^{-4/\kappa} + c_2. \tag{17}$$

Our function must be equal to 0 at the origin and hence $c_2 = 0$. We can also note that this integral converges near 0 only for $\kappa > 4$, which is our assumption. We also need $h(1) = 1$ and hence the integral must be convergent near 1. This happens if $\kappa < 8$. In this case we can choose $c_1^{-1} = \int_0^1 (1-t)^{8/\kappa - 2} t^{-4/\kappa}$. Finally, we use the optional stopping theorem to justify out assumption that $\phi \in C^2$. This proves that if $4 < \kappa < 8$, then

$$\mathbb{P}(T_x = T_y) = \frac{\int_0^t (1-s)^{8/\kappa - 2} s^{-4/\kappa}}{\int_0^1 (1-s)^{8/\kappa - 2} s^{-4/\kappa}},$$

where $t = x/y$.

For $\kappa \ge 8$ the only bounded solution with $h(0) = 0$ is $h(t) = 0$. Hence in this case $\mathbb{P}(T_x = T_y) = 0$. ∎

Note: the integral in (17) is an incomplete beta-function and the constant c_1 could be expressed in terms of gamma function

$$c_1^{-1} = \frac{\Gamma(\frac{8}{\kappa} - 1)\Gamma(\frac{\kappa - 4}{\kappa})}{\Gamma(\frac{4}{\kappa})}$$

Note: up to some irrelevant constant factor the incomplete beta-function can be written in terms of special functions

$$c_1 \frac{\kappa}{\kappa-4} t^{1-4/\kappa} {}_2F_1\left(2-\frac{8}{\kappa}, \frac{\kappa-4}{\kappa}, 1+\frac{\kappa-4}{\kappa}, t\right) + c_2,$$

where ${}_2F_1(a,b,c,z)$ is a hypergeometric function.

Definition 4.10 *Hypergeometric function ${}_2F_1(a,b,c,z)$ is defined as*

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is a Pochhammer symbol.

Alternatively ${}_2F_1$ could be defined as the solution of the hypergeometric equation

$$z(1-z)f''(z) + (c-(a+b+1)z)f'(z) - abf(z) = 0$$

which is regular at $z = 0$ and $f(0) = 1$.

Note: most of the equations in this course are of this type or very similar, hence many quantities of interest will be expressed in therms of hypergeometric functions. These are special functions that can't be expressed in terms of simpler functions (except in a few special cases), but they are well studied and many identities and asymptotic formulas are known. Site `functions.wolfram.com/HypergeometricFunctions/Hypergeometric2F1` contains nearly 112 thousands formulas for these functions. This is by far more than for any other class of functions.

Proof of Lemma 4.8. If $\gamma([0,t]) \cap [x,\infty) = \emptyset$ then, by compactness, x is not in the closure of K_t, hence $T_x > t$. On the other hand, if $\gamma(s) \in [x,\infty)$ for some $s \in [0,t]$ then $\gamma(s) \in \bar{K}_t$, hence $T_x \le T_{\gamma(s)} = s \le t$. Together, this proves that

$$\begin{aligned} \{\gamma([0,t]) \cap [x,\infty) \neq \emptyset\} =& \{T_x \le t\}, \\ \{\gamma([0,\infty]) \cap [x,y) \neq \emptyset\} =& \{T_x < T_y\}. \end{aligned}$$

If $\kappa \le 4$, then

$$\mathbb{P}(\gamma \text{ hits } \mathbb{R} \setminus \{0\}) = \lim_{n\to\infty} \mathbb{P}(\gamma \text{ hits } \mathbb{R} \setminus [-1/n, 1/n]) = 0,$$

Where the last identity follows from the fact that for every x almost surely $T_x = \infty$.

Next, consider the case $\kappa \ge 8$. Applying the same logic, for every rational x and y with probability one, there is t such that $\gamma(t) \in [x, x+y)$ (follows from $T_x < T_{x+y}$ a.s.). Since $\mathbb{Q}$ is countable, the same is true simultaneously for all rational x and y. By continuity of γ, this is true for all x and y. This implies that $[0,\infty) \subset \gamma([0,\infty))$. By symmetry $\mathbb{R} \subset \gamma([0,\infty))$.

Finally, let us consider the case $4 < \kappa < 8$. As before, using Lemma 4.9 we have

$$\begin{aligned} \mathbb{P}(\gamma \text{ hits } [x,x+y)) =& \mathbb{P}(T_x < T_{x+y}) = 1 - \frac{\int_0^{x/(x+y)} (1-s)^{8/\kappa-2} s^{-4/\kappa} \mathrm{d}\, s}{\int_0^1 (1-s)^{8/\kappa-2} s^{-4/\kappa} \mathrm{d}\, s} \\ =& \frac{\int_{1-y/(x+y)}^1 (1-s)^{8/\kappa-2} s^{-4/\kappa} \mathrm{d}\, s}{\int_0^1 (1-s)^{8/\kappa-2} s^{-4/\kappa} \mathrm{d}\, s} =: \phi\left(\frac{y}{x+y}\right). \end{aligned} \tag{18}$$

■

Next. we give some ideas about the proof of Theorem 4.7. This is just a sketch and does not cover all the cases. Complete proofs can be found in the original papers.

Sketch of proof of Theorem 4.7. **Case $\kappa \le 4$.** We know that in this case γ does not hit $\mathbb{R} \setminus \{0\}$ a.s. This also implies that $g_r(\gamma((r, \infty))$, which has the law of SLE trace, also a.s. does not intersect $\mathbb{R}$ for all rational r. Now, let us assume that γ is not simple, that is there $t < s$ such that $\gamma(t) = \gamma(s)$. Then there is a rational $r \in (s, t)$ such that $g_r(\gamma((r, \infty))$ hits the real line.

Case $\kappa < 4$. In this case the corresponding Bessel process is transient, in particular, $\inf g_t(1) - \sqrt{\kappa} B_t > 0$ a.s. It is not hard to show using complex analysis that this implies that $\inf |\gamma(t) - 1| > 0$ a.s. From this, one can get that γ eventually will leave a certain neighbourhood of the origin. By zero-one law and scaling of SLE this implies that γ eventually leaves any neighbourhood of the origin. Hence $\gamma \to \infty$.

Case $4 < \kappa < 8$. In this case we consider $\operatorname{dist}(0, \mathbb{H}_t)$ (recall that $\mathbb{H}_t$ is the unbounded component of $\mathbb{H} \setminus \gamma([0, t])$). Since $\gamma([0, 1])$ will intersect both $(-\infty, 0)$ and $(0, \infty)$, there is a neighbourhood of 0 contained in K_1. Using scaling we have $\mathbb{P}(\operatorname{dist}(0, \mathbb{H}_t) > 0, \forall t) = 1$. Using scaling again,

$$\mathbb{P}(\operatorname{dist}(0, \mathbb{H}_t) \le r) = \mathbb{P}(\operatorname{dist}(0, \mathbb{H}_1) \le r/\sqrt{t}) \to 0.$$

This implies that $\operatorname{dist}(0, H_t) \to \infty$ a.s., in particular, $\gamma(t) \to \infty$ a.s.

Finally, we already know that for $4 < \kappa < 8$ the trace 'swallows' the entire real line, but does not hit every point on it. For $\kappa \ge 8$ it hits every real point. It is possible to show that the same is true for all points in $\mathbb{H}$ as well, but it is a more complicated proof and we omit it here. ■

4.4 Locality

By Schramm's principle all conformally invariant scaling limits of lattice models interfaces can be described by SLE(κ) for some κ. To identify the value of κ we have to compute some parameter or property of the lattice model and find which SLE has this property. Surprisingly, in some cases this not only identifies κ but also proves the convergence.

For percolation this characteristic property is called *locality.* Let Ω_i, a_i, b_i, $i = 1, 2$ be two domains with marked point. Let us assume that $a = a_1 = a_2$ and that a is on the boundary of $\Omega = \Omega_1 \cap \Omega_2$, moreover domains Ω_i have common boundary near point a. In other words, our domains Ω_i look the same in some neighbourhood of a. From the definition of the percolation model one can easily see that the law of exploration process in both domains is the same up to the first time it leaves Ω or reaches b_1 or b_2. Sometimes people say that percolation exploration process "can't see the boundary or the domain or the target point until it hit the boundary".

In this section we are going to show that the only SLE curve with this property is SLE(6).

We will start by considering the local change of coordinates as on the Figure 5. For $x \in \mathbb{R}$ we say that an open subset $\mathcal{N}$ of $\mathbb{H}$ is an $\mathbb{H}$-neighbourhood of x if $B(x, \epsilon) \cap \mathbb{H} \subset \mathcal{N}$ for some $\epsilon > 0$. We say that a univalent function $\phi : \mathcal{N} \to \mathbb{H}$

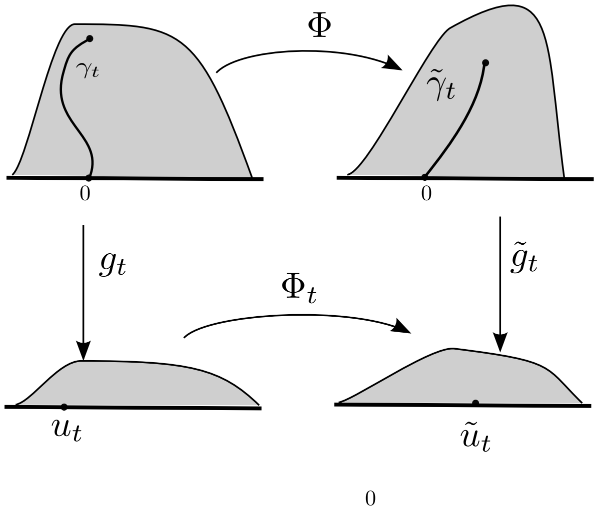


Figure 5: Local change of coordinates

is locally real at x_0 if for some $\epsilon > 0$,

$$\phi(z) = a_0 + a_1(z - x_0) + a_2(z - x_0)^2 + \dots$$

where $a_i \in \mathbb{R}$.

Let K_t be a growing family of $\mathbb{H}$-hulls given by a Loewner evolution driven by u_t. Let $x_0 = u_0$, $\mathcal{N}$ be an $\mathbb{H}$-neighbourhood of x_0 and Φ be a locally real conformal transformation of $\mathcal{N}$ into $\mathbb{H}$. We define $T = \sup\{t > 0, K_t \subset \mathcal{N}\}$. In this setup $\tilde{K}_t = \Phi(K_t)$ is a growing family of hulls for $0 \le t \le T$. Let $\tilde{g}_t$ be the corresponding Loewner evolution and $\tilde{u}_t$ be its driving function. By $\mathcal{N}_t$ we denote $g_t(\mathcal{N})$ and $\Phi_t = \tilde{g}_t \circ \Phi \circ g_t^{-1}$.

First we will need a simple statement about the change of capacity. Since under scaling the capacity changes by the square of the scale, we have

$$\partial_t \tilde{g}_t(z) = \frac{2(\Phi_t'(u_t))^2}{\tilde{g}_t(z) - \tilde{u}_t}. \tag{19}$$

Filling in all the details is a part of your home assignment.

Theorem 4.11 *Under the above assumptions we have*

$$\dot{\Phi}_t(z) = 2\left(\Phi_t'(u_t)\frac{\Phi_t'(u_t)}{\Phi_t(z) - \Phi_t(u_t)} - \Phi_t'(z)\frac{1}{z - u_t}\right)$$

and

$$\dot{\Phi}_t(u_t) = \lim_{z \to u_t} \dot{\Phi}_t(z) = -3\Phi_t''(u_2). \tag{20}$$

Proof. Let $f_t = g_t^{-1}$ be the inverse Loewner evolution, which satisfies the inverse Loewner equation. We write $\Phi_t = \tilde{g}_t \circ \Phi \circ f_t$. To obtain the first formula we differentiate the formula for Φ_t with respect to t, use the chain rule and Loewner equations for $\tilde{g}_t$ and f_t.

To obtain the second formula we have to expand both $\Phi_t(z)$ and $\Phi_t'(z)$ in Taylor series around u_t. ∎

Theorem 4.12 (Locality of SLE(6)) *Now let us assume that the curve γ is SLE(κ) (trace) and that $u_t = \sqrt{\kappa} B_t$. Then the curve $\tilde{\gamma}$ is a time-changed SLE curve if and only if $\kappa = 6$. In this case the law of $\tilde{\gamma}$ is the same as the law of (time-changed) SLE(6).*

Proof. We want to find the law of $\tilde{u}_t = \Phi_t(u_t)$. By Ito's formula

$$d\tilde{u}_t = \dot{\Phi}_t(u_t)dt + \Phi_t'(u_t)\sqrt{\kappa}dB_t + \frac{1}{2}\Phi_t''(u_t)\kappa dt$$

using (20) we can rewrite it as

$$\left(\left(\frac{\kappa}{2} - 3\right)\Phi_t''(u_t)\right) dt + \sqrt{\kappa}\Phi_t'(u_t)dB_t. \tag{21}$$

Note that $\tilde{u}_t$ is a local martingale if and only if $\kappa = 6$. In this case we have

$$d\tilde{u}_t = \sqrt{\kappa}\Phi_t'(u_t)dB_t.$$

Next we want to re-parametrize $\tilde{\gamma}_t$ so that it would have the standard capacity parametrization. The idea is very simple, we know that $\partial_t \mathrm{hcap}(\tilde{\gamma}([0,t])) = 2\Phi_t'(u_t)^2$ and we want to introduce new time $r(t)$ such that $\bar{\gamma}(t) = \tilde{\gamma}(r(t))$ would have capacity parametrization:

$$2t = \mathrm{hcap}\bar{\gamma}([0,t]) = \mathrm{hcap}\tilde{\gamma}([0,r(t)]) = \int_0^{r(t)} 2\Phi_s'(u_s)^2 ds.$$

This means that the new time should be given by

$$t = \int_0^{r(t)} \Phi_s'(u_s)^2 ds.$$

Note that this is a *random* change of time. Let us introduce $\bar{u}_t = \tilde{u}_{r(t)}$. Plugging this time change into (21) we have

$$d\bar{u}_t = \frac{\kappa - 6}{2}\frac{\Phi_{r(t)}''(u_{r(t)})}{\Phi_{r(t)}'(u_{r(t)})^2}dt + \sqrt{\kappa}d\tilde{B}_t,$$

where

$$\tilde{B}_t = \int_0^{r(t)} \Phi_t'(u_s)dB_S.$$

We claim that $\tilde{B}_t$ is a standard Brownian motion. Indeed

$$< \tilde{B}_t, \tilde{B}_t >= \int_0^{r(t)} \Phi_s'(u_s)^2 ds = t$$

and by Levy theorem $\tilde{B}_t$ is a Brownian motion.

We can easily see that for $\kappa = 6$ we have

$$d\bar{u}_t = \sqrt{\kappa}d\tilde{B}_t.$$

This proves that the curve $\tilde{\gamma}$ is a reparametrized SLE if and only if $\kappa = 6$. ∎

Corollary 4.13 *Let Ω be a simply connected domain with three marked points on the boundary a, b, c. Then the laws of SLE(6) curve (up to monotone time change) in Ω from a to b and from a to c are the same up to the first time the curve separates b and c (i.e. up to the fist time it hits the boundary arc (b, c)). This statement is also known as the locality of SLE(6).*

Proof. By the built-in conformal invariance of SLE it is sufficient to prove this result in the case $\Omega = \mathbb{H}$, $a = 0$, $b = 1$, and $c = \infty$. Let us consider the map $\phi(z) = z/(1+z)$. This is a conformal automorphism of $\mathbb{H}$ with $\phi(0) = 0$ and

By the definition, SLE $\tilde{\gamma}$ from 0 to 1 is the image of the standard chordal SLE γ under the map $z \mapsto z/(z+1)$ (note that all non-standard SLE curves are defined up to a time change). Let $T = \inf\{t > 0, \tilde{\gamma}_t \in [1, \infty)\}$. Note that this is the same as T_1 and we know that it is finite with probability one (since $6 > 4$), moreover we know that $\tilde{\gamma}(T) > 1$ (since $6 < 8$).

For every $t < T$, the curve $\gamma([0, t])$ is in some $\mathbb{H}$-neighbourhood of 0 and hence its image is a time-changed chordal SLE(6) from 0 to infinity. This proves that up to time T the chordal SLE curves from 0 towards 1 and ∞ are the same (up to reparametrization). For $t > T$ this is not true any more, since neighbourhood would have to include $\phi_{-1}(1) = \infty$.

■

Note that after time T SLE curves aiming at 1 and ∞ can not be the same, one should turn to the right and one should turn to the left. At the moment T, the curve separates two target points, so SLE has to choose its target. But before this time SLE(6) can not see the target point.

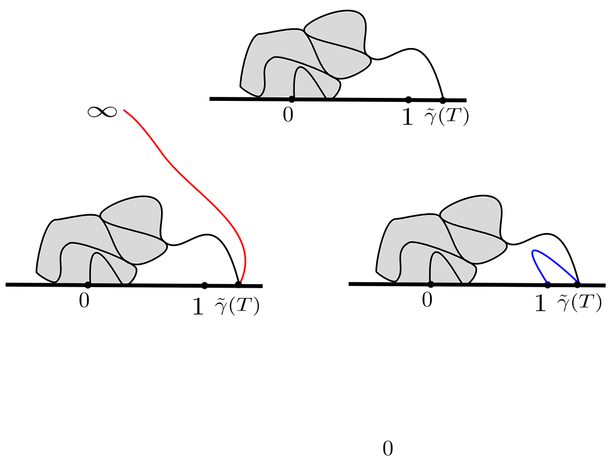


Figure 6: Top figure shows the schematic picture of SLE(6) curve up to the time T. This part is the same for SLE aiming at infinity and at 1. Bottom left shows SLE towards infinity, and bottom right shows SLE towards 1.

4.5 Restriction

Another property which is easy to formulate is the *restriction* property. Let γ be a simple random curve from 0 to ∞ and let A be an $\mathbb{H}$-hull separated from 0. By V_A we denote the event that $\gamma([0, \infty)) \cap A = \emptyset$. We say that γ has restriction property if the law of γ conditioned on the event V_A is the same as the law of γ in $\mathbb{H} \setminus A$.

It is conjectured that the scaling limit of self-avoiding random walk satisfies this property. Indeed, the self-avoiding random walk is the uniform measure on all simple trajectories. If we condition that the walk avoids some set then we get the uniform measure on all simple trajectories avoiding this set. (Rigorous justification of this argument is highly non-trivial.)

In this section we will show that the only SLE curve with restriction property is SLE(8/3). This supports the conjecture that SLE(8/3) is the scaling limit of self-avoiding random walk.

A simple curve γ considered modulo time change can be uniquely determined by specifying all $\mathbb{H}$ hulls that do not intersect $\gamma[0,\infty)$. For a random curve it suffices to give $\mathbb{P}(\gamma \cap A = \emptyset)$ for all hulls A. By symmetry of SLE, it is determined by $\mathbb{P}(\gamma \cap A = \emptyset)$ where A are *positive* hulls, i.e. the hulls such that $\bar{A} \cap \mathbb{R} \subset \mathbb{R}^+$. We denote the collection of all such hulls by Q^+.

Lemma 4.14 *Let $\kappa \leq 4$ and γ be SLE(κ) curve. For a positive hull A we denote by Φ_A the conformal map $\mathbb{H} \setminus A \to \mathbb{H}$ such that $\Phi(0) = 0$, at infinity $\Phi(z) = z + \dots$. If there is $\alpha > 0$ such that $\mathbb{P}(V_A) = \Phi'_A(0)^\alpha$ for all $A \in Q^+$, then SLE(κ) satisfies the restriction property.*

Proof. Let A and A_1 be two hulls, then

$$\mathbb{P}\left(\Phi_A(\gamma[0,\infty)) \cap A_1 = \emptyset, \gamma[0,\infty) \cap A = \emptyset\right) = \mathbb{P}\left(V_{A \cup \Phi_A^{-1}(A_1)}\right).$$

On the other hand it is easy to see that $\Phi_{A \cup \Phi_A^{-1}(A_1)} = \Phi_{A_1} \circ \Phi_A$. By the assumption of the lemma

$$\mathbb{P}\left(V_{A \cup \Phi_A^{-1}(A_1)}\right) = \left(\Phi'_{A \cup \Phi_A^{-1}(A_1)}(0)\right)^\alpha = \left(\Phi'_{A_1}(0)\right)^\alpha \left(\Phi'_A(0)\right)^\alpha.$$

We denote by $\bar{\gamma}$ the image of γ under Φ_A. We have

$$\begin{aligned} \mathbb{P}\left(\bar{\gamma} \cap A_1 = \emptyset | \gamma \cap A = \emptyset\right) &= \frac{\mathbb{P}\left(\bar{\gamma} \cap A_1 = \emptyset, \gamma \cap A = \emptyset\right)}{\mathbb{P}\left(\gamma \cap A = \emptyset\right)} \\ &= \frac{\left(\Phi'_{A_1}(0)\right)^\alpha \left(\Phi'_A(0)\right)^\alpha}{\left(\Phi'_A(0)\right)^\alpha} = \mathbb{P}(\gamma \cap A_1 = \emptyset). \end{aligned}$$

This, together with a remark before the lemma, proves that the conditional distribution of $\bar{\gamma}$ is the same as the distribution of γ, i.e. it is SLE(κ). On the other hand, by conformal invariance, pre-image of this distribution under Φ_A is SLE in $\mathbb{H} \setminus A$, hence γ conditioned not to hit A has the same distribution as SLE in $\mathbb{H} \setminus A$. This is exactly the restriction property.

■

Let A be a positive hull and let us consider the local change of variables given by Φ_A, see Figure 7.

Lemma 4.15 *Under the above assumptions*

$$\dot{\Phi}'_t(u_t) = \frac{\Phi''_t(u_t)^2}{2\Phi'_t(u_t)} - \frac{4\Phi'''_t(u_t)}{3}. \tag{22}$$

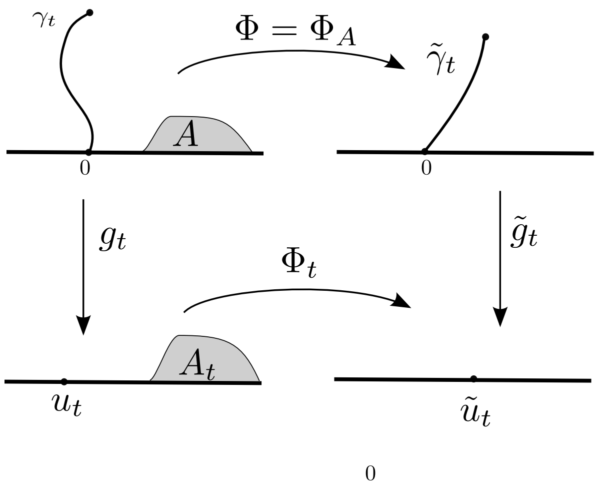


Figure 7: Removing a positive hull

The proof of this lemma is a part of your home assignment.

Theorem 4.16 *SLE(8/3) satisfies restriction property.*

Proof. Let A be a positive hull and let us define $M_t = (\Phi_t'(u_t))^\alpha \mathbf{1}_{t<T_A}$, where $T_A = \inf\{t > 0, \gamma[0,t] \cap A \neq \emptyset\}$. Let us assume that $t < T_A$, then by Ito formula and (22) we have

$$\frac{dM_t}{\alpha M_t} = \left(\frac{(\alpha-1)\kappa+1}{2}\frac{\Phi_t''(u_t)^2}{\Phi_t'(u_t)^2} + \left(\frac{\kappa}{2}-\frac{4}{3}\right)\frac{\Phi_t'''(u_t)}{\Phi_t'(u_t)}\right)dt + \frac{\Phi_t''(u_t)}{\Phi_t'(u_t)}\sqrt{\kappa}dB_t.$$

If $\kappa = 8/3$ and $\alpha = 5/8$ then M_t is a (local) martingale.

We will use without proof two facts

1. $M_t \leq 1$ and hence M_t is a bounded martingale. This is a deterministic result about boundary derivative of conformal maps.
2. M_t is continuous, i.e. if $T_A < \infty$ then $M_t \to 0$ as $t \to T_A$. This implies that $M_{t\wedge T_A} \to M_\infty$. We claim that $M_\infty = \mathbf{1}_{V_A}$.

These two facts together with optional stopping theorem give us

$$\mathbb{P}(V_A) = \mathbb{E}M_\infty = \Phi_0'(u_0)^{5/8}.$$

Lemma 4.14 implies that SLE(8/3) satisfies restriction property. ■

Finally, let us discuss the nature of these two facts. The argument below is not a complete proof, in particular, it uses some facts about Brownian excursions that are not proved in this course.

Informally, a Brownian excursion in $\mathbb{H}$ is the Brownian motion conditioned to stay in $\mathbb{H}$. Moreover. the excursion can be started at the boundary. This can be rigorously defined by starting a Brownian motion close to the boundary, stopping when its imaginary part reaches some very high level and conditioning that this happens before BM hits the real line. After that one passes to the limit as the high level goes to infinity (this gives an excursion started from inside) and then passing to the limit as the starting point goes to the boundary. It is also possible to show that the excursion can be written as $E_t = X_t + iY_t$ where X_t is a real Brownian motion and Y_t is an independent 3-dimensional Bessel process. It is easy to show that the BM started at z reaches the line $\mathrm{Im} = R$ before reaching the real line is $\mathrm{Im}\, z/R$. Using this, the conformal invariance of BM and passing to the limit as $R \to \infty$ one gets

$$\mathbb{P}^z(E_t \cap A = \emptyset) = \frac{\mathrm{Im}\, \Phi_A z}{\mathrm{Im}\, z}.$$

(We also use that $\Phi_A(\{\mathrm{Im} = R\}) \approx \{\mathrm{Im} = R\}$ for large R.

Passing to the limit $z \to x \in \mathbb{R} \setminus A$ we get

$$\mathbb{P}^x(E_t \cap A = \emptyset) = \Phi_A'(x).$$

In particular, this implies that $\Phi_A'(0) \leq 1$. Similarly, $\Phi_t'(u_t) \leq 1$.

For the second fact we use the conformal invariance of the Brownian excursion. We know that $\Phi_t'(u_t) = \mathbb{P}^{u_t}(E_s \cap A_t = \emptyset)$. Let us assume that $t < T_A$. By conformal invariance this is the same as $\mathbb{P}^{\gamma(t)}(\hat{E}_s \cap A = \emptyset)$ where $\hat{E}$ is the Brownian excursion in $\mathbb{H} \setminus \gamma([0,t])$ started from $\gamma(t)$. If $T_A < \infty$ and t is just below T_A, then we start a Brownian excursion very close to A, hence (essentially by the Beurling estimate) it will hit A with probability almost one. So in this case $M_t \to 1$ as $t \to T_A$. If $T_A = \infty$, then, for sufficiently large t, $\gamma(t)$ is very far away from A, hence the probability that a Brownian excursion from $\gamma(t)$ will hit A is almost 0. So in this case $M_t \to 0$ as $t \to T_A = \infty$. This shows that $M_\infty = \mathbf{1}_{V_A}$.

4.6 Standard SLE techniques

The goal of this section is show how some of the standard SLE techniques work. We will illustrate them by computing several SLE observables. These particular observables could be useful for various purposes but at the moment we are mostly interested in *techniques* that are frequently applicable.

4.6.1 Schramm's formula

Let us assume that $\kappa < 8$ and $z = x + iy \in \mathbb{H}$ is a fixed point. Since γ a.s. is a non-self-crossing point which does not go through z, with probability one we can say that γ passes to the left of z or to the right of z. The main result of this section is the following theorem due to Schramm.

Theorem 4.17 (Schramm's formula) *In the setting described above*

$$\mathbb{P}[\gamma \text{ passes to the left of } z] = \frac{1}{2} + \frac{x}{y} \frac{\Gamma\left(\frac{4}{\kappa}\right)}{\sqrt{\pi}\Gamma\left(\frac{8-\kappa}{2\kappa}\right)} \, {}_2F_1\left(\frac{1}{2}, \frac{4}{\kappa}, \frac{3}{2}, -\frac{x^2}{y^2}\right).$$

For some values of κ the hypergeometric function simplifies. In particular, the probability that γ passes to the left of z is

$$1 + \frac{xy}{\pi|z|^2} - \frac{\arg z}{\pi}, \quad \kappa = 2$$

$$\frac{1}{2} + \frac{x}{2|z|}, \quad \kappa = 8/3$$

$$1 - \frac{\arg z}{\pi}, \quad \kappa = 4.$$

Finally, we note that, although the theorem makes no sense for $\kappa = 8$, but the hypergeometric function does make sense and the right hand side simplifies to $1/2$ for every z.

Note: This observable with $\kappa = 6$ was used by Schramm in order to prove a certain result about percolation clusters. Detailes can be found in [9]

To prove this theorem we first need a technical lemma which allows to formulate our event in more tractable terms.

Lemma 4.18 *Let g_t be the SLE maps and define $x_t = \operatorname{Re} g_t(z) - \sqrt{\kappa} B_t$, $y_t = \operatorname{Im} g_t(z)$. Almost surely γ passes to the left of z if and only if $x_t/y_t \to \infty$ as $t \to T(z)$ and it passes to the right if and only if $x_t/y_t \to -\infty$ as $t \to T(z)$.*

Proof. First, let us consider the simpler case $\kappa \leq 4$. In this case γ is a simple transient curve and $T(z) = \infty$ a.s. Consider 2-dimensional Brownian motion W_s started from z and stopped when it hits γ or $\mathbb{R}$. Obviously, z is to the right of γ if and only if W_s hits the right side of γ or $[0, \infty)$. Let us apply g_t for some t. By conformal invariance, the image of the Brownian motion is also the Brownian motion started from $g_t(z)$. The probability that z is to the right of γ is equal to the provability that the new one hits the right side of $g_t(\gamma([t, \infty))$ or $[u_t, \infty)$ (as usual $u_t = \sqrt{\kappa} B_t$. This is bounded below by the probability that a Brownian motion in $\mathbb{H}$ started from $g_t(z)$ exits $\mathbb{H}$ through $[u_t, \infty)$. This probability can be computed explicitly:

$$\begin{aligned}\mathbb{P}^{g_t(z)}(\text{BM exits through } [u_t, \infty)) = \mathbb{P}^{z_t}(&\text{BM exits through } [0, \infty) = \\ &\frac{1}{\pi} \int_0^\infty \frac{y_t}{y_t^2 + (x_t - s)^2} \mathrm{d}\, s = \\ &\frac{1}{\pi} \left(\frac{\pi}{2} - \arctan\left(-\frac{x_t}{y_t} \right) \right).\end{aligned}$$

This goes to 1 if and only if $x_t/y_t \to \infty$ as $t \to \infty$.

Next, consider the case $\kappa \in (4, 8)$. In this case $T(z)$ is a.s. finite and z is in a bounded component of $\mathbb{H} \setminus \gamma([0, T(z)])$ but is in the unbounded component of $\mathbb{H} \setminus \gamma([0, t])$ for every $t < T(z)$. In other words, γ completes a loop around z at time $T(z)$. We say that γ is to the left of z if this is a clockwise loop and to the right if is is a counter-clockwise loop. As $t \to T(z)$, the probability that W_s will hit the right (respectively left) side of $\gamma([0, t])$ goes to 1. The same computation as above implies that in this case x_t/y_t will go to ∞ (respectively to $-\infty$). ■

Proof of Schramm's formula. By Loewner evolution

$$\begin{aligned}\mathrm{d}\, x_t &= \frac{2x_t}{x_t^2 + y_t^2} \mathrm{d}\, t - \sqrt{\kappa} B_t \\ dy_t &= -\frac{2y_t}{x_t^2 + y_t^2} \mathrm{d}\, t.\end{aligned}$$

Define $w_t = x_t/y_t$. By Ito's formula

$$\mathrm{d}\, w_t = -\frac{\sqrt{\kappa}}{y_t} \mathrm{d}\, B_t + \frac{4w_t}{x_t^2 + y_t^2} \mathrm{d}\, t.$$

We perform a time change

$$s(t) = \int_0^t \frac{\mathrm{d}\, t}{y_t^2},$$

equivalently, $\mathrm{d}\, s = \mathrm{d}\, t / y_t^2$. Define

$$\tilde{B}_s = \int_0^t \frac{\mathrm{d}\, B_t}{y_t}.$$

This is a Brownian motion. With the new time

$$\mathrm{d}\, w_s = -\sqrt{\kappa} \mathrm{d}\, \tilde{B}_s + \frac{4 w_s}{1 + w_s^2} \mathrm{d}\, s.$$

Since y_t is a decreasing function of t and it reaches 0 at $T(z)$ we have $s(t) < \infty$ if $t < T(z)$. We know that $w \to \pm\infty$, hence $s(t) \to \infty$ as $t \to T(z)$.

Consider $a < w_0 < b$ and define $h_{a,b}(w) = h(w)$ be the probability that the diffusion w_s started from w will hit b before a. Clearly $h(w_s)$ is a martingale.

As before, assuming that h is smooth we apply Ito's formulas to get

$$\frac{\kappa}{2} h''(w) + \frac{4w}{1 + w^2} h'(w) = 0, \quad h(a) = 0, \ h(b) = 1.$$

This equation has a unique solution

$$\tilde{h}(w) = \frac{f(w) - f(a)}{f(b) - f(a)},$$

where

$$f(w) = {}_2F_1 \left(\frac{1}{2}, \frac{4}{\kappa}, \frac{3}{2}, -w^2 \right).$$

By the optional stopping theorem this implies that $\tilde{h}(w_s)$ is the same martingale as $h_{a,b}(w_s)$. This gives a posteriori justification of differentiability of h.

From the behaviour of the hypergeometric functions at infinity we get

$$\lim_{w \to \pm\infty} f(w) = \pm \frac{\sqrt{\pi}\Gamma\left(\frac{8-\kappa}{2\kappa}\right)}{2\Gamma\left(\frac{4}{\kappa}\right)}.$$

This implies that $\lim_b \to \infty h_{a,b}(w) > 0$ for all $w > a$. Hence w_s is transient and

$$\mathbb{P}[\lim_{s \to \infty} w_s = \infty] = \frac{f(w) - f(-\infty)}{f(\infty) - f(-\infty)}$$

which is exactly the right hand side of the formula in Theorem 4.17. Recalling that a.s. w_s goes to $\pm\infty$ if and only if γ is to the left/right of z we complete the proof. ■

Note: This proof is a typical example of a 'martingale trick' which is one of the main tools in SLE computations. Essentially, by conditioning on g_t we construct a martingale which is closely related to the observable that we want to compute. Then Ito's formula tells us that this observable must satisfy a certain second order differential equation with known boundary conditions. Solving this equation we find the martingale (hence observable) and use the optional stopping theorem to justify the application of Ito's formula. Alternatively, the proof can be written in a different way, we can start with an explicit martingale and use the optional stopping to compute the observable.

4.6.2 One-arm exponent

In this section we give a computation of a certain observable which is used to compute the so called 'one-arm exponent'. In percolation theory, one-arm exponent is defined as λ such that the probability that the percolation cluster containing the origin has radius at least R is $R^{-\lambda+o(1)}$. In case of percolation $\lambda = 5/48$.

The computation is somewhat similar to the one before but there are important differences. As before, we use the martingale trick to write a certain differential equation, but this time, we are interested in the long-time behaviour of a time-dependent quantity. This adds one more variable and the resulting equation is a PDE instead of ODE. PDEs are much harder to analyse and they almost never admit explicit solutions.

Theorem 4.19 *Let $\kappa > 4$. Let g_t and γ be the radial SLE(κ) and its trace. For $r \in (0,1)$ we define T_r to be the first time $|\gamma(t)| = r$. Then there is a constant c such that*

$$c^{-1} r^{\lambda} < \mathbb{P}[\gamma([0,T_r]) \text{ contains no counter-clockwise loop around } 0] < c r^{\lambda},$$

where

$$\lambda = \frac{\kappa^2 - 14}{32\kappa}.$$

Proof. First of all, by Koebe 1/4 theorem the distance from the origin to $\gamma([0,t])$ is comparable to $\exp(-t)$. This means that it is sufficient to prove that

$$c^{-1} e^{-\lambda t} < \mathbb{P}[\gamma([0,t]) \text{ contains no counterclockwise loop around } 0] < c e^{-\lambda t}.$$

By A_θ we denote the arc $\{e^{is} : s \in [0,\theta]\}$ and by B_θ its counter-clockwise oriented complement. We define the event $E(\theta,t)$ to be the event that the concatenation of B_θ and $\gamma([0,t])$ contains no counter-clockwise loops around 0. By $h(\theta,t)$ we denote the probability of $E(\theta,t)$.

We make three claims that we are not going to prove

1. Function h is smooth.
2. For every t we have $\lim_{\theta\to 0} h(\theta,t) = 0$.
3. For every $t > 0$ we have $\partial_\theta h(2\pi,t) = 0$.

Note that here, unlike our previous computations, we will not be able to justify a posteriori differentiability of h since we will not be able to find an explicit formula for h. There are different way around this problem, but they are beyond the scope of this course.

The second condition is very natural and, in fact, quite easy to prove. It is possible to show that when θ is very small, the curve γ will separate $e^{i\theta}$ from 0 in a very short time with very high probability. When this happens, the event $E(\theta,t)$ happens.

For the last claim the idea behind the proof is this. The difference between events $E(2\pi,t)$ and $E(\theta,t)$ is the event that the curve $\gamma([0,t])$ goes around the origin and hits the small arc of $\exp(i[\theta,2\pi])$ and after that it does not form a counter-clockwise loop around the origin. The probability that the curve will hit this short arc is of order $2\pi - \theta$. Conditioned on this, there is no loop if the continuation of the curve after this time does not hit the beginning of the near 1. The probability of this event is $o(1)$. Overall, this gives $h(2\pi, t) - h(\theta, t) = o(2\pi - \theta)$.

Let $T(\theta)$ be the first time $\exp(i\theta)$ is separated from the origin. Equivalently, this is the first time $\exp(i\theta) \in K_t$, where K_t is the corresponding growing hull. For $t < T(\theta)$ we define

$$Y_t^\theta = Y_t = -i \log g_t(e^{i\theta}) - \sqrt{\kappa} B_t.$$

This is the arclength of the image under g_t of the union of A_θ and the right side of $\gamma([0,t])$. This is also the same as 2π times the harmonic measure from the origin of the union of A_θ and the right side of $\gamma([0,t])$. Note that after $T(\theta)$ the arc A_0 is not visible from the origin, so it does not contribute to the harmonic measure.

We can make one more important observation: at the moment when γ makes a counter-clockwise loop the harmonic measure of the right side of γ is 0. This means that $E(\theta, t)$ is the same event as $\inf_{s<t} Y_s^\theta > 0$.

In fact, using the last description, we can extend the definition of Y_t beyond $T(\theta)$. It is a diffusion defined on $[0, 2\pi]$ which instantaneously reflects at 0 and 2π.

Let us fix some s and assume that $t < \min\{s, T(\theta)\}$. Then by the standard argument (condition on $\mathcal{F}_t$ and applying g_t) we can see that $h(Y_t^\theta, s-t)$ is a martingale.

Recall that g_t satisfies the radial Loewner evolution

$$\partial_t g_t(z) = -g_t(z) \frac{g_t(z) + e^{i\sqrt{\kappa}B_t}}{g_t(z) - ie^{i\sqrt{\kappa}B_t}}.$$

From this, by Ito's formula we get

$$\mathrm{d}\, Y_t = \cot(Y_t/2) \mathrm{d}\, t - \sqrt{\kappa} \mathrm{d}\, B_t.$$

Again by Ito's formula

$$\begin{aligned} \mathrm{d}\, h(Y_t, s-t) = \Big(&\frac{\kappa}{2} \partial_\theta^2 h(Y_t, s-t) + \cot(Y_t/2) \partial_\theta h(Y_t, s-t) - \partial_s h(Y_t, s-t) \Big) \mathrm{d}\, t \\ &- \sqrt{\kappa} \partial_\theta h(Y_t, s-t) \mathrm{d}\, B_t. \end{aligned}$$

Since $h(Y_t^\theta, s-t)$ is a martingale, the drift must vanish for all t. Taking $t = 0$ and using that $Y_0^\theta = \theta$ we get that h satisfies the following PDE

$$\frac{\kappa}{2} \partial_\theta^2 h(\theta, s) + \cot(\theta/2) \partial_\theta h(\theta, s) - \partial_s h(\theta, s) = 0.$$

We also know that h satisfies the Dirichlet-Neumann boundary condition $h(0, s) = \partial_\theta h(2\pi, s) = 0$.

This is where this computation deviates from the computations before. We can not explicitly solve the initial value problem for this PDE. On the other hand, we only want to know the asymptotic behaviour as $s \to \infty$.

There are two different ways to deal with this problem. We will discuss both of them, but will skip some technicalities.

PDE approach. Let us consider the function

$$H(\theta, t) = \left(\sin\left(\frac{\theta}{4}\right)\right)^q e^{-\lambda t},$$

where

$$q = \frac{\kappa - 4}{\kappa}, \quad \text{and} \quad \lambda = \frac{\kappa^2 - 16}{32\kappa}.$$

The first factor in H is the main eigenfunction of an elliptic operator

$$\frac{\kappa}{2}\partial_\theta^2 + \cot(\theta/2)\partial_\theta$$

and λ is its main eigenvalue. The main idea is that for a parabolic (heat) equation we can expand the solution in series $\sum \phi_i(\theta)\exp(-\lambda t)$ where ϕ_i and λ_i are eigenfunctions and eigenvalues of the corresponding elliptic operator. It is clear that for generic initial condition the series decays like $\phi_0(\theta)\exp(-\lambda_0 t)$.

This is usually done by application of the Maximum Principle which implies that since both h and H are positive solutions of the same PDE with the same boundary condition, there is a constant $c > 0$ such that

$$c^{-1}H(\theta, t) < h(\theta, t) < cH(\theta, t).$$

In this particular case, the situation is a bit more complicated since the coefficients of PDE have singularity at $\theta = 0$. This requires some rather standard modification of the Maximum Principle. More details can be found in the original paper [5].

Diffusion approach Let us consider

$$Z_t = \left(\sin\left(\frac{Y_t^\theta}{4}\right)\right)^q e^{\lambda t}.$$

An explicit computation (the same as the one showing that $\sin^q$ is an eigenfunction) shows that Z_t is a local martingale. Moreover, it is clearly uniformly bounded on every bounded time-interval. Using the optional stopping theorem and total probability formula we obtain

$$\left(\sin\left(\frac{\theta}{4}\right)\right)^q = Z_0 = \mathbb{E}Z_t = e^{\lambda t}\mathbb{P}\left[\inf_{s<t} Y_s > 0\right]\mathbb{E}\left[\left(\sin\frac{Y_t^\theta}{4}\right)^q \mid \inf_{s<t} Y_s > 0\right]$$

Recall that $E(\theta, t)$ is the same as the event $\inf_{s<t} Y_s^\theta > 0$.

Let us consider the diffusion Y stopped when it hits 0. We claim that conditioned on the event $\inf_{s<t} Y_s^\theta > 0$ it has a uniformly bounded distribution. General theory of Markov processes implies that this distribution converges to the stationary distribution. This means that

$$\mathbb{P}\left[Y_t^\theta \in [\pi/2, 2\pi] \mid \inf_{s<t} Y_s^\theta > 0\right] > c,$$

where c is some positive universal constant. This implies

$$\mathbb{E}\left[\left(\sin\frac{Y_t^\theta}{4}\right)^q \mid \inf_{s<t} Y_s > 0\right] > c > 0,$$

where c is some other universal constant. This implies

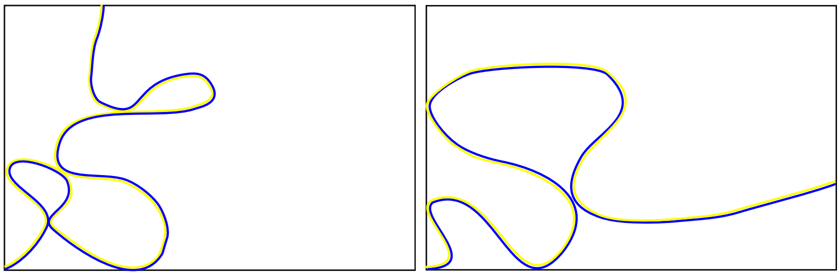
$$h(\theta, t) = \mathbb{P}\left[\inf_{s<t} Y_s > 0\right] \asymp \left(\sin\left(\frac{\theta}{4}\right)\right)^q e^{-\lambda t}.$$

■

5 Convergence to SLE

5.1 General scheme

Let us consider a rectangle with vertices $0, L, i\pi$ ans $L + i\pi$. Let us consider a curve starting from 0 and aiming at $L + i\pi$. We colour its left side yellow and right side blue. We have two options: our curve hits the right side or the top side first. In the first case there is a yellow path connecting left and right sides, in the second case there is a blue path connecting top to bottom. See Figure 8.



(a) Curve hits the top side and creates a blue vertical crossing

(b) Curve hits the right side and creates a yellow horizontal crossing

Figure 8: Two crossing types

If we think in terms of percolation (i.e. our curve is an exploration process from 0), where the bottom is coloured blue and the left yellow, then these two types of crossing give us two mutually excluding events: there is a blue cluster connecting top to bottom or there is a yellow cluster connecting left to right. The same picture is valid for many other cluster models including Ising model. Assuming that this curve is SLE(κ) from 0 to $L + i\pi$ we would like to compute the probabilities of these two events. By Riemann theorem there is a conformal map from the rectangle onto $\mathbb{H}$. By choosing three parameters we can map $0 \mapsto 0$, $i\pi \mapsto \infty$, and $L + i\pi \mapsto 1$. After that the image of L is uniquely determined and we call it $1 - u$. There is one-to-one correspondence between $u \in (0, 1)$ and $L \in (0, \infty)$. In this setup, the probability of horizontal crossing will be the same as the probability in the half-plane to hit $[1 - u, u]$ before $[1, \infty)$. It is easy to see that this event is the same as $T_{1-u} < T_1$. We have already computed this probability and it is equal to

$$\mathbb{P}(T_{1-u} < T_1) = 1 - \frac{\Gamma\left(\frac{8}{\kappa} - 1\right)\Gamma\left(\frac{\kappa-4}{\kappa}\right)}{\Gamma\left(\frac{4}{\kappa}\right)} \int_0^{1-u} (1-x)^{8/\kappa - 2} x^{-4/\kappa} dx$$

For percolation J. Cardy predicted that the crossing probability is given by:

$$F(u) = \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{4}{3}\right)} u^{1/3} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, u\right) \tag{23}$$

Locality suggest that percolation corresponds to SLE(6). It also turns out that for $\kappa = 6$ Cardy's formula is the same as SLE crossing formula. L. Carleson noticed that the function in (23) is the same as the conformal map from the half-plane onto the equilateral triangle with side-length 1. Hence in the triangle the crossing probability is given by very simple formula: Figure 9

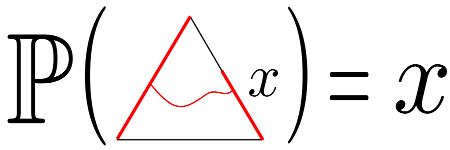


Figure 9: Crossing probability in the equilateral triangle

It is believed that many two-dimensional critical lattice models have conformally invariant scaling limits. By Schramm's principle, these limits must be SLE curves. This gives not only the description of conjectural limits, but also gives some tools to prove this convergence. The main strategy follows these steps:

1. Show that some observable of the model has a conformally invariant scaling limit. Here and later on, by observable we mean a real (or complex) valued functional of the model. Some examples are: probability that the interface passes to the left of a given point, probability that there is some kind of crossing etc. In most cases we first show that the observable is a solution (or an almost solution) to some discrete boundary value problem. Then we show that as the mesh of the lattice goes to zero, the discrete problem converges to a continuous boundary problem and the discrete solution converges to the continuous solution. If the boundary value problem is conformally invariant (many problems involving harmonic or holomorphic functions are conformally invariant), then the solution is also conformally invariant.

2. Use some a priori estimates to show that the space of discrete interfaces if not too wild. In other words we need some pre-compactness or tightness argument. This shows that there are sub-sequential limits of interfaces.

3. Convergence of an observable will show that all sub-sequential limits are the same, hence we have a limit.

We start by giving an example showing how to do the last step. This is a rather general method which works if there is an explicit formula for the scaling limit of one observable.

5.2 Convergence of an observable implies convergence of driving function

In this section we sketch how to show the convergence of the driving function of percolation exploration to the Brownian motion (with speed 6).

Let us assume that the probability that in $\mathbb{H}$ there is a crossing connecting $(-\infty, 0)$ with (x, rx) is given by some explicit function $F(r)$ (see (23)). We will show that this is indeed true later on.

Note that this probability is the same as the probability that the exploration process hits (x, rx) before (rx, ∞). Let's parametrize exploration path by capacity and write the corresponding Loewner evolution $g_t = g_{t,\delta}$. Fix t and condition the crossing probability on g_t (i.e. on the the initial segment of the exploration). By Markov property this is the same as the crossing probability in $\mathbb{H} \setminus K_t$, and by conformal invariance of the crossing probability this is the same as the crossing probability from $(-\infty, u_t)$ to $(g_t(x), g_t(rx))$ (see Figure 10). Applying the Mobius transformation and using conformal invariance this probability is given by

$$F\left(\frac{g_t(rx) - u_t}{g_t(x) - u_t}\right).$$

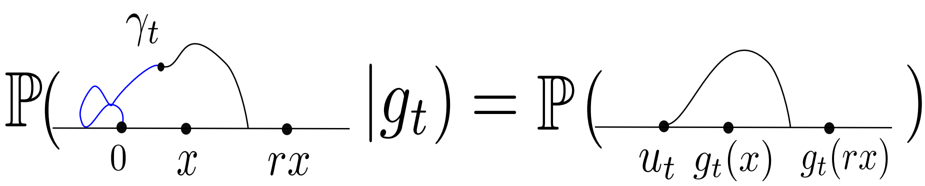


Figure 10: Conformal invariance of crossing probability

Taking expectation with respect to g_t we have

$$\mathbb{E}F\left(\frac{g_t(rx) - u_t}{g_t(x) - u_t}\right) = F(r).$$

Assuming that x is large compared to t and u_t we have

$$\begin{aligned} g_t(x) &= x + \frac{2t}{x} + O(x^{-2}), \\ g_t(rx) &= rx + \frac{2t}{rx} + O(x^{-2}). \end{aligned}$$

We can expand F using explicit formula for it and obtain

$$\begin{aligned} F\left(\frac{g_t(rx) - u_t}{g_t(x) - u_t}\right) =& F(r) + c_1 \left(\frac{r-1}{r^2}\right)^{1/3} \frac{u_t}{x} + \\ & c_2 \frac{(r+1)(r-1)^{1/3}}{r^{5/3}} (u_t^2 - 6t) \frac{1}{x^2} + O(x^{-2}) \end{aligned}$$

Taking the expectation we have $\mathbb{E}u_t = 0$ and $\mathbb{E}u_t^2 = 6t$. By Markov property the same is true for all increments. This proves that $u_t = \sqrt{6}B_t$ where B_t is a Brownian motion.

5.3 Percolation: Cardy formula

In this section we show that the percolation crossing probability indeed converges to Cardy's formula. Initially it was proved by S. Smirnov [10]. Later, his argument was simplified by V. Beffara [2]. We will show an eval simpler argument by M. Khristoforov and S. Smirnov (unpublished).

5.3.1 Discrete setup

Our first goal is to reformulate the percolation observable in terms of a different loop model. It is similar to the loop representation of percolation, but we introduce certain disorders.

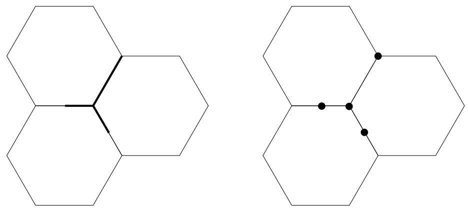


Figure 11: The set E contains 4 half-edges (left). Its boundary is made of 2 mid-edges and two vertices (right).

Let Ω be a simply connected domain made of hexagons belonging a fixed hexagonal lattice. In the percolation model we consider all possible configurations that are collections of 'open' hexagons. We define the uniform measure on the set of all percolation configurations. The percolation probability measure is denoted by $\mathbb{P}^{\text{perc}}$. It is easy to see that choosing a percolation configuration is the same as independently choosing with probability $1/2$ whether each hexagon is 'open' or 'closed'. A percolation model with given boundary condition is defined in the same way, except that we fix which hexagons on the boundary are open and configurations only involve internal hexagons.

Beyond vertices and edges, we also consider mid-edges and half-edges. For a set of half-edges E we defined its boundary ∂E as the set of all vertices and mid-edges that are adjacent to an odd number of half-edges from E (see Figure 11).

Let $U = \{u_1, \ldots u_k\}$ be a collection of distinct mid-edges or vertices. We define $W(u_1, \ldots, u_k)$ to be the collection of all sets of half-edges E in a hexagonal domain Ω such that $\partial E = U$. Such E is called a configuration with disorders at marked points u_i.

It is clear that all connected components of E are either loops or paths connecting two distinct u_i and u_j. The important part of E, denoted $IP(E)$ is the union of components of E that are not loops. In particular, this implies that W is non-empty only if k is even. From now on we always assume that this is the case.

The connectivity pattern of E is the homotopy class of $IP(E)$. The loop model is defined as the uniform measure on $W(U)$. The corresponding probability measure is denoted by $\mathbb{P}^{\text{loop}}$.

5.3.2 Loop representation

When all u_i are on the boundary, there is a natural connection bijection between percolation configurations with certain boundary condition and loop configurations.

We will consider only the case $k = 4$. We also assume that u_i are in the counter-clockwise order and we use cyclic notation, namely $u_{k+4n} = u_k$ for every n.

Let us consider a percolation configuration where we assume that the (counter-clockwise) boundary arcs $[u_1, u_2]$ and $[u_3, u_4]$ are open. This gives rise to a collection of mid-edges that separate open hexagons from closed ones. It is easy to see that this is a bijection. Since both loop and percolation measures are uniform, this is a measure preserving bijection.

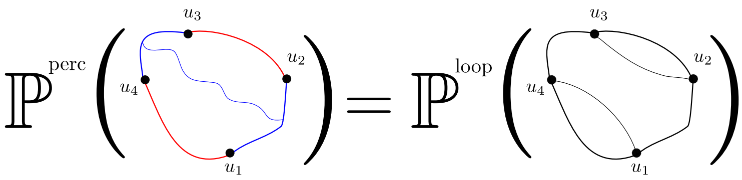


Figure 12: The probability that there is an open (blue) percolation cluster connecting two boundary arcs is the same as the probability that in the loop model the important part of E connects u_1 to u_4 and u_2 to u_3.

It is easy to see that the event that there is an open percolation cluster connecting $[u_1, u_2]$ and $[u_3, u_4]$ corresponds to the event that in the loop configuration u_1 is connected to u_4 and u_2 is connected to u_3. This proves our first result. It's graphical representation is given by Figure 12.

5.3.3 Discrete holomorphicity

Next, we assume that u_1, u_2 and u_3 are on the boundary and the fourth marked point, which we now denote by z is an interior mid-edge. There are three topological types of loop configurations: z is connected to u_1 (in this case u_2 must be connected to u_3), it is connected to u_2 or to u_3. We denote these events by $[z \leftrightsquigarrow u_i]$.

Let $\tau = \exp(2\pi i/3)$ and define the function F on interior mid-edges by

$$F(z) = \mathbb{E}^{\text{loop}} H(E) = \sum \tau^k H_k(z),$$

where

$$H(z) = \sum \tau^k \mathbb{1}_{[z \leftrightsquigarrow u_k]}$$

and

$$H_k(z) = \mathbb{P}^{\text{loop}}[z \leftrightsquigarrow u_k].$$

Our next result is a discrete version of Cauchy-Riemann equations.

Lemma 5.1 *Let z_1, z_2 and z_3 be three mid-edges adjacent to the same vertex v. We assume that they are indexed in the counter-clockwise order. Then*

$$\sum \tau^k F(z_k) = 0 \tag{24}$$

Proof. By the definition

$$\sum_{k=1,2,3} \tau^k F(z_k) = \sum_{k=1,2,3} \tau^k \sum_{j=1,2,3} \tau^j H_j(z_k) = \sum_{k,j} \tau^{k+j} \mathbb{P}[z_k \leftrightsquigarrow u_j].$$

The probability $\mathbb{P}[z_k \leftrightsquigarrow u_j]$ is proportional to the number of configurations where $z_k \leftrightsquigarrow u_j$.

We split all configurations in triplets. Let us consider a configuration where z_1 is connected to u_1. There are three types of them: the line connecting u_2 to u_3 is away from v and there is no loop passing through v (first line in Figure 13), the line connecting u_2 to u_3 is away from v and there is a loop passing through v (second line line in Figure 13) and when the line connecting

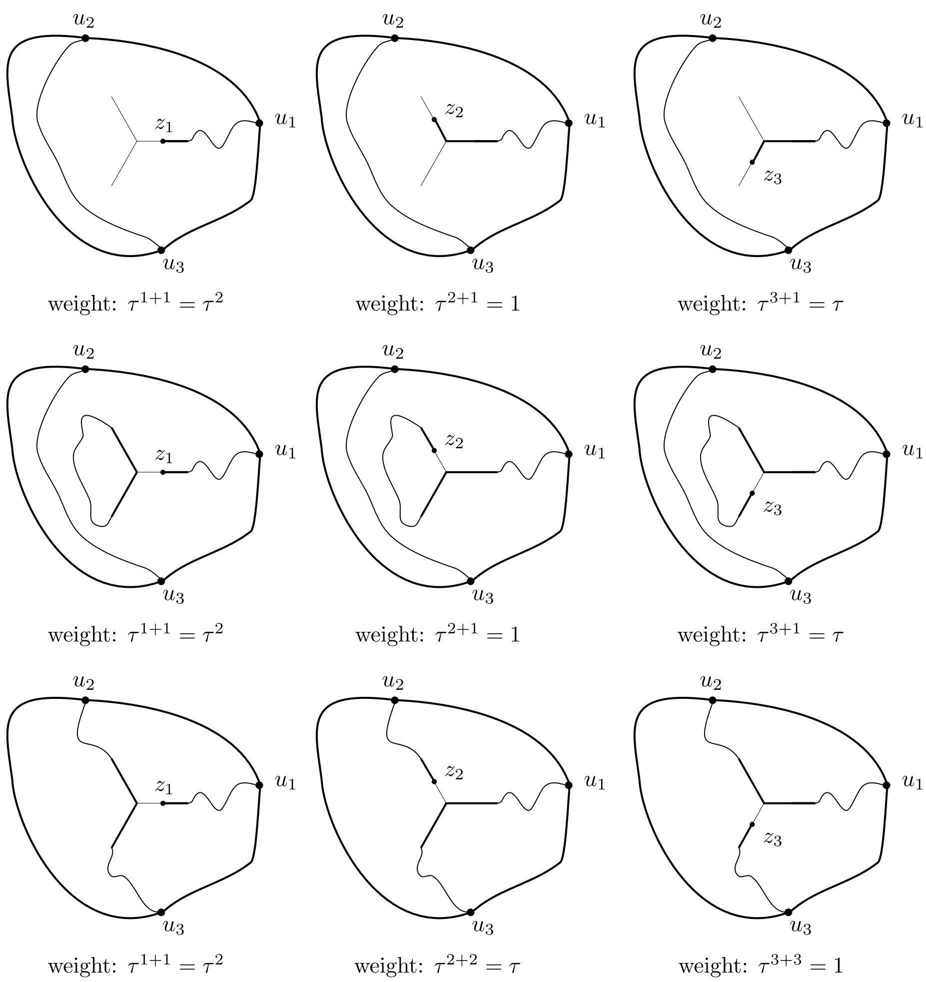


Figure 13: All possible modifications around v of configurations of $[z_1 \leftrightsquigarrow u_1]$ type.

u_2 and u_3 passes through v. For each configuration, we consider two other configurations that differ exactly by two half-edges adjacent to v and belong to one of $W(u_1, u_2, u_3, z_j)$. In Figure 13 we list all possibilities.

In each row the number of configurations in each column is the same. hence their probabilities are equal. In below each type we list its weight in the double sum above. It is easy to see that in each row the waits are 1, τ and τ^2 in some order. This means that the sum in each row is 0. All rows must be repeated for configurations of $[z_1 \leftrightsquigarrow u_2]$ and $[z_1 \leftrightsquigarrow u_3]$ types. Eventually, all configurations are split in triplets and the sum in each triplet is 0. Hence the entire sum vanishes. ■

As mentioned above, this lemma is a discrete version of the Cauchy-Riemann equations. So F is discrete holomorphic. Similar to the usual complex analysis we can show that a contour integral of a discrete holomorphic function vanishes.

Let γ be a closed contour, that is a sequence of hexagons $w_1, w_2, \ldots, w_{n+1}$ where $w_1 = w_{n+1}$ and for each j hexagons w_j and w_{j+1} share an edge which we denote e_j. The center of e_j is denoted z_j. Let F be a function defined on mid-edges. In this case we define the discrete integral

$$\int_\gamma^{\#} F(z)\mathrm{d}\,z = \sum_{j=1}^{n} F(z_j)(w_{j+1}^\circ - w_j^\circ),$$

where $w_j^circ \in \mathbb{C}$ denotes the point in the center of the hexagon w_j.

Corollary 5.2 (Discrete Cauchy theorem) *Let γ be a closed contour inside Ω and F be a function satisfying* (24)*, then*

$$\int_\gamma^{\#} F(z)\mathrm{d}\,z = 0.$$

Proof. First, we consider the simplest case. Let w_1, w_2 and w_3 be three faces in counter-clockwise order around a vertex v. Then the integral is

$$\int_\gamma^{\#} F(z)\mathrm{d}\,z = F(z_1)(-\tau) + F(z_2)(-\tau^2) + F(z_3)(-1).$$

By (24) this sum vanishes.

Any closed contour can we written as a sum of simplest triangular contours. It is easy to see from the definition that the integral along the contour is equal to the sum of integral along small contours. But all these integrals are equal to 0. ■

Now we investigate boundary behaviour of F. Let us assume that z is a mid-edge on the arc $[u_{j+1}, u_{j-1}]$ (this is the arc opposite to u_j).

First of all, by topological reasons there are no loop configurations such that $[z \leftrightsquigarrow u_j]$. This proves that $H_j(z) = 0$.

Using the bijection between the percolation and loop configurations we have

$$\begin{aligned} F(z) = &\mathbb{P}^{\text{perc}}[[u_{j+1}, z] \text{ is connected to } [u_{j-1}, u_j]]\tau^{j-1} + \\ &\mathbb{P}^{\text{perc}}[[u_j, u_{j+1}] \text{ is connected to } [z, u_{j-1}]]\tau^{j+1}. \end{aligned}$$

In particular, when $z \in [u_{j+1}, u_{j-1}]$, $F(z)$ is a convex linear combination of τ^{j-1} and τ^{j+1}, so $F([u_{j+1}, u_{j-1}]) \subset [\tau^{j-1}, \tau^{j+1}]$.

5.3.4 Convergence of F

Let Ω be a simply connected domain with four marked points A, B, C and D. Let $\delta > 0$ be small enough and Ω_δ be an approximation of Ω by a hexagonal domain with mesh δ. Let A_δ, B_δ, C_δ and D_δ be mid-edges that approximate A, B, C and D.

We define for each mid-edge we define $f_\delta(z) = F(z)$, where F is the function defined above for a loop model on Ω_δ with marked points A_δ, B_δ and C_δ. For each vertex we define $f_\delta(v)$ to be the average of a values at adjacent mid-edges. For each center of a face it is also the average of a values at adjacent mid-edges. Inside each small triangle formed by a vertex, mid-edge and a face center we define f_δ as a linear interpolation of the values at the corners of this triangle. This allows us to extend the definition of F to the entire domain in the complex plane.

Standard results from percolation theory (Russo-Seymour-Welsh estimate) implies that function f_δ are uniformly Holder continuous. By Arzela-Ascoli theorem, there is $\delta \to 0$ such that $f_{\delta_n} \to 0$ uniformly on compact subsets of Ω.

It is easy to see that the discrete integral converges to the usual contour integral. Corollary 5.2 implies that all contour integrals of f vanish. By Morera's theorem this implies that f is holomorphic.

The boundary conditions for F_δ imply that f maps the boundary of Ω onto the boundary of the triangle with vertices 1, τ and τ^2. By the standard Argument Principle argument this implies that f is the unique conformal map from Ω onto this triangle such that $f(A) = \tau$, $f(B) = \tau^2$ and $f(C) = 1$.

Since the limit of every convergent subsequence is equal to f, the entire family $f_\delta \to f$ as $\delta \to 0$.

The argument in the proceeding subsection shows that

$$\mathbb{P}^{\text{perc}}[[A_\delta, B_\delta] \text{ is connected to } [C_\delta, D_\delta]] = \frac{\tau^3 - f_\delta(D_\delta)}{\tau^3 - \tau}.$$

Passing to the limit as $\delta \to 0$ we get

$$\mathbb{P}^{\text{perc}}[[A_\delta, B_\delta] \text{ is connected to } [C_\delta, D_\delta]] = \frac{f(C) - f(D)}{f(C) - f(A)}$$

which is exactly Carleson's version of Cardy's formula.

Note, that since the limit of crossing probabilities is defined in terms of conformal maps, it is conformally invariant by definition.

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