Introduction to SLE

Example Random Walk on $\mathbb{Z}^2$ $\longrightarrow$ Brownian Motion in $\mathbb{R}^2$

$\mathbb{R}^2 = \mathbb{C}$

Sketch of the proof

Levy characterization Z_t is a cont complex martingale s.t $\langle Z_t, \bar{Z}_t \rangle = 2t$ then Z_t is a complex BM

$Z_t = X_t + i Y_t$, X_t, Y_t indep. 1-d BM

Complex Ito Z_t is a local complex mart.
f is an analytic function

the $f(Z_t) - f(Z_0) = \int_0^t f'(Z_t)\, dZ_t$

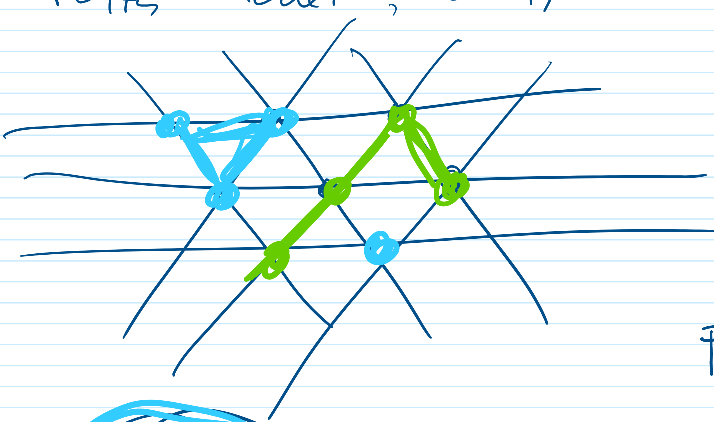
Ex $\langle f(Z_t), \overline{f(Z_t)} \rangle = \int_0^t |f'(Z_t)|^2 d\langle Z_t, \bar{Z}_t \rangle$

this is a positive increasing function.

After a time-chang $f(Z_t)$ is a time-changed complex B.M

Other models Percolation, Ising model

Potts model, $O(n)$ model

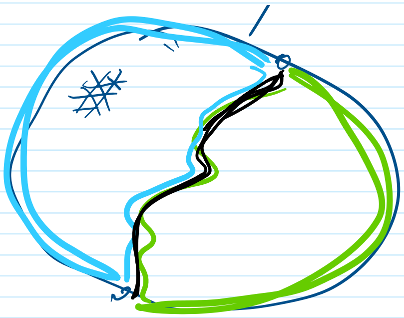


$\mathbb{P}$(vertex is open) $= p$

$\mathbb{P}$(closed) $= 1-p$

$p_c = 1/2$

Black curve is an interface



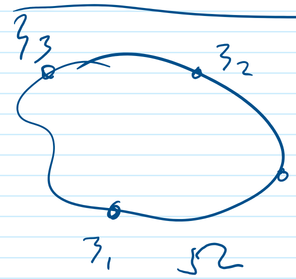
Black curve is an interface between two clusters

Complex Analysis

Riemann mapping theorem Let Ω be a proper domain in $\mathbb{C}$, $z_0 \in \Omega$ then there is a unique conformal map $\varphi: \Omega \to \mathbb{D}$ s.t

$\varphi(z_0) = 0$, $f'(z_0) > 0$ $(\arg f'(z_0) = 0)$

Other normalizations



① $\exists$ $\varphi: \Omega \to \mathbb{H}$

$\exists!$ $\varphi:$ s.t $\varphi(\zeta_1) = 0$
$\varphi(\zeta_2) = 1$
$\varphi(\zeta_3) = \infty$

② $\exists!$ $\varphi(\zeta_1) = 0$
$\varphi(\zeta_2) = \infty$
$|\varphi'(\zeta_1)| = 1$

Basic properties

$$S = \left\{ f \text{ s.t } \begin{array}{l} f \text{ is univalent in } \mathbb{D} \\ f(0) = 0, \quad f'(0) = 1 \\ f(z) = z + a_2 z^2 + \cdots \end{array} \right\}$$

$$f \longleftrightarrow f(\mathbb{D}) = \Omega$$

- Growth theorem $f \in S$ $z \in \mathbb{D}$, $|z| = r$

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}$$

equality for some $z \Rightarrow f = e^{i\theta} K(e^{-i\theta} z)$

$K(z) = z + 2z^2 + 3z^3 + \cdots$ Koebe function

- Distortion theorem

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}$$

$(1+r)^3 \quad \ldots \quad (1-r)^3$

- **Schwarz lemma** f is analytic in $\mathbb{D}$

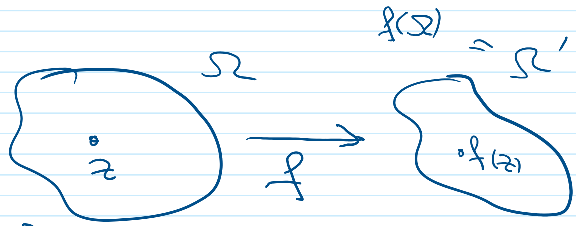
$f(0)=0 \,, \quad |f| \le 1$

then $|f(z)| \le |z| \,, \quad |f'(0)| \le 1$

Equality for some $z \neq 0$, or $|f'(0)| = 1$

then $f(z) = e^{i\theta} z$

- Koebe distortion



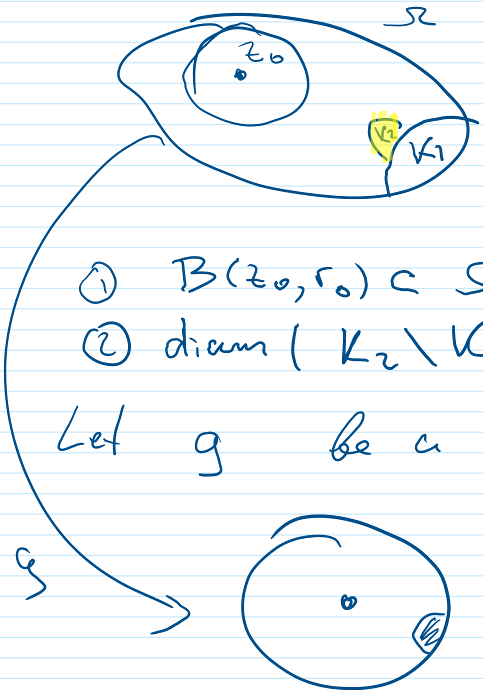
$$\frac{1}{4} \le |f'(z)| \frac{\operatorname{dist}(z, \partial\Omega)}{\operatorname{dist}(f(z), \partial\Omega')} \le 4$$

Beurling estimate

K_1 is a compact in Ω s.t $\Omega \setminus K_1$ is a s.c domain containing z_0

K_2 comp in Ω

$K_1 \subset K_2$



① $B(z_0, r_0) \subset \Omega \setminus K_2$

② $\operatorname{diam}(K_2 \setminus K_1) < \varepsilon < \frac{r_0}{10}$

Let g be a conf. map $\Omega \setminus K_1 \to \mathbb{D}$

$g(z_0) = 0$

$A \overset{\text{def}}{=} g(K_2 \setminus K_1)$

Then $\exists\, c = c(r_0, \frac{1}{10})$ s.t

$\operatorname{diam}(A) \le c\sqrt{\varepsilon}$

Carathéodory convergence $f_n : \mathbb{D} \to \Omega_n$

$f_n \to f$ if $f_n \to f$ unif on all compacts in $\mathbb{D}$

Th $f_n \to f \; (\mathbb{D} \to \Omega) \iff \Omega_n \to \Omega$

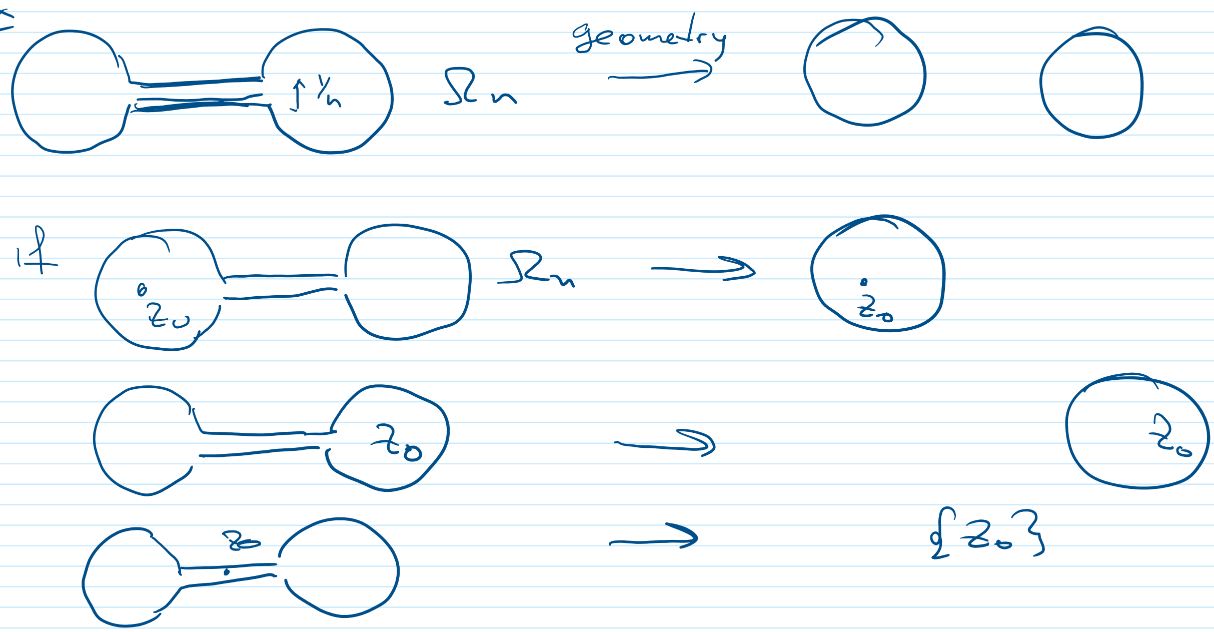
Th $f_n \to f$ $(\mathbb{D} \to \Omega) \iff \Omega_n \to \Omega$ in sense of kernels

Def $\{\Omega_n\}_n$ Kernel w.r.t. z_0 is the largest s.c domain Ω s.t $z_0 \in \Omega$ and every closed subset of Ω belongs to all but finitely many Ω_n

(if there is no such Ω then Kernel $= \{z_0\}$)

$\Omega_n \to \Omega$ if Ω is the kernel of every subsequence

Ex



Poisson kernel

- $\mathbb{D}$ $\quad u(z) = \dfrac{1}{2\pi} \displaystyle\int_{\partial \mathbb{D}} u(\zeta)\, P(\zeta, z)\, |d\zeta|$

where $P_{\mathbb{D}}(\zeta, z) = \dfrac{1 - |z|^2}{|\zeta - z|^2} = \mathrm{Re}\, \dfrac{\zeta + z}{\zeta - z}$

Poisson kernel in $\mathbb{D}$

- $\mathbb{H}$ if $u \to 0$ as $|z| \to \infty$ (sufficiently fast)

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$$u(z) = \frac{1}{\pi} \int_{\mathbb{R}} u(t) \frac{y}{y^2 + (x-t)^2} dt = \int u(t) P(t,z)\, dt$$

$$P_{\mathbb{H}}(t,z) = \frac{1}{\pi} \frac{y}{|z-t|^2} = \operatorname{Im} \frac{1}{\pi(t-z)} \qquad \boxed{z = x+iy}$$

Complexity <u>Schwarz kernels</u> / integral formulas

$\mathbb{D}$ f is analytic in $\mathbb{D}$, cont in $\overline{\mathbb{D}}$ then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \operatorname{Re} f(e^{i\theta})\, d\theta + ic \qquad \{ c = \operatorname{Im} f(0)$$

$$= \frac{1}{2\pi} \int \frac{\zeta + z}{\zeta - z} \operatorname{Re} f(e^{i\theta}) |d\zeta| + ic$$

$\mathbb{H}$ f is analytic in $\mathbb{H}$, cont $\overline{\mathbb{H}}$, $\underline{f(z) = O(|z|^{-\alpha})}_{z \to \infty}$

$\alpha > 0$ <u>then</u>

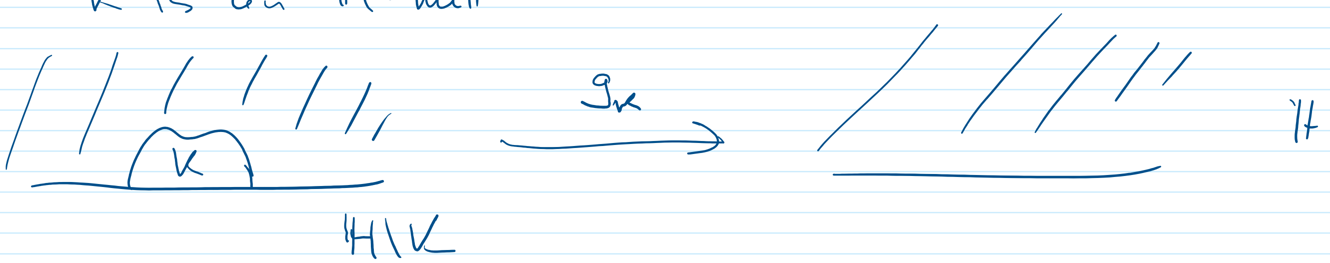
$$f(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Im} f(t)}{t - z} dt$$

<u>Ex</u> $f(z) = z$ then on $\mathbb{R}$ $\operatorname{Im} f(t) = 0$

<u>Mapping out functions</u>

K is a compact in $\mathbb{H}$ s.t $\mathbb{H} \setminus K$ is simply connected

K is an $\mathbb{H}$-hull



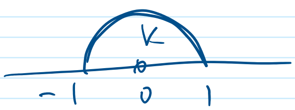
There ∃ unique g s.t. $g: \mathbb{H} \setminus K \to \mathbb{H}$

$g(\infty) \to \infty$

$$g(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots \quad \text{at } \infty$$

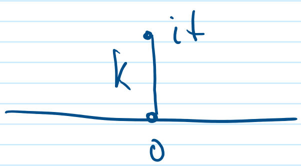
<u>Def</u> $\operatorname{hcap}(K) = b_1$ half-plane capacity

Ex



$$\operatorname{hcap}(K) = 1$$

Ex



$$\operatorname{hcap}(K) = t^2/2$$

Properties

- $\operatorname{hcap}(\lambda K) = \lambda^2 \operatorname{hcap}(K)$
- $\operatorname{hcap}(K + x) = \operatorname{hcap}(K)$
- hcap is cont w.r.t kernel convergence

Lemma $\Omega = \mathbb{H} \setminus K$ $\quad f_K : \mathbb{H} \to \Omega$ $\quad f_K = g_K^{-1}$

Then (1) $f_K(z) = z - \dfrac{\operatorname{hcap}(K)}{z} + \dots$ and

(2) $\operatorname{hcap}(K) = \dfrac{1}{\pi} \displaystyle\int_{-\infty}^{\infty} \operatorname{Im} f_K(x)\, dx$

Proof (1) follows from $g_K = z + \dfrac{\operatorname{hcap}(K)}{z} + \dots$

(2) Consider $f_K(z) - z$ $\quad (\to 0 \text{ as } z \to \infty)$

Apply Schwarz formula

$$f_K(z) - z = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\operatorname{Im}(f_K(x) - x)}{x - z}\, dx = \frac{1}{\pi} \int \frac{\operatorname{Im} f_K(x)}{x - z}\, dx$$

multiply by z and take the limit as $z \to \infty$

$$z(f_K(z) - z) = \frac{1}{\pi} \int \left(-1 + \frac{x}{x - z}\right) \operatorname{Im} f_K(x)\, dx$$

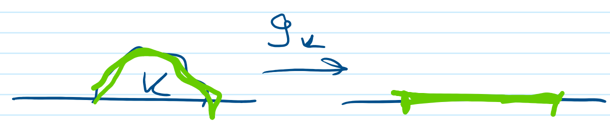
$\operatorname{Im} f_K(x) \neq 0$ only on a compact set.

$\parallel$

$\cancel{z}(\cancel{z} - \frac{\operatorname{hcap} K}{\cancel{z}} + \dots - \cancel{z})$

$\downarrow z \to \infty$ $\qquad\qquad \downarrow z \to \infty$

$-\operatorname{hcap} K \qquad = -\dfrac{1}{\pi} \displaystyle\int \operatorname{Im} f\, dx$



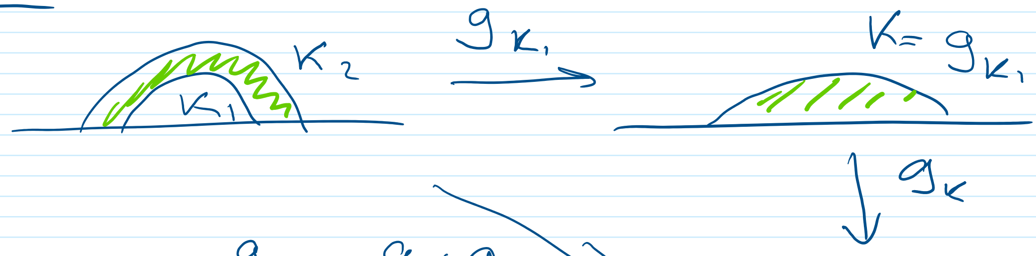
$\blacksquare$

Corollary • $\operatorname{hcap}(K) > 0$ if K is non-trivial.

Corollary • $\mathrm{hcap}(K) > 0$ if K is non-trivial.

• $K_1 \subsetneq K_2$ then $\mathrm{hcap}(K_1) < \mathrm{hcap}(K_2)$

Proof



$$g_{K_2} = g_K \circ g_{K_1}$$

expand at infinity

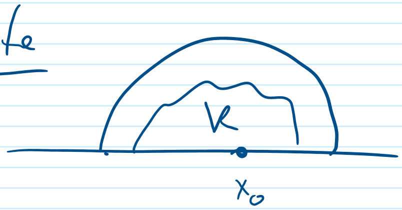
compare $\frac{1}{z}$ terms

hcap. is additive

$$\Rightarrow \mathrm{hcap}(K_2) = \mathrm{hcap}(K_1) + \underbrace{\mathrm{hcap}(K)}_{>0}$$

$$\mathrm{hcap}(K_2) > \mathrm{hcap}(K_1)$$

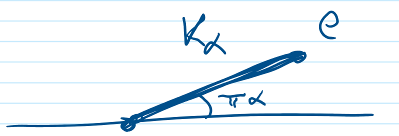
Note



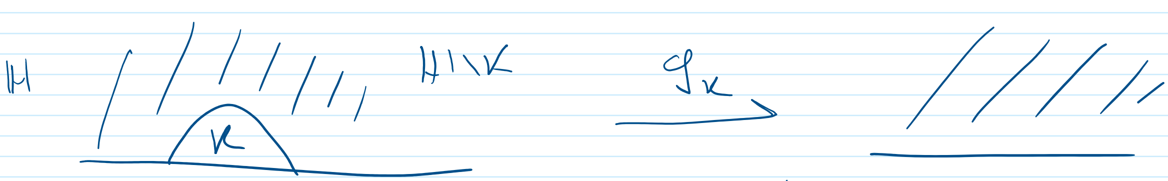
$K \subset \{z : z \in \mathbb{H} \ |z - x_0| \le R\}$

$\mathrm{hcap}\, K \le \mathrm{hcap}(\text{half disc}) = R^2$

No lower Bound

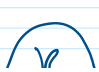


$$\mathrm{hcap}(K_\alpha) = \frac{1}{2}\alpha^{1-2\alpha}(1-\alpha)^{2\alpha-1} \longrightarrow 0 \quad \text{as } \alpha \to 0 \text{ or } \alpha \to 1$$



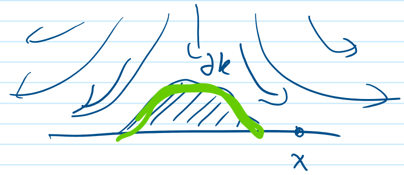
$$g_K(z) = z + \frac{\mathrm{hcap}(K)}{z} + \cdots$$

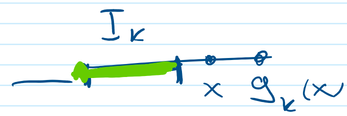
thermodynamical normalization.

Lemma  Let $x \in \mathbb{R}$ be to the right

Lemma

Let $x \in \mathbb{R}$ be to the right of K





Then $g_K(x) > x$

Proof Schwarz formula for $f_K(z) - z$

$$f_K(x) - x = \frac{1}{\pi}\int_{I_K} \frac{\operatorname{Im} f_K(t)}{t - x}\, dt$$

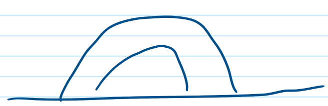
x to the right of K $\Rightarrow$ $g_K(x)$ is to the right of I_K

Take x to the right of I_K

$\Rightarrow f_K(x) - x < 0 \Rightarrow x > f_K(x)$ (apply g_K)

$\Rightarrow g_K(x) > x$

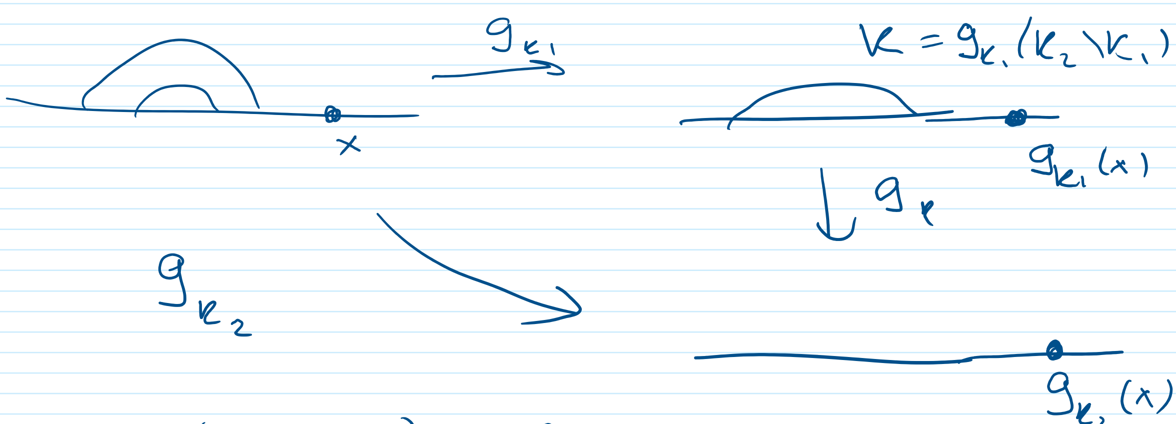
Lemma



$K_1 \subset K_2$ x to the right of K_2

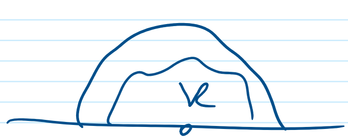
then $g_{K_1}(x) < g_{K_2}(x)$

Proof

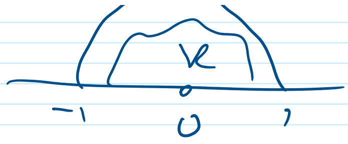


$g_{K_2}(x) = g_K(g_{K_1}(x)) > g_{K_1}(x)$ ∎

Lemma



$K \subset \mathbb{D}$ then if $x > 1$

Lemma  $K \subset \mathbb{D}$ then if $x > 1$

then $x < g_K(x) < x + \frac{1}{x}$

Proof $K_1 = K, \quad K_2 = \overline{\mathbb{D}} \cap \mathbb{H}$

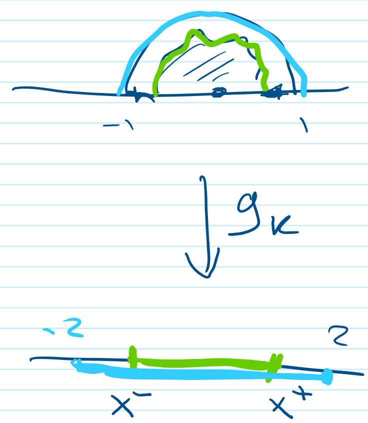
$$g_{K_2} = z + \frac{1}{z}$$

$$g_{K_2}(x) = x + \frac{1}{x} > g_{K_1}(x) > x$$

Lemma $\mathbb{H} \setminus K \xrightarrow{g_K} \mathbb{H}$ then

$$|g_K(z) - z| \leq 3\,\mathrm{rad}(K)$$

Proof Assume that $K \subset \mathbb{D}$



Previous lemmas $\Rightarrow$
$[x^-, x^+] \subset [-2, 2]$

- if $x > 1$ then
$$0 \leq g_K(x) - x \leq \frac{1}{x} \leq 1$$
$\Rightarrow |g_K(x) - x| \leq 1 \Rightarrow |f_K(x) - x| \leq 1$ if $x > 2$

- $f_K((-2,2)) \subset \mathbb{D} \Rightarrow |f_K(x) - x| \leq 3$ on $[-2, 2]$

- if $x < 1 \cdots \Rightarrow$ if $x < -2 \Rightarrow |f_K(x) - x| \leq 1$

- $|f_K(z) - z| \to 0$ as $z \to \infty$

Maximum modulus $\Rightarrow |f_K(z) - z| \leq 3$ in $\mathbb{H}$

$\Rightarrow |g_K(z) - z| \leq 3$ in $\mathbb{H} \setminus K$

Lemma Let us assume $K \subset \{|z| \leq R\}$
$z \in \mathbb{H}, \quad |z| \geq 10R$

Then
$$\left| g_K(z) - z - \frac{\mathrm{hcap}\,K}{z} \right| \leq \frac{10R\,\mathrm{hcap}(K)}{|z|^2}$$

Proof.

Proof $\underbrace{g_K(z) - z}\ - \underbrace{\frac{\mathrm{hcap}\, K}{z}} = \frac{1}{\pi}\int_{\mathbb{R}} \underbrace{\mathrm{Im}\, f_K}\left(\frac{1}{g_K(z)-x} - \frac{1}{z}\right)dx$

Since $|z| > 10R$

$|g_K(z) - z| \le 3R$

$$\left|\frac{1}{g_K(z)-x} - \frac{1}{z}\right| = \frac{|g_K(z) - x - z|}{|z|\,|g_K(z) - x|} \le \frac{5R}{|z|\,|z - 5R|} \le \frac{10R}{|z|^2}$$

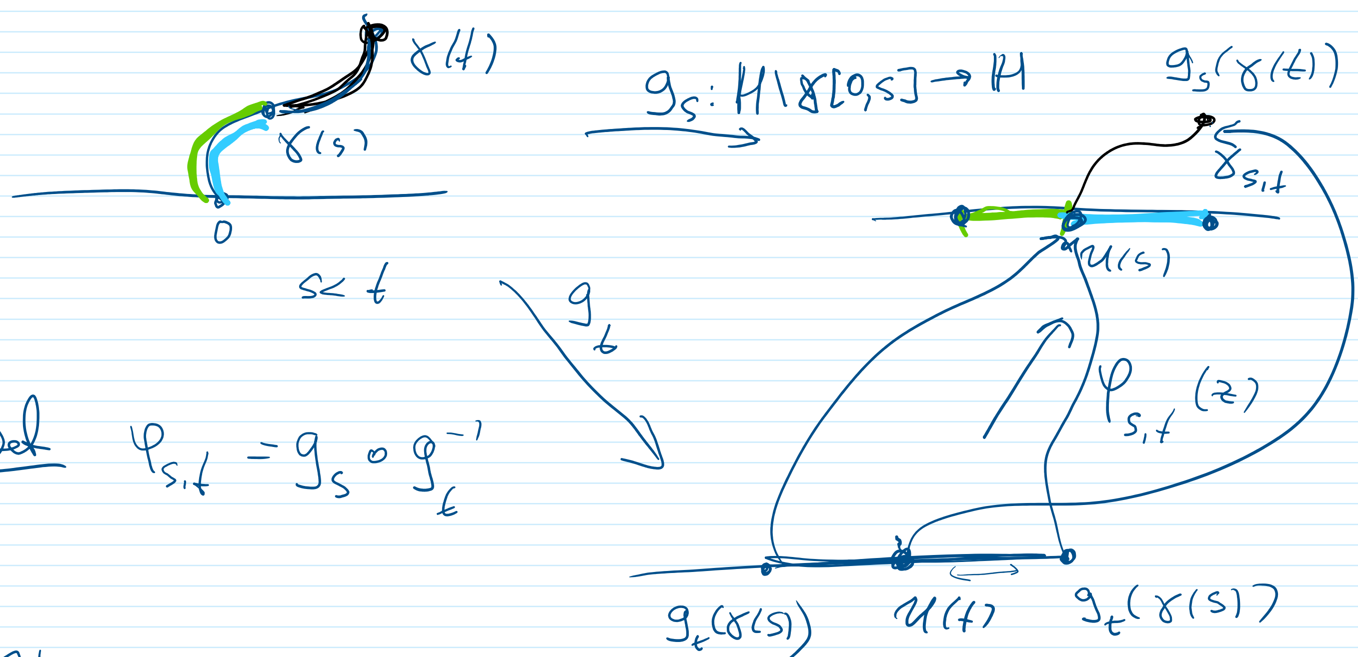
if $|x| \le 2R$

Since $K \subset \{|z| \le R\} \Rightarrow \mathrm{Im}\, f_K = 0$ outside of $[-2R, 2R]$

Then $\left|g_K(z) - z - \frac{\mathrm{hcap}\, K}{z}\right| \le \frac{10R}{|z|^2}\frac{1}{\pi}\int \mathrm{Im}\, f_K = \frac{10R\,\mathrm{hcap}\, K}{|z|^2}$ $\blacksquare$

Loewner Evolution

Chordal version $\gamma(t)$ is a simple curve in $\mathbb{H}$ s.t. $\gamma(0) = 0$, $\gamma(t) \notin \mathbb{R}$ for $t > 0$



Def $\varphi_{s,t} = g_s \circ g_t^{-1}$

$g_t(\gamma(s)) \quad U(t) \quad g_t(\gamma(s))$

Claim: $U(t)$ is continuous in t

Fix T large $0<s<t<T$

$\forall \varepsilon>0 \ \exists \delta$ s.t $|s-t|<\delta$ then

diam $\gamma([s,t]) < \varepsilon$

By Beurling estimate

$$\operatorname{diam}(\gamma_{s,t}) = \operatorname{diam}(g_s(\gamma([s,t]))) \lesssim c\sqrt{\varepsilon}$$

$$\Rightarrow |\varphi_{s,t}(z)-z| \le c\sqrt{\varepsilon}$$

$$\Rightarrow |U(t)-U(s)| = \underbrace{|U(t)-g_t(\gamma(s))|}_{\lesssim c\sqrt{\varepsilon}} + \underbrace{|g_t(\gamma(s))-U(s)|}_{\lesssim c\sqrt{\varepsilon}}$$

$$\le c\sqrt{\varepsilon}$$

∎

Claim g_t satisfies a certain DE

$$\varphi_{s,t}(z)-z = \frac{1}{\pi}\int \frac{1}{x-z} \operatorname{Im}\varphi_{s,t}(x)\,dx$$

$g_s = \varphi_{s,t}(g_t)$, plug in $g_t(z)$ instead of z

$$g_s(z)-g_t(z) = \frac{1}{\pi}\int \frac{1}{x-g_t(z)} \operatorname{Im}\varphi_{s,t}(x)\,dx$$

$\operatorname{Im}\varphi_{s,t} = 0$ outside of an interval around $U(t)$ which shrinks as $s \nearrow t$

$$\partial_t g_t(z) = \lim \frac{g_s(z)-g_t(z)}{s-t} = \frac{1}{U(t)-g_t(z)} \lim \frac{1}{s-t} \underbrace{\int \frac{1}{\pi}\operatorname{Im}\varphi_{s,t}(x)\,dx}_{=\operatorname{hcap}\gamma_{s,t}}$$

$$\operatorname{hcap}\gamma[0,t] - \operatorname{hcap}\gamma[0,s] = \operatorname{hcap}\gamma_{s,t}$$

$$= \frac{1}{g_t(z)-U(t)}\,\partial_t \operatorname{hcap}\gamma[0,t]$$

$$= \frac{1}{g_t(z)-u(t)} \partial_t \operatorname{hcap} \gamma[0,t]$$

Loewner Evolution or Loewner DE

$$\dot{g}_t(z) = \frac{\partial_t \operatorname{hcap} \gamma[0,t]}{g_t(z) - u(t)}$$

Consider $f_t(z) = g_t^{-1}(z)$

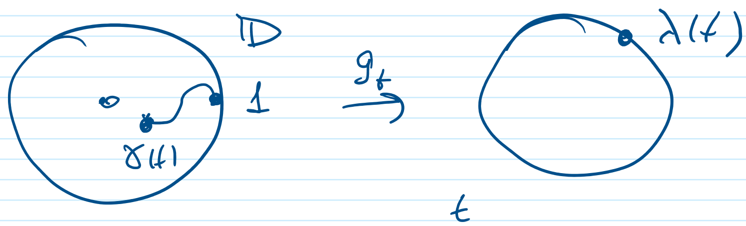
$f_t(g_t(z)) = z$, differentiate w.r.t t

$$\dot{f}_t(z) = -f_t'(z) \frac{\partial_t \operatorname{hcap} \gamma[0,t]}{z - u(t)}$$

Note We assumed that $\partial_t \operatorname{hcap} \gamma[0,t]$ exists

Def If γ is parametrized s.t $\operatorname{hcap} \gamma[0,t] = 2t$ then we say that γ is parametrized by capacity

Radial version

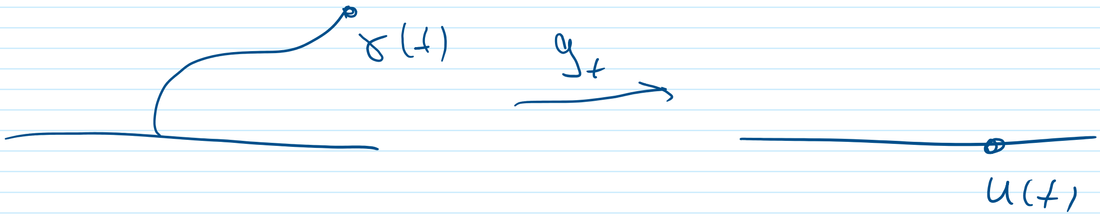


Parametrize γ s.t $g_t(z) = e^t z + \cdots$

$$\dot{g}_t(z) = g_t(z) \frac{\lambda(t) + g_t(z)}{\lambda(t) - g_t(z)}$$

$$\dot{f}_t(z) = -z f_t'(z) \frac{\lambda(t) + z}{\lambda(t) - z}$$

Solving Loewner DE



$$\begin{cases} \partial_t g_t(z) = \dfrac{2}{g_t(z) - U(t)} \\ g_0(z) = z \end{cases} \qquad \left[\gamma \text{ is param. by cap} \right]$$

Given $\gamma \longrightarrow \{g_t\} \rightsquigarrow U(t)$

$\overset{?}{\longleftarrow}$

Th Let μ_t be a family of non-negative Borel measures on $\mathbb{R}$ s.t $\forall\, T\ \exists\, R(T)$ s.t $\forall\, t \in [0,T]$

- $\operatorname{supp} \mu_t \subset [-R, R]$
- $\mu_t(\mathbb{R}) < R$

Assume that μ_t is cont in t w.r.t weak topology. For each $z \in \mathbb{H}$ let $g_t(z)$ be the solution

$$\dot{g}_t(z) = \int_{\mathbb{R}} \frac{\mu_t(dx)}{g_t(z) - x}, \qquad g_0(z) = z$$

Define $T(z)$ to be the supremum of t s.t $g_t(z)$ exists and $g_s(z) \in \mathbb{H}$ for all $s < t$

$K_t \overset{\text{def}}{=} \{ z \in \mathbb{H} \text{ s.t } T(z) \le t \}$

$\mathbb{H}_t = \{ z : \quad T(z) > t \} = \mathbb{H} \setminus K_t$

$\mathbb{H}_t = \{z : \quad T(z) > t\} = \mathbb{H} \setminus K_t$

<u>Then</u> g_t is the conformal map $\mathbb{H}_t \to \mathbb{H}$ with the thermodynamical norm.

$$g_t(z) = z + \frac{b(t)}{z} + \cdots \qquad \text{at } \infty$$

$$b(t) = \int_0^t \mu_s(\mathbb{R})\,ds$$

<u>Note</u> if $\mu_t = 2\delta_{u(t)}$ then

$$\int \frac{\mu_t(dx)}{g_t(z) - x} = \frac{2}{g_t(z) - u(t)}$$

$u(t)$ is the driving function

$\{\mu_t\}$ is the driving family of measures

<u>Proof</u> $\partial_t \left(\operatorname{Im} g_t(z) \right) = -\int \frac{\operatorname{Im} g_t(z)}{|g_t(z) - x|^2} \mu_t(dx) < 0$

$\Rightarrow \operatorname{Im} g_t(z) \searrow \quad \Rightarrow T(z) = \sup\{t : \operatorname{Im} g_t > 0\}$

Let w, v be two trajectories started from z_1 and z_2

Assume that both exist up to t

$$\partial_t (w - v) = -(w - v) \int \frac{\mu_t(dx)}{(w - x)(v - x)}$$

$$\Rightarrow \quad w - v = (z_1 - z_2) \exp\left[-\int_0^t \int_{\mathbb{R}} \frac{\mu_s(dx)}{(w - x)(v - x)}\, ds \right]$$

$w - v = 0 \iff z_1 = z_2$

divide by $z_1 - z_2$ and pass to the limit $z_2 \to z_1$

trivial way $z_1 - z_2$ and pass to the limit $z_2 \to z_1$

$$g_t'(z) = \exp\left(- \int_0^t \int_{\mathbb{R}} \frac{\mu_s(dx)}{(g_s(z)-x)^2} ds \right)$$

- $g_t(z)$ is differentiable in z
- $g_t'(z) \neq 0$

Claim: g_t is onto $\mathbb{H}$

Proof take $w \in \mathbb{H}$, $t \geq 0$

Consider $\dot{h}_s(w) = - \int_{\mathbb{R}} \frac{\mu_{t-s}(dx)}{h_t(w) - x}$, $h_0(w) = w$

$\operatorname{Im} h_s(w) \nearrow \Rightarrow$ solution exists for all $s \leq t$

Take $z = h_t(w)$, then $g_t(z) = g_t(h_t(w)) = w$

Claim g_t is properly normalized

$$\dot{g}_t = \int \frac{\mu_t(dx)}{g_t(z)-x} = \frac{\mu_t(\mathbb{R})}{g_t(z)} + o\left(\frac{1}{g_t(z)}\right)$$

$$= \frac{\mu_t(\mathbb{R})}{z} + o\left(\frac{1}{z}\right) \quad \left[\text{use uniform bound on } g_t \text{ at } \infty\right]$$

$$g_t(z) = z + \left(\int_0^t \mu_s(\mathbb{R}) ds\right) \frac{1}{z} + \cdots$$

$\blacksquare$

Radial case $\{\mu_t\}$ family of measures on $[0, 2\pi]$, $\sup_{t<T} \mu_t([0,2\pi]) < \infty \ \forall T$

solve
$$\begin{cases} \dot{g}_t(z) = g_t(z) \int_0^{2\pi} \frac{e^{i\theta} + g_t(z)}{e^{i\theta} - g_t(z)} \mu_t(d\theta) \\ g_0(z) = z \end{cases}$$

$T(z)$ the time of maximal existence of g_t

$\left(\; g_0(z) = z \right.$

$T(z)$ the time of maximal existence of g_t in $\mathbb{D}$

$\Omega_t = \mathbb{D}_t = \{ z \in \mathbb{D} \, , \, T(z) > t \}$

<u>Then</u> $g_t : \Omega_t \to \mathbb{D}$ s.t

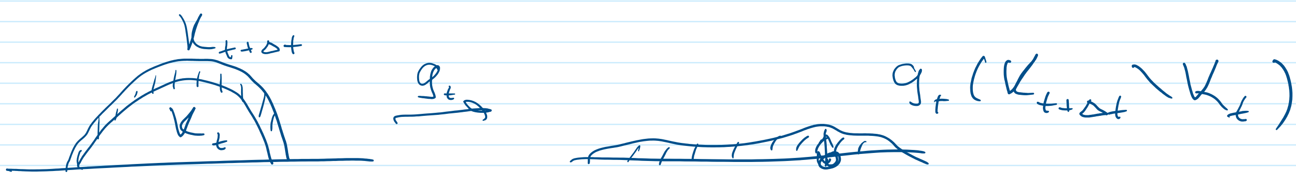
$g_t(0) = 0$ and

$g_t'(0) = \exp\left(\int_0^t \mu_t([0, 2\pi]) \, dt \right)$

Standard parametrization $\mu_t([0, 2\pi]) = 1$

<u>When K_t is a curve?</u>

$$\dot{g}_t(z) = \int \frac{\mu_t(dx)}{g_t(z) - x}$$



For K_t to be a cont curve

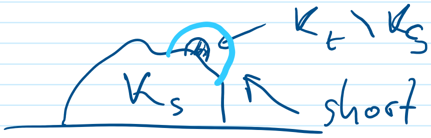
μ_t must be $\delta_{u(t)}$

<u>Th</u> g_t, K_t driving function of g_t is $u(t)$

Then the following are equivalent

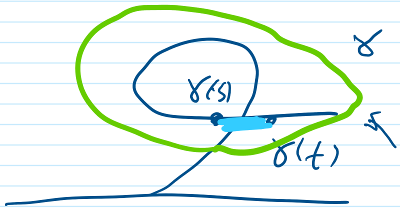
1. $u(t)$ is cont.
2. $\forall \varepsilon > 0 \; \exists \delta$ s.t $|s - t| < \delta$ then
   $\exists$ cross-cut in H_s of diam $\leq \varepsilon$
   which separates $K_t \setminus K_s$ from ∞

$\exists$ cross-cut in $\mathbb{H}_s$ of diam. $\leq \varepsilon$
which separates $K_t \setminus K_s$ from ∞



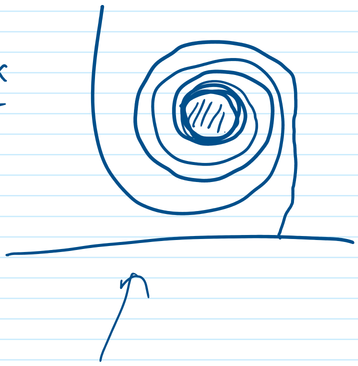
short cross cut in $\mathbb{H}_s$ sep. $K_t \setminus K_s$ from ∞

$\Rightarrow u(t)$ is cont.



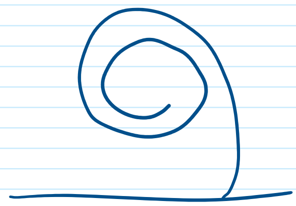
almost shortest cross-cut.

Ex

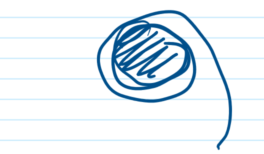


$u(t)$ is cont.

$K_t \quad t < 1$



K_1



but at time $t = 1$

$K_1 \setminus \bigcup_{t<1} K_t$ = disc

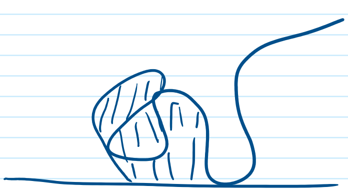
$K_t \quad t > 1$

$\Rightarrow K_t$ is not generated by a curve

($\exists\, \gamma$ s.t. $\mathbb{H}_t = \mathbb{H} \setminus K_t$ = unbounded comp of $\mathbb{H} \setminus \gamma[0,t]$

Def $\gamma(t)$ is the trace of K_t

$K_t = \gamma[0,t]$ + all loops

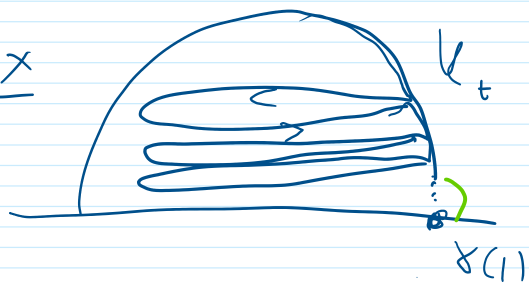


Th Let us assume that

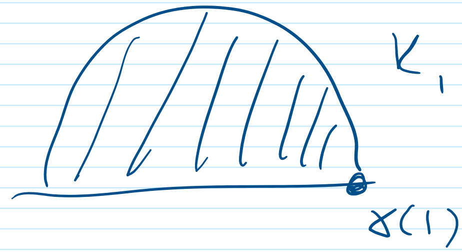
$$\lim_{y \to 0} g_t^{-1}(u(t) + iy) \overset{\text{def}}{=\!=} \gamma(t) \quad \text{exists}$$

Then γ is right cont. function and it is the trace of the Loewner Evolution

Ex



$t < 1$



$t = 1$

γ is not left cont at $t = 1$

K_t is generated by a Loewner Evolution with cont $u(t)$

Th if $u(t)$ is Hölder-$\frac{1}{2}$ cont. with small norm. then there is a cont. simple trace

$$\sup_{s \neq t} \frac{|u(t) - u(s)|}{|t-s|^{1/2}} \overset{\text{def}}{=} \|u\|_{1/2} < 4$$

Basic Properties of LE

Ex $u(t) = 0$

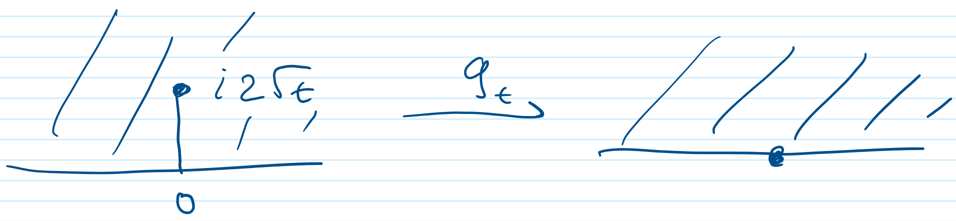
$$\dot{g}_t(z) = \frac{2}{g_t} \quad \Rightarrow \quad 2 g_t \dot{g}_t = 4 \quad \Rightarrow$$

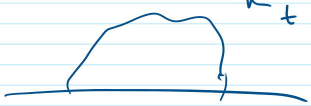
$$\dot g_t(z) = \frac{2}{g_t} \quad \Rightarrow \quad \underbrace{2 g_t \dot g_t}_{} = 4 \quad \Rightarrow$$

$$\Rightarrow \quad \partial_t (g_t^2) = 4 \quad \Rightarrow \quad g_t^2 = 4t + \mathrm{const}(z)$$

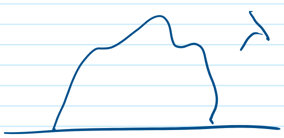
At $t=0$ $\qquad z^2 = \mathrm{const} \quad \Rightarrow$

$$g_t(z) = \sqrt{z^2 + 4t}$$





normalized $\quad \mathrm{hcap}\, K_t = 2t$



$$\lambda K_t = \tilde K_s \qquad \mathrm{hcap}(\lambda K_t) = 2t\lambda^2$$

suggests time change $\quad s = \lambda^2 t$

$$\mathrm{hcap}(\tilde K_s) = 2s$$

$$\tilde g_s = \lambda\, g_{s/\lambda^2}(z/\lambda) \qquad \text{diff. w.r.t } s$$

$$\partial_s \tilde g_s = \frac{1}{\lambda}\, \dot g_{s/\lambda^2}(z/\lambda) = \frac{1}{\lambda}\, \frac{2}{g_{s/\lambda^2}(z/\lambda) - u(s/\lambda^2)}$$

$$= \frac{2}{\underbrace{\lambda g_{s/\lambda^2}(z/\lambda)}_{\tilde g_s} - \underbrace{\lambda u(s/\lambda^2)}_{\stackrel{\text{def}}{=}\ \tilde u(s)}} = \frac{2}{\tilde g_s(z) - \tilde u(s)}$$

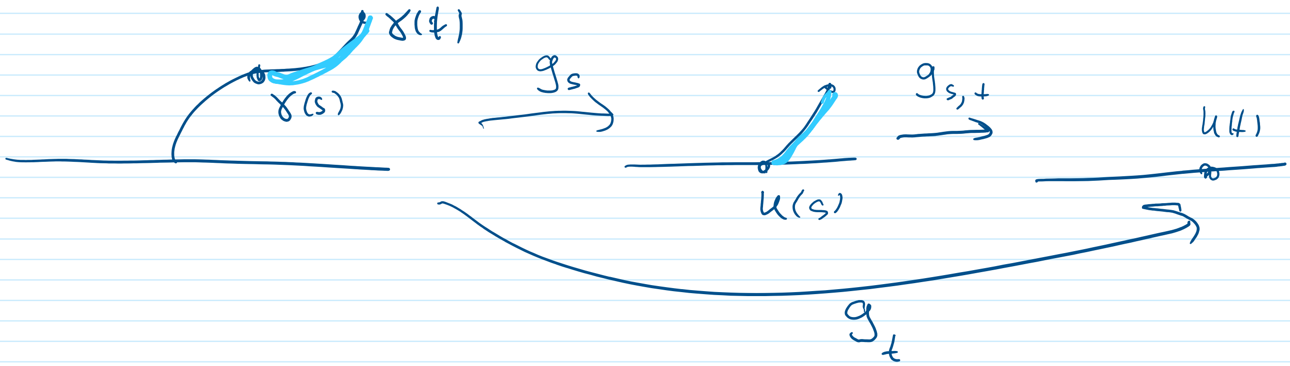
$$s = \lambda^2 t$$

g_s $u(s)$

Scaling K_t by $\lambda \longleftrightarrow$ $s = \lambda^2 t$
$\tilde{u}(s) = \lambda u(s/\lambda^2)$

Note if $u(t)$ is BM then $\tilde{u}$ is also BM

Semi-group property (Markov)



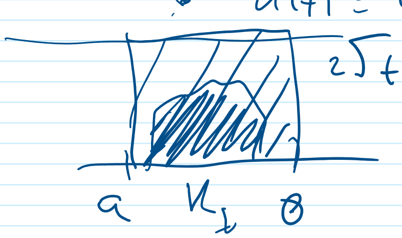
$$g_t = g_{s,t} \circ g_s$$

Consider $\tilde{g}_t = g_{s,s+t}$ then

$$\partial_t \tilde{g}_t = \frac{2}{\tilde{g}_t - u(s+t)}$$ $\longleftarrow$ LE with the driver $\tilde{u}(t) = u(s+t)$

Lemma Suppose $u(t) \in [a,b]$ for all t then $K_t \subset [a,b] \times [0, 2\sqrt{t}]$

Ex $u(t) \equiv a$ then $K_t = \{a\} \times [0, 2\sqrt{t}]$
$u(t) \equiv b$ $K_t = \{b\} \times [0, 2\sqrt{t}]$



Proof Let $y_t = \mathrm{Im}\, z_t = \mathrm{Im}\, g_t(z)$

$$\dot{y}_t = -\frac{2y_t}{|g_t(z)-u(t)|^2} \geq -\frac{2}{y_t}$$ [Note: this is sharp when $\mathrm{Re}\, z \equiv u(t)$]

$$y_t^2 \geq y_0^2 - 4t$$

y_t can not hit 0 before $t = \frac{y_0^2}{4}$

Or if $y_t = 0 \Rightarrow 4t \geq y_0^2 \Rightarrow y_0 \leq 2\sqrt{t}$

$\Rightarrow$ if $y_0 > 2\sqrt{t}$ then $z \notin K_t$

(2) $$\dot{x}_t = \frac{2(x_t - u(t))}{|g_t(z)-u(t)|^2}$$

If $x = x_0 > b$ then $\dot{x}_t > 0 \quad \forall t$

$\Rightarrow x_t > b \Rightarrow z_t$ can not hit $u(t)$

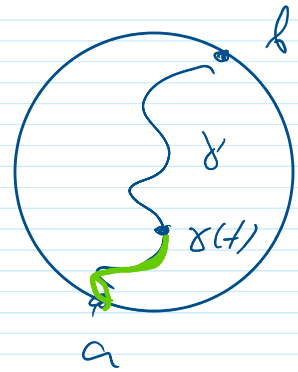
$$\dot{y}_t \geq -\frac{2y_t}{(x_0-b)^2} \Rightarrow y_t \geq y_0 e^{-2t(x_0-b)^{-2}}$$

$\Rightarrow T(z) = \infty$

The same holds when $x_0 < a$ $\blacksquare$

Schramm Loewner Evolution

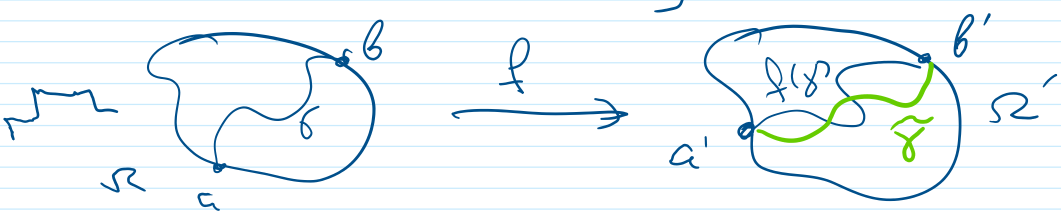
<u>Def</u> Let $\mu(\Omega, a, b)$ be a measure on curves/hulls from a to b inside Ω



① μ satisfies the Domain Markov property if

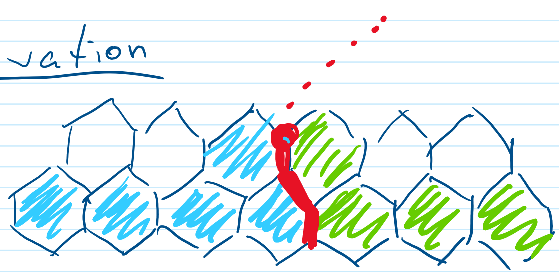
$$\mu(\Omega, a, b \mid \gamma[0,t]) = \mu(\Omega \setminus \gamma([0,t]), \gamma(t), b)$$

② μ is conformally invariant if



$$f(\mu(\Omega, a, b)) = \mu(\Omega', a', b')$$

<u>Motivation</u>



Condition that red curve starts like this

For percolation, we get DMP for free

Also true for Ising model

True for all models with short-range (nearest neighbour) interaction

- <u>Conf inv</u> inv under rotations

  inv under inversion

inv under inversion

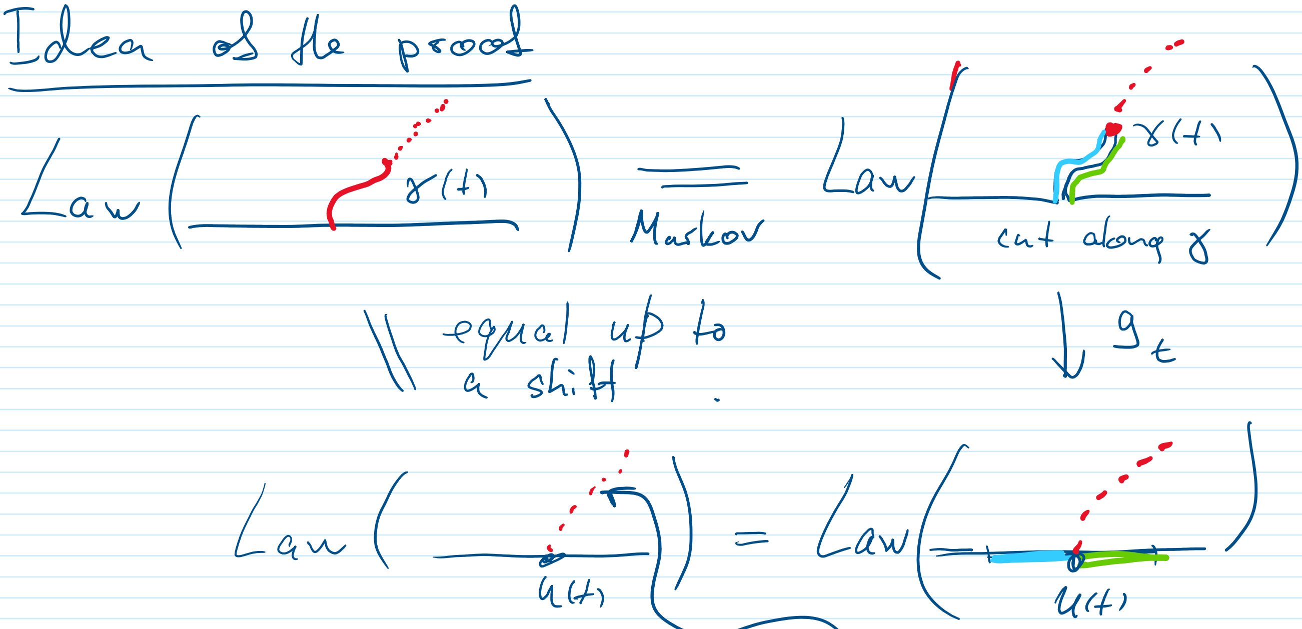
suggests that on large scales the model is inv. under conf. transformations

Th (Schramm's Principle)

Let μ be a conf. invariant family of measures satisfying the Domain Markov property. Let γ be a random curve in $\mathbb{H}$ given by $\mu(\mathbb{H}, 0, \infty)$. Let us parametrize γ by capacity, γ can be described by a Loewner Evolution with the driving function u. Then $u(t) \overset{d}{=} c B_t$ where B_t is a standard Brownian Motion.

Def Let $\varkappa \geq 0$ then $SLE(\varkappa)$, $SLE_\varkappa$ is the Loewner Evolution driven by $u(t) = \sqrt{\varkappa} B_t \overset{d}{=} B_{\varkappa t}$

Idea of the proof



Driving function for this is $u(t+s)$?

$\tilde{u}(s) = u(t+s)$

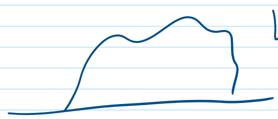
$\Rightarrow \text{Law}(u(t+s) - u(t)) = \text{Law}(u(s))$

Driving function has stationary indep. increments

$\Rightarrow$ multiple of BM

Properties of SLE

- SLE is scale invariant.



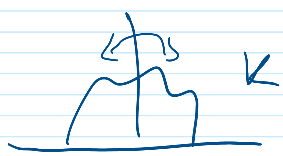
K_t - $SLE_\varkappa$ hull

$\lambda K_t \leftarrow$ this is given by LE driven by $\tilde{u} = \lambda u(t/\lambda^2) = \lambda\sqrt{\varkappa}\, B_{t/\lambda^2} \overset{d}{=}$

$\overset{d}{=} \sqrt{\varkappa}\, B_t$

Scaled $SLE_\varkappa \overset{d}{=}$ time changed $SLE_\varkappa$

- SLE is symmetric



K_t is $SLE_\varkappa$ hull

$\hat{K}_t = \{x+iy \in \mathbb{H} \text{ s.t } -x+iy \in K_t\}$

driving function for $\hat{K}_t$ is $-u(t) =$

$= -\sqrt{\varkappa}\, B_t \overset{d}{=} \sqrt{\varkappa}\, B_t$

Th (Rohde - Schramm) Let g_t be SLE maps

Then a.s $\exists\, \gamma(t)$ s.t

$\gamma(t) = \lim g_t^{-1}(u(t) + iy)$

$$\gamma(t) = \lim_{y \to 0} g_t^{-1}(U(t) + iy)$$

$\mathbb{H}_t = \mathbb{H} \setminus K_t$ = the unbounded comp of $\mathbb{H} \setminus \gamma[0,t]$.

In other words a.s. there is an SLE trace.

Phase Transitions

Th 4.7 Let γ be SLE_κ trace then

1. $\kappa \in [0,4]$ then $\gamma(t)$ is a.s a simple curve s.t $\gamma(t) \to \infty$ as $t \to \infty$

2. $\kappa \in (4,8)$ then γ is not a simple curve, $\bigcup K_t \supset \overline{\mathbb{H}}$ and $\mathrm{dist}(0, \mathbb{H}_t) \to \infty$ (and $\gamma \to \infty$)

3. $\kappa \geq 8$ then γ is a space-filling curve $\forall z \in \overline{\mathbb{H}}$ $\exists t$ s.t $\gamma(t) = z$

   $\gamma(t) \to \infty$ as $t \to \infty$

Lemma (Boundary phase transition) [4.8 Lecture notes]

Let γ be SLE_κ trace

1. $\kappa \leq 4$ then $\gamma((0,\infty)) \cap \mathbb{R} = \emptyset$
2. $4 < \kappa < 8$ then for all $x, y > 0$ γ intersects $[x, x+y)$ a.s

   $$0 < \mathbb{P}(\gamma \text{ hits } [x, x+y)) = \varphi\left(\frac{y}{x+y}\right) < 1$$

3. $\kappa \geq 8$ then $\mathbb{R} \subset \gamma([0,\infty))$

Lemma $X_t = g_t(x) - \sqrt{\varkappa}\, dB_t$

$$dX_t = \frac{2}{X}\,dt - \sqrt{\varkappa}\, dB_t \quad , \quad X_0 = x \in \mathbb{R}$$

$$0 < r < x < y < R < \infty$$

Denote $T_x = \inf\{s : X_s = 0\}$, $\tau = \inf\{s : X_s \notin [r, R]\}$

$T(R) = \inf\{s : X_s = R\}$

Then

$\varkappa \le 4$ then $\mathbb{P}(T_x = \infty) = 1$ ✓

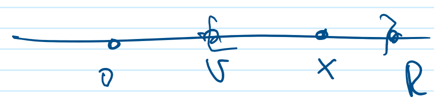
$\varkappa > 4$ then $\mathbb{P}(T_x = \infty) = 0$ ✓

$4 < \varkappa < 8$

$$\mathbb{P}(T_x = T_y) = \frac{\int_0^{x/y} (1-s)^{8/\varkappa - 2} s^{-4/\varkappa}\, ds}{\int_0^1 (1-s)^{8/\varkappa - 2} s^{-4/\varkappa}\, ds}$$

$\varkappa \geqslant 8$ $\mathbb{P}(T_x = T_y) = 0$

Proof $F(x) = \mathbb{P}^x(X_\tau = r)$



Markov property $\Rightarrow$ $F(X_{t\wedge\tau})$ is a martingale

$$\mathbb{P}^x(X_\tau = r \mid \mathcal{F}_t) = F(X_{t\wedge\tau})$$

Assume that $F \in C^2$

Ito $\Rightarrow$

$$dF(X_t) = \underbrace{\left(\frac{2}{X_t} F'(X_t) + \frac{\varkappa}{2} F''(X_t)\right)}_{\substack{\parallel \leftarrow \text{Martingale} \\ 0}} dt - \sqrt{\varkappa}\, F'(X_t)\, dB_t$$

$$\left(\frac{\varkappa}{2} F''(x) + \frac{2}{x} F'(x) = 0\right)$$

$$\begin{cases} \frac{\varkappa}{2} F''(x) + \frac{2}{x} F'(x) = 0 \\ F(r) = 1, \quad F(R) = 0 \end{cases}$$

$\varkappa \neq 4$
$$\tilde{F}(x) = \frac{R^{1-4/\varkappa} - x^{1-4/\varkappa}}{R^{1-4/\varkappa} - r^{1-4/\varkappa}}$$

$\varkappa = 4$
$$\tilde{F}(x) = \frac{\ln R - \ln x}{\ln R - \ln r}$$

Consider $M_t = \tilde{F}(X_{t \wedge \tau})$. This is a martingale

Optional stopping theorem. $\Rightarrow M_t = F(X_t)$

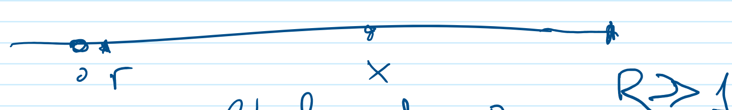
$\Rightarrow F(x) = \tilde{F}(x)$

Case $\varkappa \leq 4$ $\quad \mathbb{P}(X_\tau = r) \to 0$ as $r \to 0$

$\Rightarrow \forall R \quad \mathbb{P}(T(R) < T_x) = 0$

$\Rightarrow \quad \mathbb{P}(T_x = \infty) = 1$

$\varkappa = 4$ $\quad$ fix r, then $F(x) \to 1$ as $R \to \infty$



as X_t will visit any nbhd of 0

Case $\varkappa > 4$ $\quad$ Consider the limit $r \to 0$

$$\mathbb{P}(T(R) < T_x) = 1 - \lim_{r \to 0} F(x) = \left(\frac{x}{R}\right)^{1-4/\varkappa} \xrightarrow[R \to \infty]{} 0$$

$$\Rightarrow \quad \mathbb{P}(T_x = \infty) = 0$$

$$\mathbb{P}(T_x < \infty) = 1$$

$$\varphi(x,y) = \mathbb{P}(T_x = T_y) \qquad \text{a.s}$$

By monotonicity $X_t < Y_t \Rightarrow T_x \le T_y < \infty$

$\varphi(X_t, Y_t)$ is a martingale

$$\left.\begin{aligned} d\varphi(X_t, Y_t) &= \left(\frac{2}{X_t}dt + \sqrt{\varkappa}\,dB_t\right)\varphi_x + \left(\frac{2}{Y_t}dt + \sqrt{\varkappa}\,dB_t\right)\varphi_y \\ &\quad \frac{\varkappa}{2}\left(\varphi_{xx} + \varphi_{yy} + 2\varphi_{xy}\right)dt \end{aligned}\right] \quad \text{requires } \varphi \in C^2$$

Martingale $\Rightarrow$ drift $= 0$ $\Rightarrow$ PDE

$$\frac{2}{x}\varphi_x + \frac{2}{y}\varphi_y + \frac{\varkappa}{2}\varphi_{xx} + \frac{\varkappa}{2}\varphi_{yy} + \varkappa\varphi_{xy} = 0$$

SLE is scale invariant $\Rightarrow$

$\varphi(x,y) = h(x/y)$, change variables $t = x/y$

$$h''(t)\,\varkappa\, t(t-1) = h'(t)\, 2\left(t(\varkappa-2) - 2\right)$$

$$(\ln h')' = \frac{2}{\varkappa}\left(-\frac{2}{t} - \frac{4-\varkappa}{1-t}\right)$$

$$h(t) = c_1 \int_0^t (1-s)^{8/\varkappa - 2} s^{-4/\varkappa}\, ds + c_2$$

$$h(0) = 0 \quad \Rightarrow \quad c_2 = 0$$

$$\underbrace{h(1) = 1 \quad \Rightarrow \quad \frac{1}{c_1} = \int_0^1 (1-s)^{8/\varkappa - 2} s^{-4/\varkappa}\, ds}_{\text{requires } \varkappa < 8}$$

If $\varkappa \ge 8$ the only Bounded solution

If $\kappa \geq 8$ the only bounded solution with $h(0)=0$ is $h(t)=0$

$$\Rightarrow \mathbb{P}(T_x = T_y) = 0$$

∎

$$\int_0^1 \dots = \frac{\Gamma\left(\frac{8}{\kappa}-1\right)\Gamma\left(\frac{\kappa-4}{\kappa}\right)}{\Gamma\left(4/\kappa\right)}$$

Solution to ODE

$$c_1 \frac{\kappa}{\kappa-4} t^{1-4/\kappa} \, {}_2F_1\left(2-\frac{8}{\kappa}, \frac{\kappa-4}{\kappa}, 1+\frac{\kappa-4}{\kappa}, t\right) + c_2$$

${}_2F_1(a,b,c,z)$ Hypergeometric function ${}_2F_1$

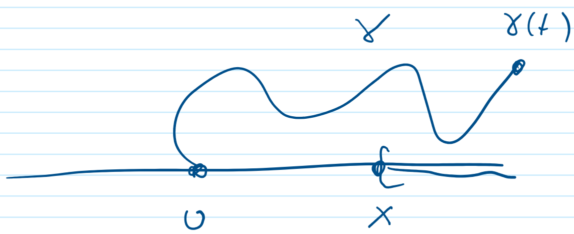
$$= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} z^n, \qquad (a)_n = a(a+1)\cdots(a+n-1)$$

$$z(1-z)f''(z) + (c-(a+b+1)z)f'(z) - ab\,f(z) = 0$$

which is regular at $z=0$ with $f(0)=1$

∎

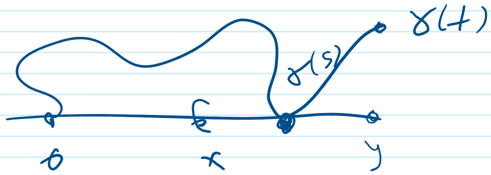
Proof of the Boundary phase transition



If $\gamma([0,t]) \cap [x,\infty) = \emptyset$ then by compactness $[x,\infty)$ is not in a nbhd of K_t

$\Rightarrow x \notin K_t \Rightarrow \underline{T_x > t}$

If



if $\gamma(s) \in [x, \infty)$

then $T_x \le T_{\gamma(s)} = s \le t$

$$\{\gamma([0,t]) \cap [x,\infty) \neq \emptyset\} = \{T_x \le t\}$$
$$\{\gamma([0,t]) \cap [x,y] \neq \emptyset\} = \{T_x < T_y\}$$

If $\varkappa \le 4$

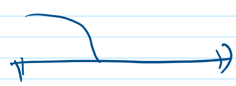
$$\mathbb{P}(\gamma \text{ hits } \mathbb{R}\setminus\{0\}) = \lim_{n\to\infty} \mathbb{P}(\gamma \text{ hits } \mathbb{R}\setminus[-\tfrac{1}{n}, \tfrac{1}{n}]) = 0$$

$T_x = \infty$ a.s $\forall x$

If $\varkappa \ge 8$ a.s $\forall x, y \in \mathbb{Q}$ $\exists t$ s.t. $\gamma(t) \in [x,y]$

$T_x < T_y$

Since γ is continuous the same is true for all $x, y \in \mathbb{R}$



$\Rightarrow \mathbb{R} \subset \gamma([0,\infty))$

If $4 < \varkappa < 8$

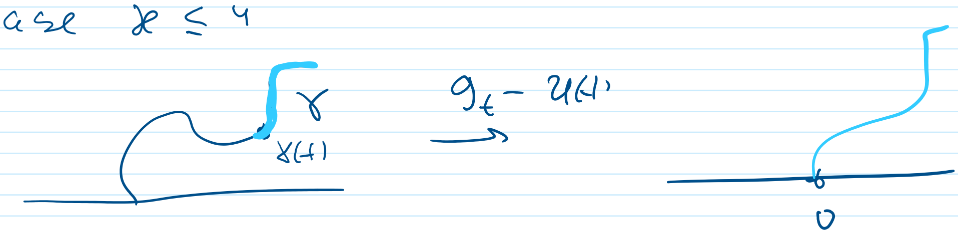
$$\mathbb{P}(\gamma \text{ hits } [x, x+y)) = \mathbb{P}(T_x < T_{x+y})$$

$$= 1 - \frac{\int_0^{\frac{x}{x+y}} (1-s)^{8/\varkappa - 2} s^{-4/\varkappa} ds}{\int_0^1 \cdots\cdots ds} = \frac{\int_{1-\frac{y}{x+y}}^1 \cdots}{\int_0^1 \cdots}$$

$$= \varphi\left(\frac{y}{x+y}\right)$$

"Proof" of the phase transition.

Case $\kappa \le 4$

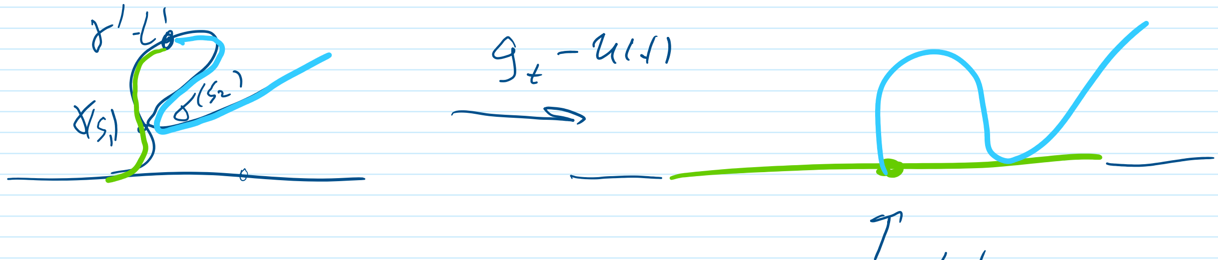


$g_t(\gamma[t,\infty)) - u(t)$
$\overset{d}{=}$
SLE_κ curve γ

This is true simultaneously for all rational t

Assume that $\gamma(s_1) = \gamma(s_2)$ $\quad s_1 < s_2$

$\exists$ rational $t \in (s_1, s_2)$

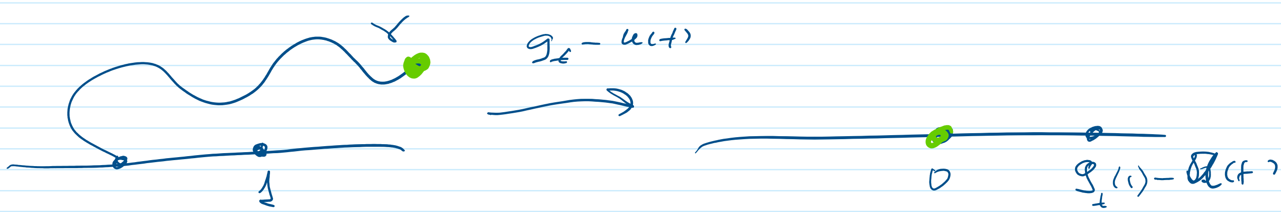


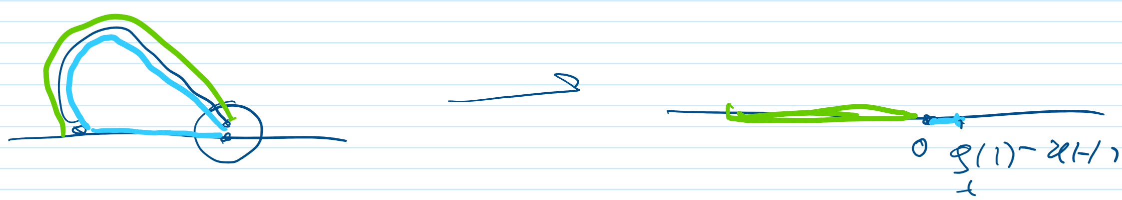
probability $= 0$

$\Rightarrow$ γ is simple with $\mathbb{P} = 1$

If $\kappa < 4$ $\quad \inf_{t>0} g_t(1) - \sqrt{\kappa} B_t > 0$ $\quad$ a.s

$\Rightarrow$ $\inf |\gamma(t) - 1| > 0$ $\quad$ a.s





γ will leave a nbhd of the origin.

and by scaling $\Rightarrow$ γ will leave any

γ will leave a nbhd of the origin.

0-1 law / scaling $\Rightarrow$ γ will leave any nbhd of 0

$\Rightarrow \gamma \to \infty$

<u>If $4 < \kappa < 8$</u> Consider $\text{dist}(0, \mathbb{H}_t)$

$\gamma([0,1]) \leftarrow$ will a.s hit $(0,\infty)$ and $(-\infty, 0)$

0

$\exists$ nbhd of 0 s.t this nbhd is inside K_1

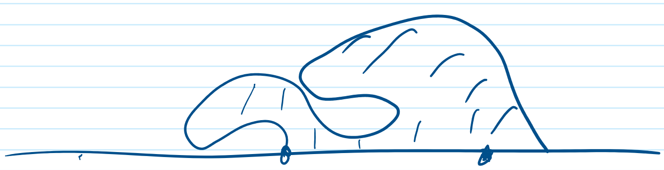
$$\mathbb{P}(\text{dist}(0, \mathbb{H}_t) > 0 \ \forall t) = 1$$

$$\mathbb{P}(\text{dist}(0, \mathbb{H}_t) \le r) = \mathbb{P}\left(\text{dist}(0, \mathbb{H}_1) \le \frac{r}{\sqrt{t}}\right) \xrightarrow[t\to\infty]{} 0$$

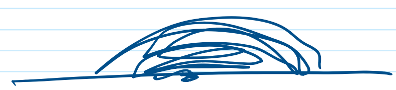
$\Rightarrow \text{dist}(0, \mathbb{H}_t) \to \infty$ as $t \to \infty$ a.s.

$\Rightarrow \gamma(t) \to \infty$ as $t \to \infty$ a.s.

γ is space-filling



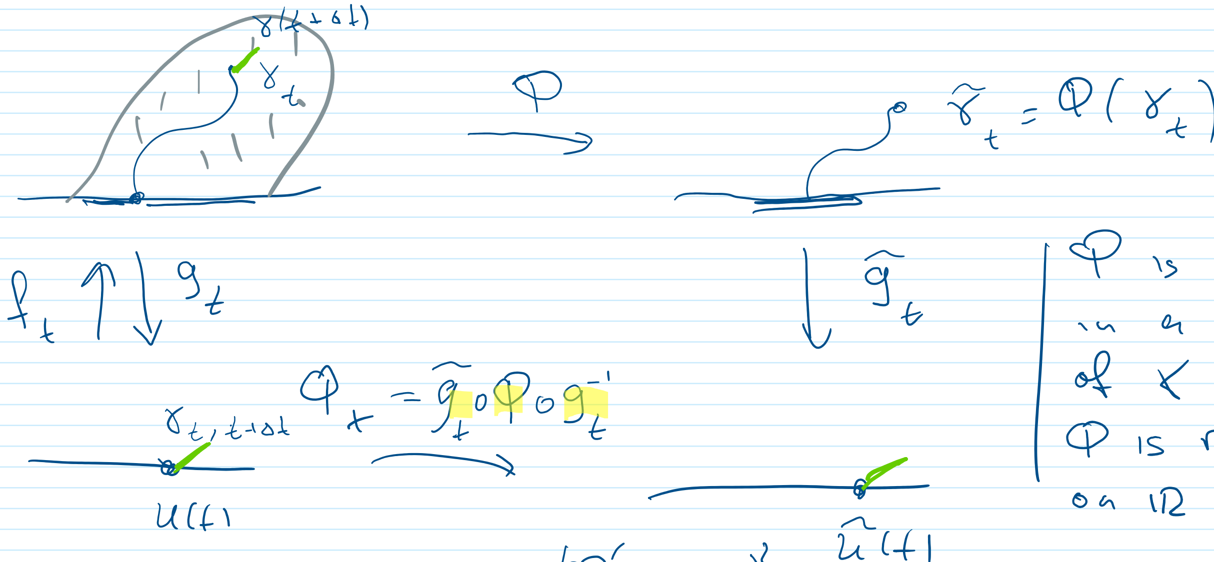
$4 < \kappa < 8$



$\kappa \geq 8$

Locality

Change of local coordinates in LE



φ is univalent in a nbhd of K

φ is real on $\mathbb{R}$

<u>Claim</u> $$\partial_t \tilde{g}_t(z) = \frac{2\left(\varphi_t'(u(t))\right)^2}{\tilde{g}_t - \tilde{u}(t)}$$

$$\Longleftrightarrow \quad \partial_t \operatorname{hcap} \tilde{\gamma}[0,t) = 2\left(\varphi_t'(u(t))\right)^2$$

$$\operatorname{hcap}(g_t(\gamma[t, t+\Delta t])) = 2\Delta t$$

$$\operatorname{hcap}(\tilde{g}_t(\tilde{\gamma}[t,t+\Delta t])) = \operatorname{hcap}(\varphi_t(\gamma_{t,t+\Delta t}))$$

$$\varphi_t \approx \varphi_t(u(t)) + \varphi_t'(u(t))(z-u(t)) + \dots$$

$$\approx \left(\varphi_t'(u(t))\right)^2 2\Delta t$$

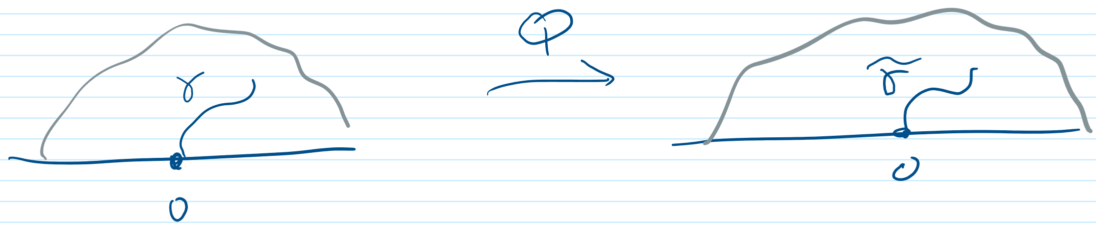
<u>Claim</u> $$\dot{\varphi}_t(z) = 2\left(\frac{\left(\varphi_t'(u(t))\right)^2}{\varphi_t(z) - \varphi_t(u(t))} - \varphi_t'(z)\frac{1}{z-u(t)}\right)$$

$$\dot{\varphi}_t(u(t)) = \lim_{z \to u(t)} \dot{\varphi}_t(z) = -3\varphi_t''(u(t))$$

Th (Locality of SLE(6))

Let γ be an SLE($\varkappa$) trace, $u(t) = \sqrt{\varkappa} B_t$

Let Φ be as above



Then $\tilde{\gamma}_t$ is a time-changed SLE($\varkappa$) curve if and only if $\varkappa = 6$

Proof $\tilde{\gamma}$ is given by a LE with the driver $\tilde{u}(t) = \Phi_t(u(t))$

By Ito

$$d\tilde{u}(t) = \dot{\Phi}_t(u(t))dt + \Phi_t' \sqrt{\varkappa}\, dB_t + \frac{1}{2}\Phi_t'' \varkappa\, dt$$

$$= \left(\left(\frac{\varkappa}{2} - 3\right)\Phi''\right)dt + \sqrt{\varkappa}\, \underbrace{\Phi' \, dB_t}_{d\tilde{B}_s}$$

$\tilde{u}$ is a mart. $\Leftrightarrow$ $\varkappa = 6$

$\Updownarrow$

$\tilde{u}$ is a time-changed BM

$\partial_t \, \mathrm{hcap}\, \tilde{\gamma}[0,t] = 2(\Phi_t'(u_t))^2$

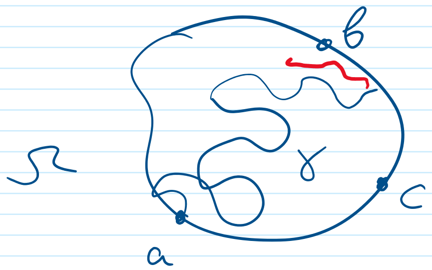
new time $s(t)$ given by

$$s = \int_0^s \Phi_\tau'(u(\tau))^2 d\tau \quad \leftarrow \text{random change of time}$$

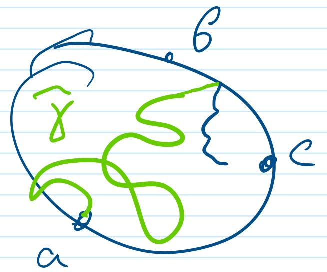
If $\varkappa = 6$ $\quad d\tilde{u}(s) = \sqrt{\varkappa}\, d\tilde{B}_s$, $\tilde{B}_s$ – BM $\blacksquare$

Corollary SLE(6) is target independent.

Corollary SLE(6) is target independent.

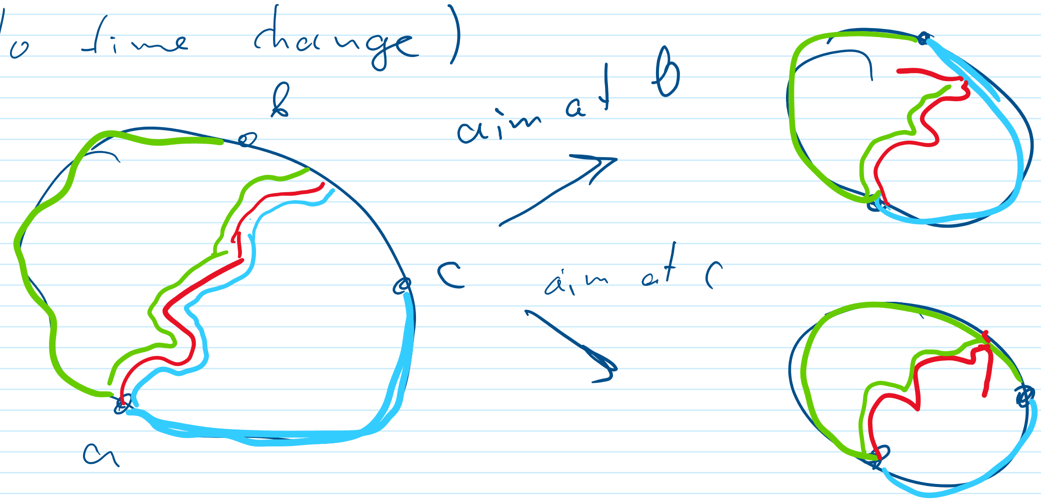


γ is SLE(6) trace in Ω from a to b

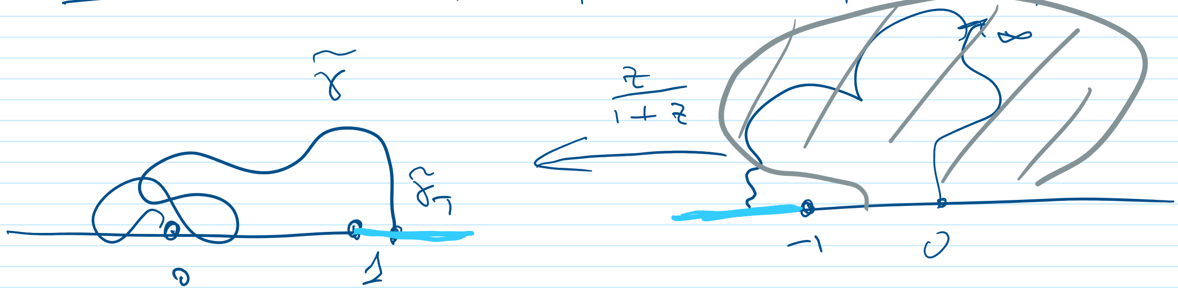


$\hat{\gamma}$ is SLE(6) trace in Ω from a to c

Law of γ = Law of $\hat{\gamma}$ up to the first time they separate b and c (up to time change)



Proof $\Omega = \mathbb{H}$, $a = 0$, $b = 1$, $c = \infty$



By def $\tilde{\gamma}$ = (time changed) SLE$(\mathbb{H}, 0 \to 1)$

$T = \inf\{t > 0, \tilde{\gamma}_t \in [1, \infty)\}$

$T = \inf\{t > 0 \;\; \gamma_t \in [-\infty, 1]\}$

$$T = \inf\{t>0 \;\; \gamma_t \in [-\infty, 1]\}$$

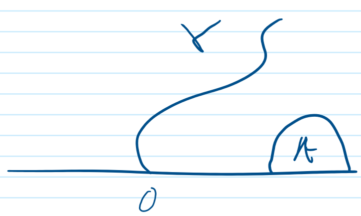
$\kappa = 6 > 4 \Rightarrow T < \infty$ a.s

$\kappa = 6 < 8 \Rightarrow \tilde{\gamma}_T > 1$

For $t < T$ $\gamma[0,t]$ is in some s.c nbhd of 0. By our theorem applied to $\varphi = \frac{z}{1+z}$ in this nbhd $\tilde{\gamma}$ is a time changed SLE(6) [in $\mathbb{H}$ from 0 to ∞] $\blacksquare$

Restriction

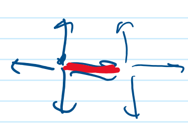
Let γ be a simple curve from 0 to ∞ in $\mathbb{H}$



Let A be an $\mathbb{H}$-hull separated from 0. V_A is the event $\gamma([0,\infty)) \cap A = \emptyset$

Def We say that γ has the restriction property if the law of γ conditioned on V_A is the same as the law of γ in $\mathbb{H} \setminus A$

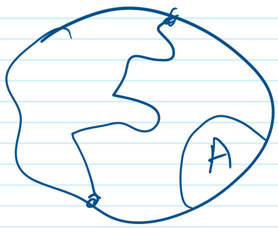
Self-avoiding random walk (SAW)

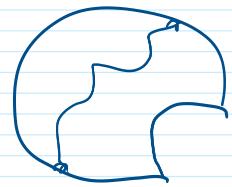


RW of length n is the uniform measure on all lattice paths of length n

SAW of length n is the unif. measure on all simple lattice paths of length n

simple lattice paths ~~of~~ length n

Law 

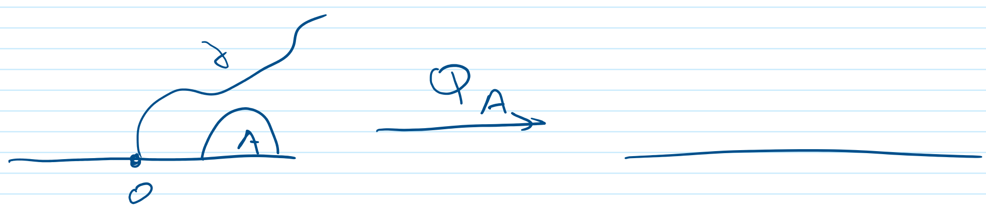
$=$ Law 

If $\forall A$ we know whether $\gamma \cap A = \emptyset$ or $\neq \emptyset$ the we know γ

If γ is random then knowing $\mathbb{P}(\gamma \cap A = \emptyset)$ for all A gives us full info. about the law of γ

Lemma Let $\kappa \leq 4$, γ is $SLE(\kappa)$ trace

Let A be a positive hull $\left(A \cap \mathbb{R} \subset (0, \infty) \right)$



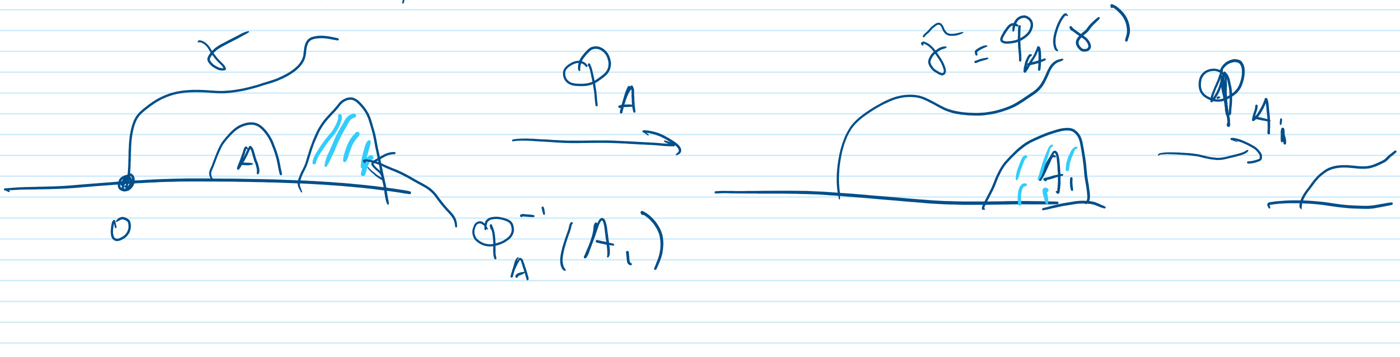
$\varphi_A : \mathbb{H} \setminus A \to \mathbb{H}$ s.t. $\varphi_A(0) = 0$, $\varphi_A(\infty) = \infty$

$\varphi_A'(\infty) = 1$ $\quad \left(\varphi(z) = z + \cdots \text{ at } \infty \right)$

If $\exists\, \alpha > 0$ s.t. $\mathbb{P}(V_A) = \varphi_A'(0)^\alpha$ $\forall A$

then $SLE(\kappa)$ has the restriction property

Proof Let A, A_1 be two hulls



$$\mathbb{P}(\varphi_A(\gamma)\cap A_1=\emptyset,\ \gamma\cap A=\emptyset)=$$

$$=\mathbb{P}(\gamma\cap(A\cup\varphi_A^{-1}(A_1))=\mathbb{P}(V_{A\cup\varphi_A^{-1}(A_1)})=$$

$$\left[\begin{array}{l}\varphi_{A\cup\varphi_A^{-1}(A_1)}=\varphi_{A_1}\circ\varphi_A \quad \text{by chain rule}\\ \varphi'_{A\cup\varphi_A^{-1}(A_1)}(0)=\varphi'_{A_1}(0)\,\varphi'_A(0)\end{array}\right]$$

$$=\left(\varphi'_{A\cup\varphi_A^{-1}(A_1)}(0)\right)^\alpha=\left(\varphi'_{A_1}(0)\right)^\alpha\left(\varphi'_A(0)\right)^\alpha$$

$$\mathbb{P}(\tilde\gamma\cap A_1=\emptyset\mid\gamma\cap A=\emptyset)=\frac{\mathbb{P}(\tilde\gamma\cap A_1=\emptyset,\ \gamma\cap A=\emptyset)}{\mathbb{P}(\gamma\cap A=\emptyset)}$$

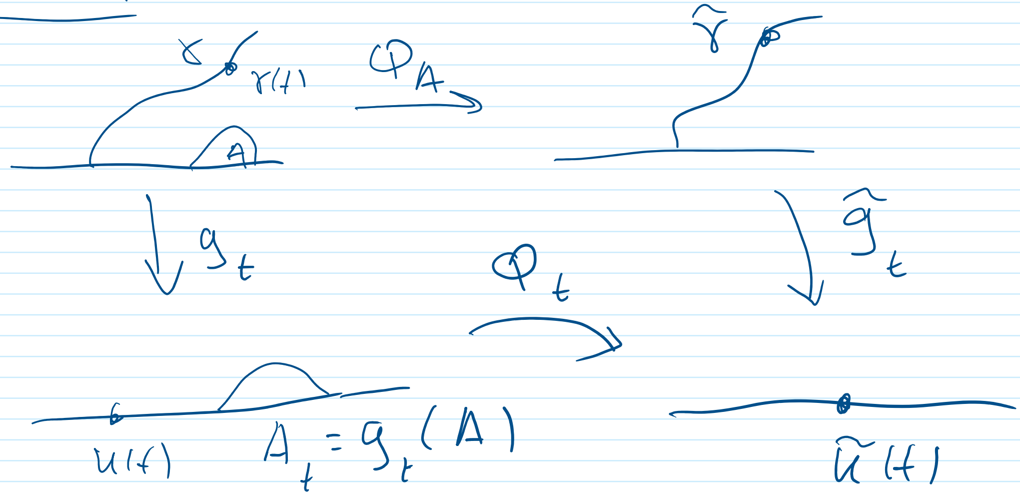
$$=\frac{(\varphi'_{A_1}(0))^\alpha(\varphi'_A(0))^\alpha}{(\varphi'_A(0))^\alpha}=(\varphi'_{A_1}(0))^\alpha=\mathbb{P}(\gamma\cap A_1=\emptyset)$$

$\Rightarrow$ Law $\tilde\gamma|V_A$ = Law γ = SLE in $\mathbb{H}$

$\Rightarrow$ Law $\gamma|_{V_A}$ = Law $\varphi_A^{-1}(SLE)$ = Law of SLE

$\blacksquare$

Lemma



$u(t) \qquad A_t = g_t(A) \qquad \tilde{u}(t)$

<u>Then</u> $$\dot{\varphi}_t'(u(t)) = \frac{\varphi_t''(u(t))^2}{2\varphi_t'(u(t))} - \frac{4}{3}\varphi_t'''(u(t))$$

<u>Th</u> SLE(8/3) satisfy the restriction property

<u>Proof</u> $M_t = (\varphi_t'(u(t)))^\alpha \mathbb{1}_{t<T}$

where $T_A = \inf\{t>0 \quad \gamma[0,t] \cap A \neq \emptyset\}$

$V_A = \{T_A = \infty\}$

Assume that $t < T$

By Ito $\quad = 0 \text{ if } \alpha = 5/8,\ \varkappa = 8/3$

$$\frac{dM_t}{\alpha M_t} = \left[\frac{(\alpha-1)\varkappa+1}{2}\left(\frac{\varphi_t''(u(t))}{\varphi_t'(u(t))}\right)^2 + \underbrace{\left(\frac{\varkappa}{2}-\frac{4}{3}\right)}_{=0 \text{ if } \varkappa=8/3}\frac{\varphi_t'''(u(t))}{\varphi_t'(u(t))}\right]dt$$

$$+ \frac{\varphi_t''(u(t))}{\varphi_t'(u(t))}\sqrt{\varkappa}\, dB_t$$

if $\varkappa = 8/3$, $\alpha = 5/8 \to M_t$ is a local martingale

<u>Claims</u>

- $M_t \le 1$ ✓
- M_t is cont $\quad M_{t\wedge T_A} \to M_\infty = \mathbb{1}_{V_A}$

$$\mathbb{E} M_{T_A} = M_0 = (\varphi_A'(0))^{5/8}$$

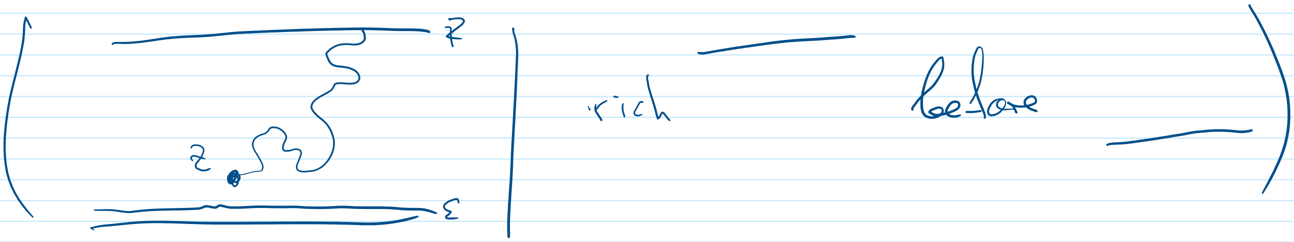
$\quad \shortparallel$

$P(V_A) \qquad$ by the previous lemma

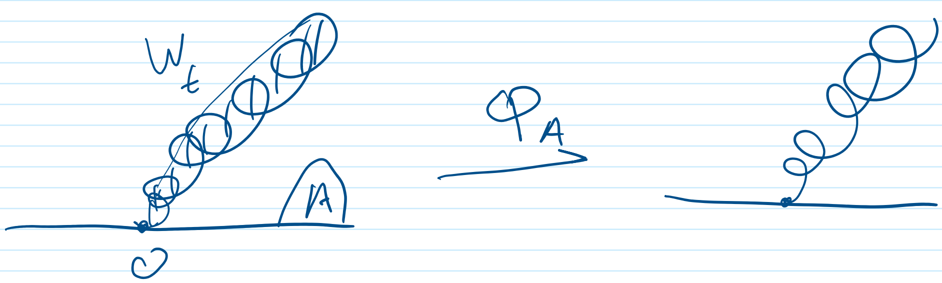
$\mathbb{P}(V_A)$ by the previous lemma
there is the restriction property ∎

About these claims

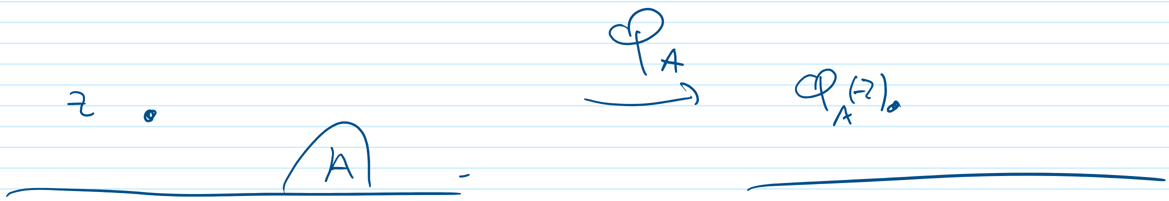
Brownian excursion in $\mathbb{H}$ = BM in $\mathbb{H}$ conditioned to stay in $\mathbb{H}$



Pass to the limit as $\varepsilon \to 0$ and $R \to \infty$
With one more limit we can define an excursion started on the boundary



$$\underbrace{\mathbb{P}^z(W_t \text{ does not hit } A)}_{R} = \underbrace{\frac{\operatorname{Im}\varphi_A(z)}{\operatorname{Im} z}}_{\approx \operatorname{Im} = R}$$

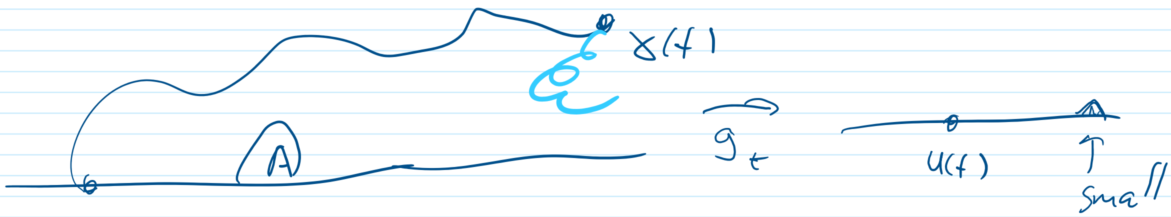


$$\mathbb{P}^x(W_t \text{ does not hit } A) = \varphi_A'(x)$$

$$\mathbb{P}^0(W_t \text{ does not hit } A) = \varphi_A'(0) \leq 1$$

$$M \to 1_{V_A}$$

if $T_A = \infty$ $\qquad t \gg 1$



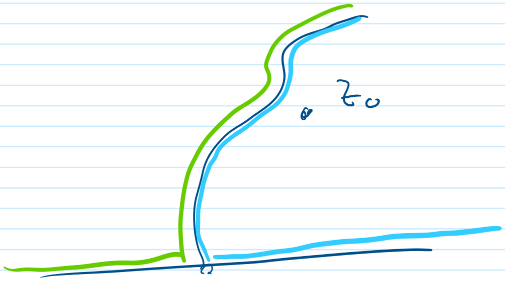
$$\mathbb{P}^{\gamma(t)}(W \cap A = \emptyset) \approx 1$$

if $T_A < \infty$, take $t = T_A - \varepsilon$

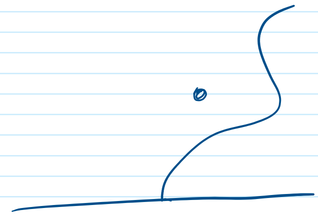
$$\mathbb{P}^{\gamma(t)}(W_s \cap A = \emptyset) \approx 0$$

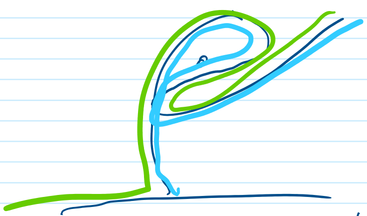
Standard SLE techniques

Schramm's formula $\kappa < 8$, γ is $SLE(\kappa)$ curve, $z_0 \in \mathbb{H}$



or





clockwise $\Rightarrow$ to the left

to the right.

Th

$$\mathbb{P}[\gamma \text{ passes to the left of } z_0] =$$

$$= \frac{1}{2} + \frac{\Gamma\left(\frac{4}{\kappa}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{8-\kappa}{2\kappa}\right)} \frac{x_0}{y_0} \, {}_2F_1\left(\frac{1}{2}, \frac{4}{\kappa}, \frac{3}{2}, -\frac{x_0^2}{y_0^2}\right)$$

when $\kappa = 2$ $\qquad \mathbb{P} = 1 + \frac{x_0 y_0}{\pi |z_0|^2} - \frac{\arg z_0}{\pi}$

$\kappa = 8/3 \qquad \mathbb{P} = \frac{1}{2} + \frac{x_0}{2|z_0|}$

$\kappa = 4 \qquad \mathbb{P} = 1 - \frac{\arg z_0}{\pi}$

$$\boxed{\kappa = 8 \qquad \text{r.h.s} = 1/2}$$

Lemma a.s γ passes to the left of z_0

Lemma a.s γ passes to the left of z_0 if and only if $\lim_{t \to T(z_0)} \frac{x_t}{y_t} = \infty$

where $x_t + i y_t = z_t = g_t(z) - \sqrt{\kappa} B_t$

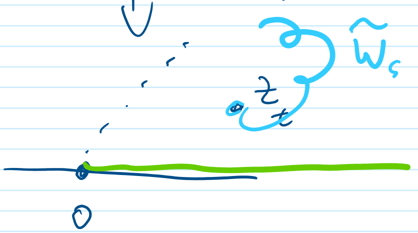
γ passes to the right $\iff \lim \frac{x_t}{y_t} = -\infty$

Proof $\boxed{\text{Case } \kappa \le 4}$ In this case $T(z_0) = \infty$ a.s

W_s - BM started from z_0 stopped when it hits γ or $\mathbb{R}$

γ is to the left if and only if W_s a.s hits the right side of γ or $[0,\infty)$

$\downarrow g_t - \sqrt{\kappa} B_t$



$1 = \mathbb{P}\Big(\tilde{W}_s$ hits the right side of $\underline{\quad\quad}$ or $[0,\infty)\Big)$

$\approx \mathbb{P}\big(\tilde{W}$ hits $[0,\infty)\big)$

$$= \frac{1}{\pi}\int_0^\infty \frac{y_t}{y_t^2 + (x_t - s)^2}\, ds = \frac{1}{\pi}\left(\frac{\pi}{2} - \arctan\left(-\frac{x_t}{y_t}\right)\right)$$

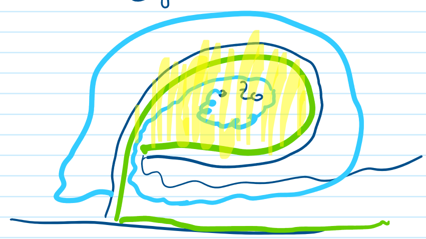
$\downarrow \iff \frac{x_t}{y_t} \to \infty$

1

$\boxed{\text{Case } 4 < \kappa < 8}$ $T(z_0) < \infty$ a.s.

z_0 is in unbounded comp. of $\mathbb{H} \setminus \gamma[0,t]$ $\forall t < T(z_0)$

z_0 is in a bdd comp of $\mathbb{H} \setminus \gamma([0, T(z_0)])$



W_s stopped when it hits $\gamma[0,t]$ or $\mathbb{R}$

γ is to the left $\iff$ clockwise loop

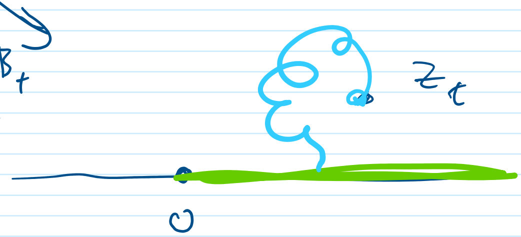
$\mathbb{P}[W_s$ hits the right side of $\gamma[0,t] \cup \mathbb{R}_+]$

$\Leftrightarrow$ clockwise loop

$\mathbb{P}[W_s$ hits the right side of $\gamma[0,t] \cup \mathbb{R}_+]$

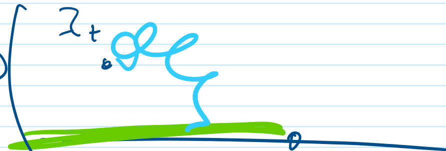
$\to \mathbb{P}[W_s$ hits the right side of $\gamma[0,\infty]] = 1$

$g_t - \sqrt{\kappa} B_t$

$\mathbb{P}\Big($  $\Big) \longrightarrow 1 \iff \frac{x_t}{y_t} \to \infty$

$\mathbb{P}\Big($  $\Big) \longrightarrow 1$

$=$

$\mathbb{P}\Big($  $\Big)$

$\iff \frac{x_t}{y_t} \to -\infty$

$\blacksquare$

Proof [Schramm's formula]

$$dx_t = -\frac{2x_t}{x_t^2+y_t^2}\,dt - \sqrt{\kappa}\,dB_t \qquad dy_t = -\frac{2y_t}{x_t^2+y_t^2}\,dt$$

$$w_t = \frac{x_t}{y_t}, \quad \text{Itô} \quad dw_t = -\frac{\sqrt{\kappa}\,dB_t}{y_t} + \frac{4w_t\,dt}{x_t^2+y_t^2}$$

Time change $ds(t) = \frac{dt}{y_t^2}$ $\quad s(t) = \int_0^t \frac{dt}{y_t^2}$

$\tilde{B}_s = \int_0^t \frac{dB_t}{y_t}$ $\leftarrow$ this is a BM

$$\boxed{dw_s = -\sqrt{\kappa}\,d\tilde{B}_s + \frac{4w_t}{1+w_t^2}\,ds}$$

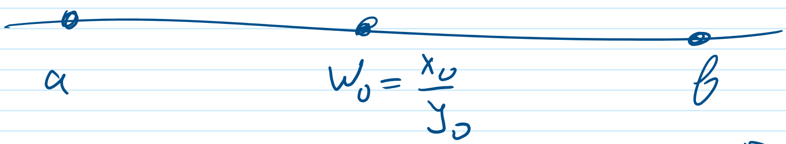
Note: $y_t \searrow$, $y_t > 0 \Rightarrow s(t) < \infty$ if $t < T(z_0)$

Note • $y_t \searrow$. $y_t > 0$ $\Rightarrow$ $s(t) < \infty$ iff $t < T(z_0)$

• a.s $w \to \pm\infty$ as $t \to T(z_0)$

$\Rightarrow$ $s(t) \to \infty$ as $t \nearrow T(z_0)$

Take $a < w_0 < b$.



Define $h(w_0) = h_{a,b}(w_0) = \mathbb{P}[\, w_t$ hits b before $a\,]$

$h(W_s)$ is a martingale.

assume that $h \in C^2$

Itô $\Rightarrow$ $\begin{cases} \frac{\kappa}{2} h''(w) + \frac{4w}{w^2+1} h'(w) = 0 \\ h(a) = 0, \quad h(b) = 1 \end{cases}$

The unique solution $\tilde{h}(w) = \dfrac{f(w) - f(a)}{f(b) - f(a)}$ where

$$f(w) = {}_2F_1\left(\tfrac{1}{2}, \tfrac{4}{\kappa}, \tfrac{3}{2}, -w^2\right) w$$

$\tilde{h}(W_s)$ is a loc. mart + bounded $\Rightarrow$ martingale.

Optional stopping $\Rightarrow$ $\tilde{h}(W_s) = h(W_s)$

When $\kappa < 8$ then $\displaystyle\lim_{w \to \pm\infty} f(w) = \pm \frac{\sqrt{\pi}\, \Gamma\left(\frac{8-\kappa}{2\kappa}\right)}{2\Gamma(4/\kappa)}$

$\Rightarrow$ $\displaystyle\lim_{b \to \infty} h_{a,b}(w) > 0$ $\quad \forall\, w > 0$

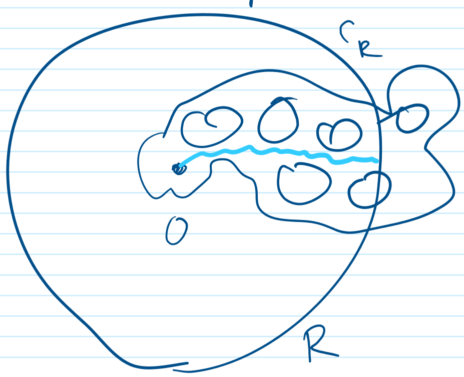
$\Rightarrow$ W_s is transient

$$\mathbb{P}\left[\lim W_s = +\infty\right] = \frac{f(w_0) - f(-\infty)}{f(\infty) - f(-\infty)}$$

$$\mathbb{P}\left[\lim_{s\to\infty} W_s = +\infty\right] = \frac{f(W_0) - f(-\infty)}{f(\infty) - f(-\infty)}$$ ☒

One-arm exponent

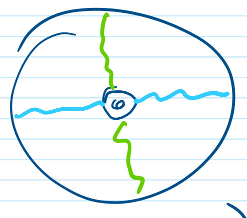
Consider percolation cluster containing the origin

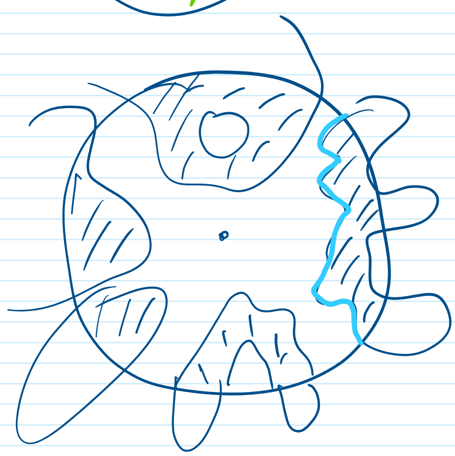


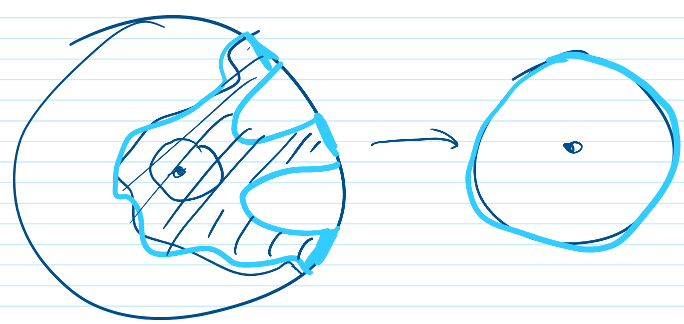
$$\mathbb{P}(0 \leftrightarrow C_R) = R^{-5/48 + o(1)}$$

one-arm exponent

$$\mathbb{P}\Big(\quad\Big) = R^{-(4\text{-arm exponent}) + o(1)}$$



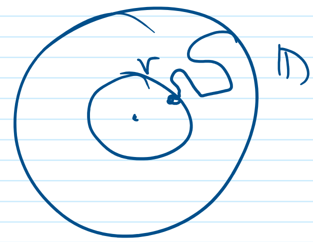




Th For every $\kappa > 4$ there is $c > 0$ s.t. the radial SLE_κ path γ for $\forall r \in (0,1)$ satisfies

$$c^{-1} r^{\lambda} \le \mathbb{P}\left[\gamma[0, T_r] \text{ contains no counter-clockwise loop around } 0\right] \le c\, r^{\lambda}$$

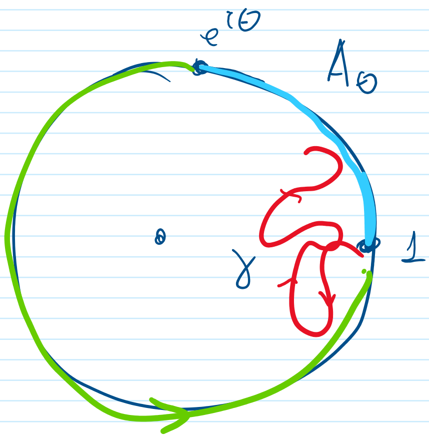
$c\, r^{\lambda} \le \mathbb{P}\big[\, \gamma[0, T_r]$ contains no counter-clockwise loop around $0 \,\big] \le c\, r^{\lambda}$



$$T_r = \inf\{ t : |\gamma(t)| = r \}$$

$$\lambda = \frac{\varkappa^2 - 16}{32\varkappa}$$

Proof



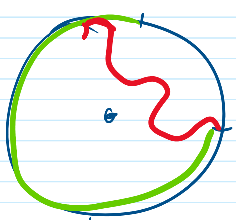
at time t dist 0 to K_t is $\asymp e^{-t}$ ($g_t'(0) = e^{t}$)

(Koebe 1/4 theorem)

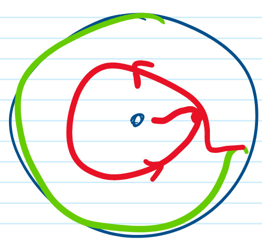
$\mathbb{P}(\gamma[0,t]$ has no ccw loops$)$ $\asymp e^{-\lambda t}$

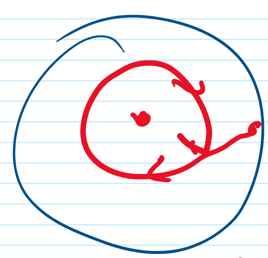
$$\gamma_\theta^t = (\mathbb{T} \setminus A_\theta) \cup \gamma([0,t])$$

$E(\theta,t)$ = event that γ_θ^t contains no counter-clockwise loop around θ



counter clockwise loop

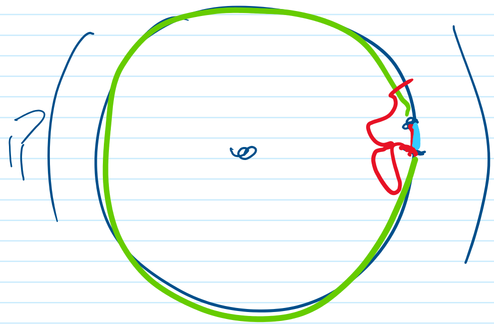




$\in E(\theta, t)$

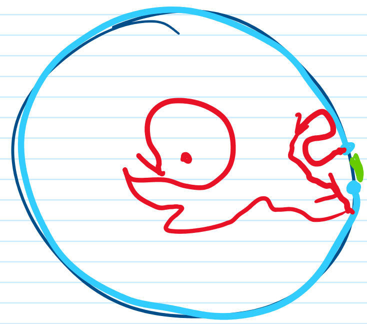
Define $h(\theta,t) = \mathbb{P}[E(\theta,t)]$

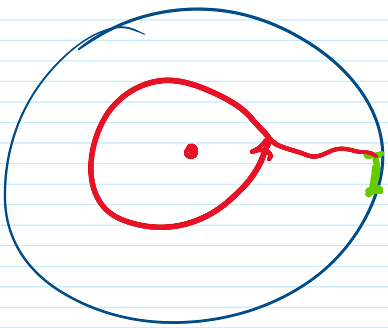
Claim
- $h \in C^2$
- $h(\theta,t) \to 0$ as $\theta \to 0$ $\quad \forall t > 0$
- $\partial_\theta h(2\pi, t) = 0$ $\quad \forall t$



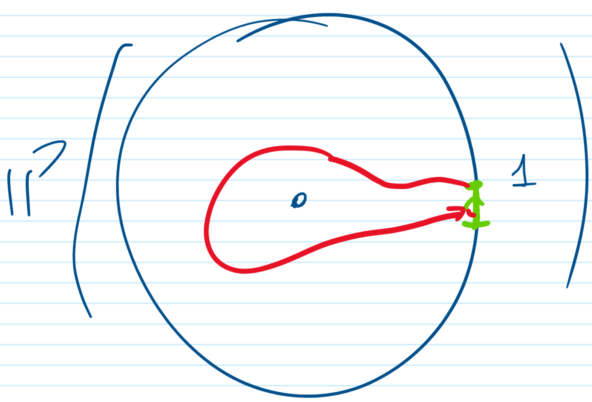
$\approx 1 \quad \Rightarrow \quad \exists$ counter-clockwise loop with $\mathbb{P} \approx 1$

$$\lim \frac{h(2\pi,t) - h(\theta,t)}{2\pi - \theta}$$

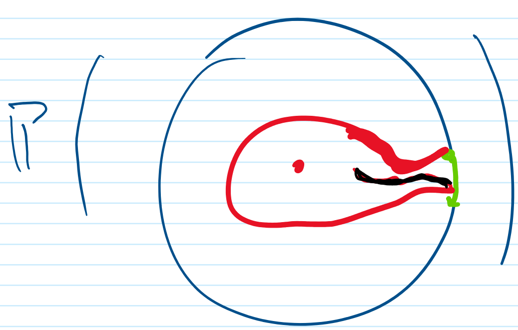




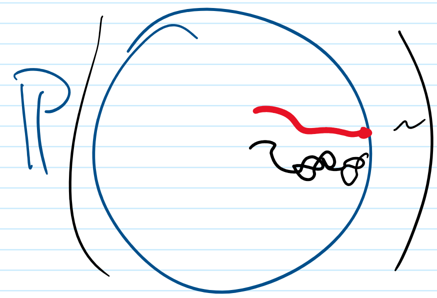
$\leftarrow$ counts in both $E(2\pi,t)$, $E(\theta,t)$



$\approx (2\pi - \theta)$



$\approx (2\pi - \theta) \cdot$



$\sim O(1)$

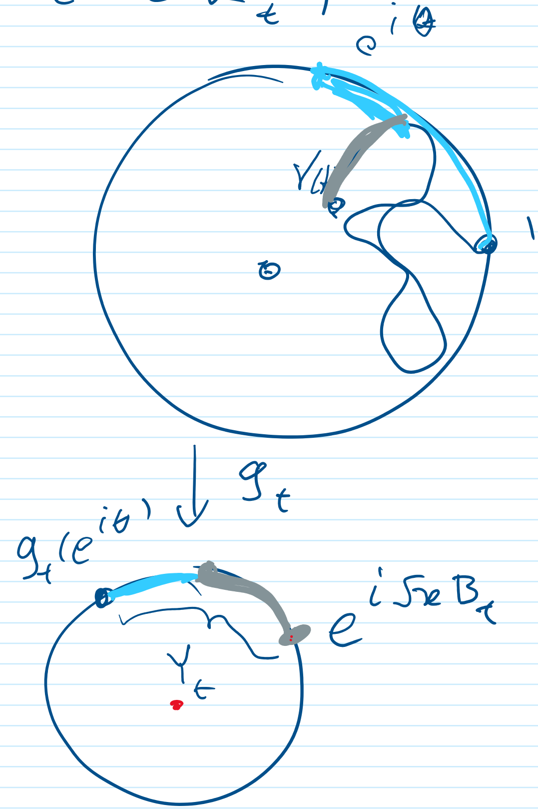
$T(\theta)$ the first time $e^{i\theta}$ is separated from 0 (time when $e^{i\theta} \in K_t$) $e^{i\theta}$

from 0 (time when $e^{i\theta} \in K_t$)

$t < T(\theta)$

$$Y_t^\theta = \underline{-i \log g_t(e^{i\theta}) - \sqrt{\kappa} B_t}$$

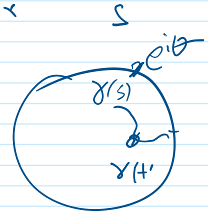
$= 2\pi$ harmonic measure of



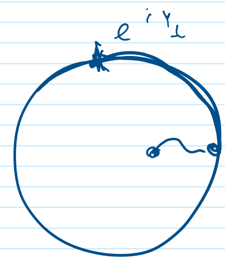
= arclength of the image under g_t of A_θ and the right side of $\gamma[0,t]$

Fix s assume that $t < T \wedge s$

$\mathbb{P}($



$\mid \mathcal{F}_t)$



$E(\theta, s)$

$g_t e^{-i\sqrt{\kappa}B_t}$

$h(Y_t^\theta, s-t)$ this is a martingale on $t < T \wedge s$

Radial SLE

$$\partial_t g_t(z) = -g_t(z) \frac{g_t(z) + e^{i\sqrt{\kappa}B_t}}{g_t(z) - e^{i\sqrt{\kappa}B_t}}$$

Itô $\Rightarrow$ $dY_t^\theta = \cot\left(\frac{Y_t}{2}\right) dt - \sqrt{\kappa}\, dB_t$

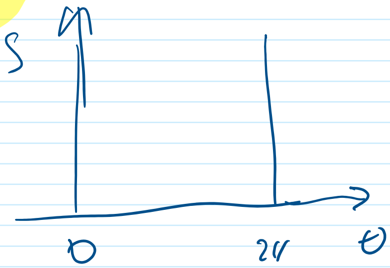
Itô $\Rightarrow$ $dh(Y_t^\theta, s-t) = \left[\frac{\kappa}{2}\partial_y^2 h + \cot\left(\frac{Y}{2}\right)\partial_y h - \partial_s h\right] dt$

Itô $\Rightarrow$ $dh(Y_t^\theta, s-t) = \left[\frac{\kappa}{2}\partial_\theta^2 h + \cot\left(\frac{Y}{2}\right)\partial_\theta h - \partial_s h\right] dt$

$$- \sqrt{\kappa}\, \partial_\theta h \, dB_t$$

Martingale $\Rightarrow$ drift is 0 $\forall t$, use $t=0$ $\left[Y_t^\theta \big|_{t=0} = \theta \right]$

$$(\star) \quad \begin{cases} \frac{\kappa}{2} h_{\theta\theta}(\theta, s) + \cot\left(\frac{\theta}{2}\right) h_\theta(\theta, s) - h_s(\theta, s) = 0 \\ h(0, s) = 0 & \forall s \\ h_\theta(2\pi, s) = 0 & \forall s \end{cases}$$



We want to understand how solutions of $(\star)$ ~~xx~~ behave as $s \to \infty$

Two approaches

① $\boxed{\text{PDE}}$. Consider $H(\theta, t) = \left(\sin\frac{\theta}{4}\right)^q e^{-\lambda t}$

$$\lambda = \frac{\kappa^2 - 16}{32\kappa} \qquad q = \frac{\kappa - 4}{\kappa}$$

H is a positive solution of $\Lambda H = 0$ + boundary cond

$$\Lambda = \underbrace{\frac{\kappa}{2}\partial_\theta^2 + \cot\frac{\theta}{2}\,\partial_\theta}_{\hat{\Lambda}} - \partial_t$$

$\left(\sin\frac{\theta}{4}\right)^q$ is a positive e.f of $\hat{\Lambda}$ with e.v λ [the main e.f]

$$\sum a_i \varphi_i(\theta) e^{-\lambda t} \sim a_1 \varphi_1(\theta) e^{-\lambda_1 t}$$

Maximum Principle for parabolic PDE

$\Rightarrow$ $h(\theta, t) \asymp H(\theta, t)$ [There are some complications. Coefficients have sing.

$\Rightarrow \quad h(\theta,t) \asymp H(\theta,t)$ [There are some complications. Coefficients have sing.

② Optional stopping

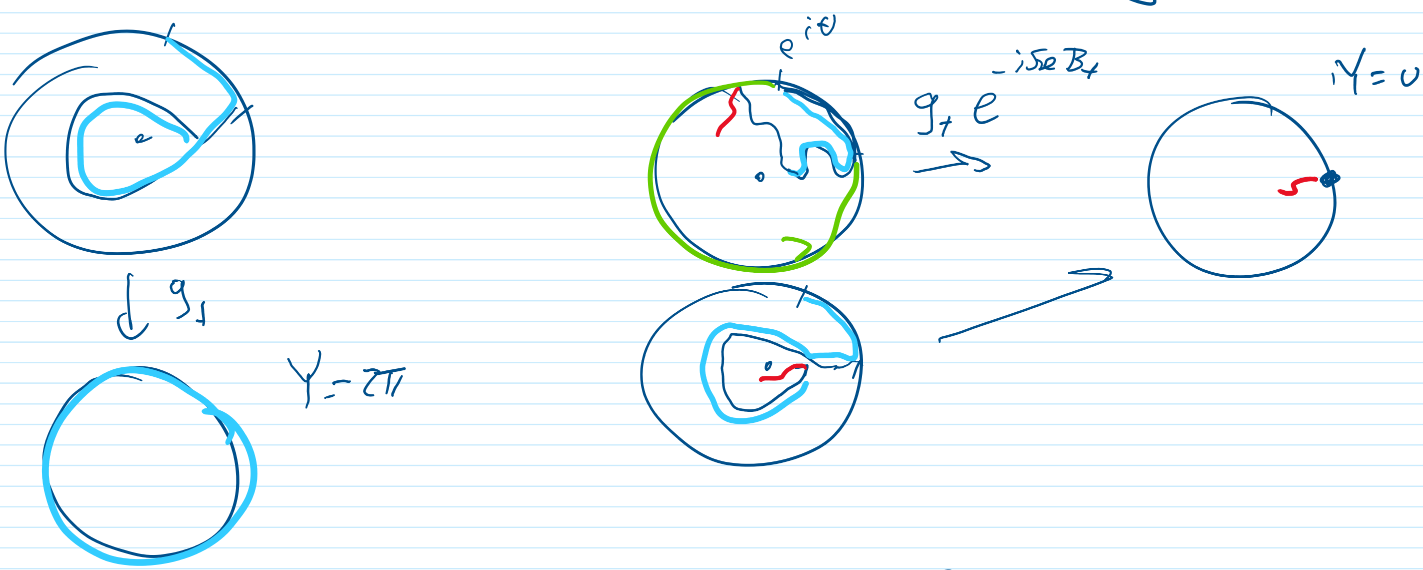
Consider $Z_t = \left(\sin \frac{Y_t^\theta}{4}\right)^q e^{\lambda t}$

Explicit computation $\Rightarrow$ Z_t is a local mart.

Z_t is unif bounded on every $[0, t_0]$

$\Rightarrow$ Z_t is a martingale

$$\left(\sin\frac{\theta}{4}\right)^q = Z_0 = \mathbb{E} Z_t = e^{\lambda t}\, \mathbb{E}\left[\left(\sin\frac{Y}{4}\right)^q \,\middle|\, \inf_{s<t} Y_s > 0\right]$$
$$\times\, \mathbb{P}[\inf Y_s > 0]$$



Claim $\mathbb{P}\left[Y \in [\tfrac{\pi}{2}, 2\pi] \,\middle|\, \inf Y > 0\right] > 0$

$\Rightarrow \mathbb{E}\left[\left(\sin\frac{Y}{4}\right)^q \,\middle|\, \inf Y > 0\right] \asymp 1$

$\Rightarrow \mathbb{P}[\inf Y > 0] \asymp \left(\sin\frac{\theta}{4}\right)^q e^{-\lambda t}$

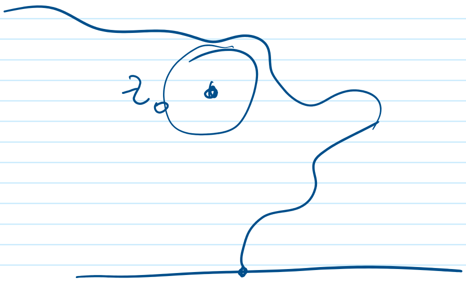
$\qquad\qquad \shortparallel$

$\qquad h(\theta, t)$

$\boxtimes$

$h(\theta, t)$

$\blacksquare$



$\kappa < 8$

$$\mathbb{P}(B(z_0,\varepsilon) \cap \gamma \neq \emptyset) \asymp \left(\frac{\varepsilon}{\operatorname{Im} z_0}\right)^{1-\frac{\kappa}{8}} \left(\sin(\arg z_0)\right)^{8/\kappa - 1}$$

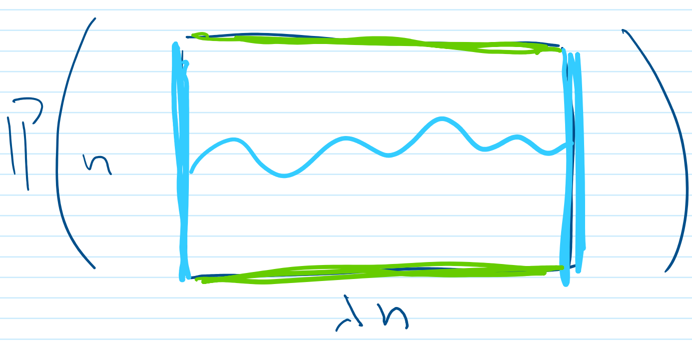
<u>Corollary</u> $\dim_H \gamma \leq 2 - \left(1 - \frac{\kappa}{8}\right) = \frac{\kappa}{8} + 1$ a.s

<u>Th</u> $\dim_H \gamma = \frac{\kappa}{8} + 1$ a.s

Towards conformal invariance

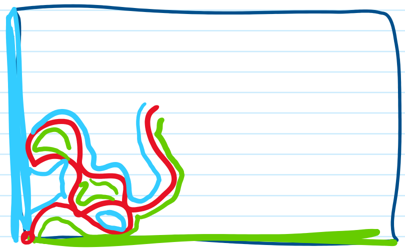
- Prove some weak estimates that imply that interfaces are not too wild.
  Perc: certain observables are Hölder functions.
  $\Rightarrow$ interfaces have subsequential scaling limits
- [Show that one observable has a conf. inv. scaling limit.] There are subseq. limits. Identify this limit $\Rightarrow \exists$ limit and we know it
  This limit is a solution of some equation
  $\Rightarrow$ it is conf. invariant
- γ_δ describe γ_δ in terms of LE
  $\gamma_\delta \rightsquigarrow u_\delta$
  using an observable $\Rightarrow \lim u_{\delta_n} = BM$
  $\Rightarrow \gamma_\delta \to \gamma = SLE$

Crossing probability

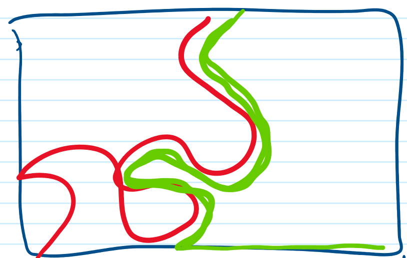


$$= f(\lambda, n) \xrightarrow[n\to\infty]{} f(\lambda)$$

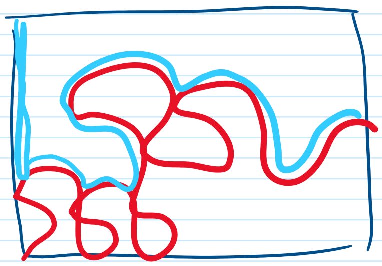
green vertical connection



Or

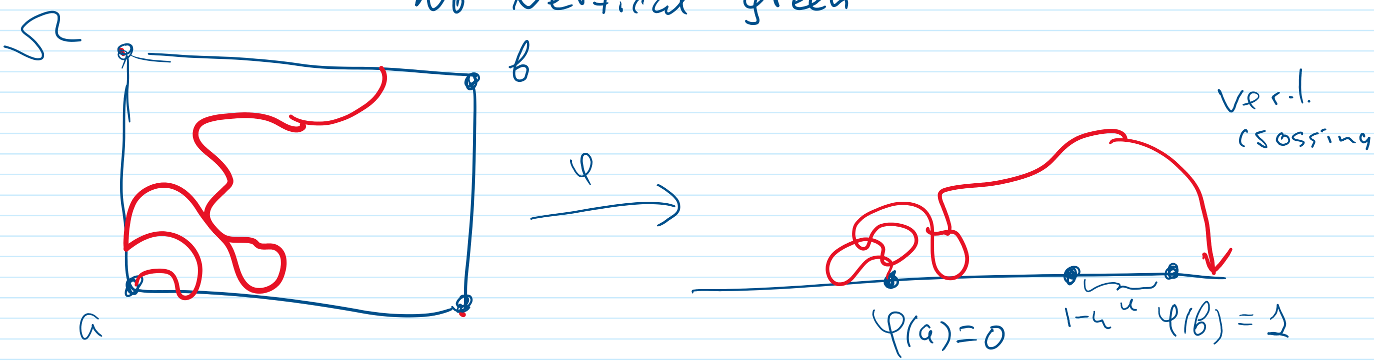


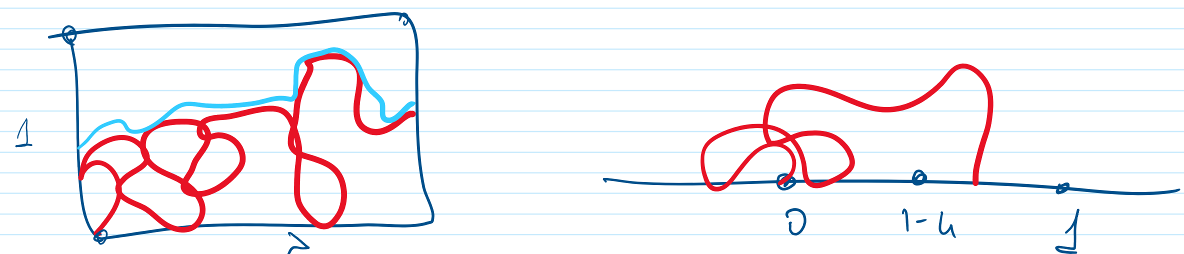
no horizontal Blue connection



There is horizontal Blue connection

No vertical green





$$\left\{\begin{array}{l}\text{Vertical green}\\ \text{crossing}\end{array}\right\} = \left\{\begin{array}{l}SLE(6)\ \text{in}\ \mathbb{H}\\ \text{hits}\ [1,\infty)\ \text{before}\ [1-u,1]\end{array}\right\}$$

$$= \{ T(1-u) = T_1 \}$$

$$\left\{\begin{array}{l}\text{Horizontal blue}\\ \text{crossing}\end{array}\right\} = \{ SLE(6)\ \text{hits}\ [1-u,1] \}$$

$$= \{ T(1-u) < T(1) \}$$

$4 < \kappa = 6 < 8$

$$\mathbb{P} = \boxed{1 - \frac{\Gamma(\frac{8}{\kappa}-1)\,\Gamma(\frac{\kappa-4}{\kappa})}{\Gamma(\ldots)} \int^{1-u} (1-x)^{\frac{8}{\kappa}-2}\, x^{-\frac{4}{\kappa}}\, dx}$$

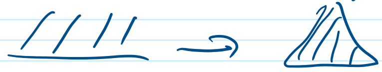
$$\mathbb{P} = \boxed{1 - \frac{\Gamma\left(\frac{8}{\kappa} - 1\right)\Gamma\left(\frac{\kappa-4}{\kappa}\right)}{\Gamma\left(\frac{\kappa}{4}\right)} \int_0 (1-x)^{\cdots} x^{\cdots}\, dx}$$

in terms of u, but u is a function of λ

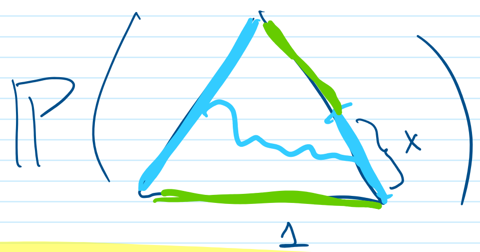
$\boxed{\text{Cardy's formula}}$ Using Conformal Field Theory

Carleson: This expression $\dfrac{\Gamma(2/3)}{\Gamma(1/3)\Gamma(4/3)} u^{1/3} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, u\right)$

is a conf map



Carleson-Cardy formula

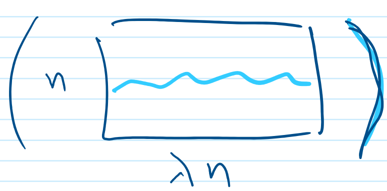


$= x$

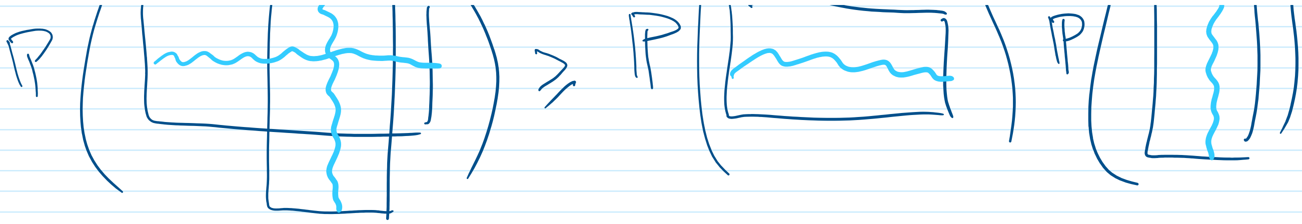
Russo-Seymour-Welsh estimate

RSW $\quad 0 < c < \mathbb{P}\left(n \right) < 1 - c < 1$

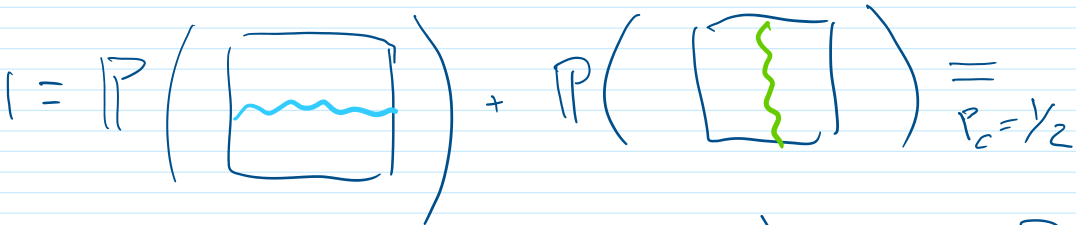
$\exists c = c(\lambda) > 0$

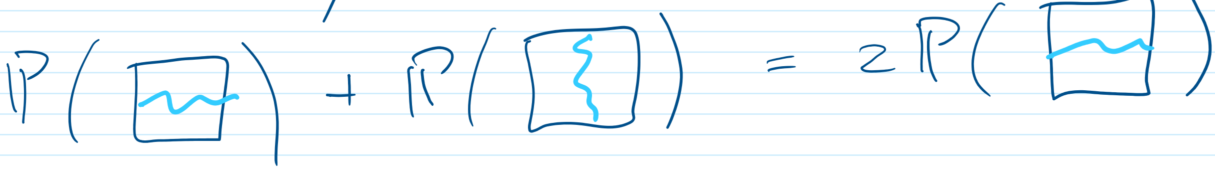


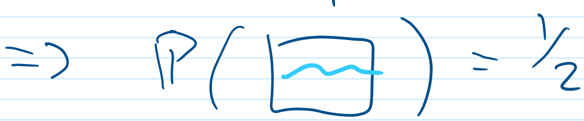
Crossing events are positively correlated

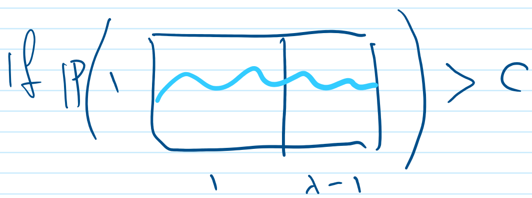


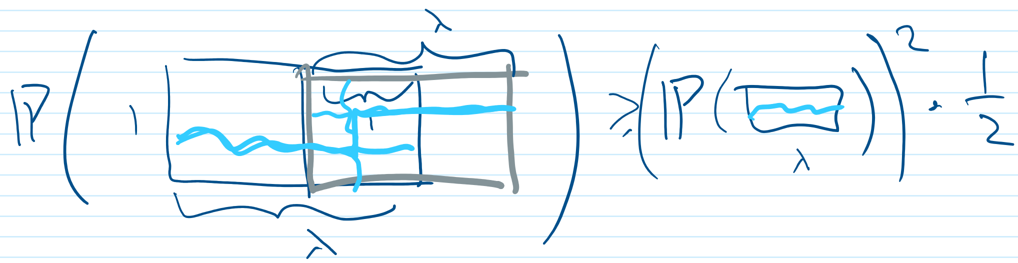
RSW for some $\lambda \neq 1$ then we have SSW for all λ

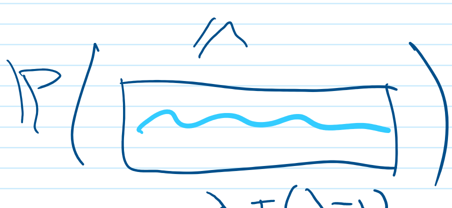


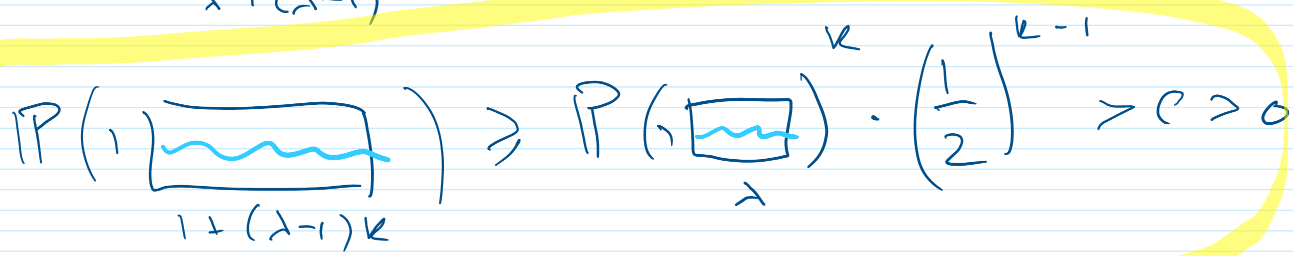


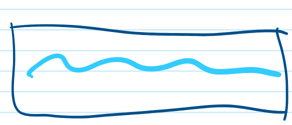






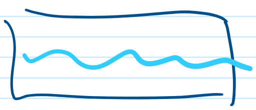




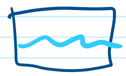
$\mathbb{P}\Big($  $\Big)$

$\lambda+(\lambda-1)$

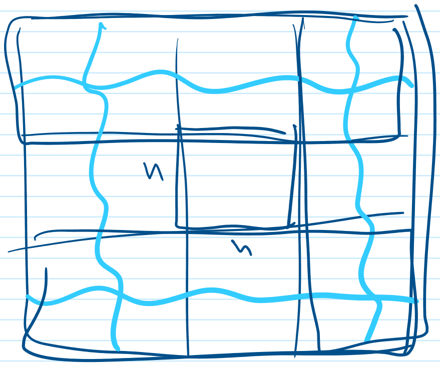
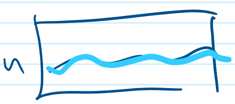
$$\mathbb{P}\Big(1\ \text{}\Big) \geq \mathbb{P}\big(1,\ \cdot\ \big)^{k}\cdot\left(\frac{1}{2}\right)^{k-1} > c > 0$$



$1+(\lambda-1)k$

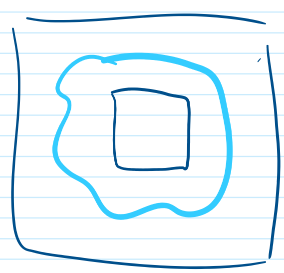


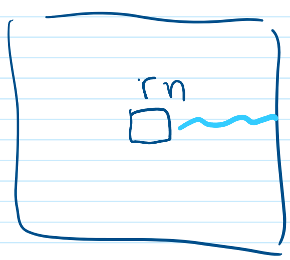
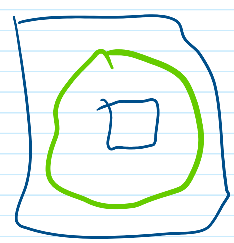
λ

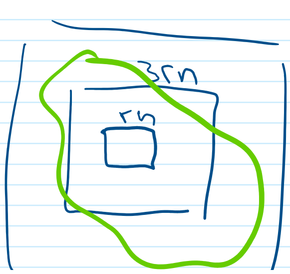
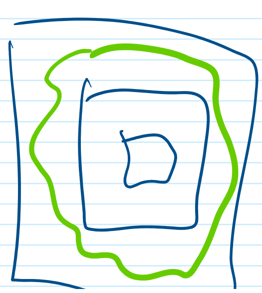
$\mathbb{P}\ 3n\ \Big($  $\Big) \geq \Big(\mathbb{P}\ n$  $\Big)^{4} > c$

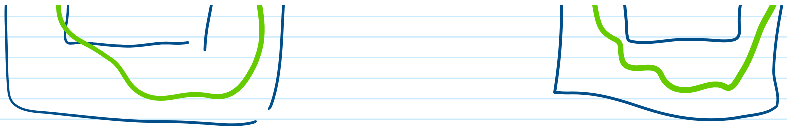
$3n$ $\qquad$ $3n$

rn

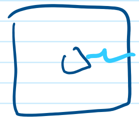
$\mathbb{P}$ 

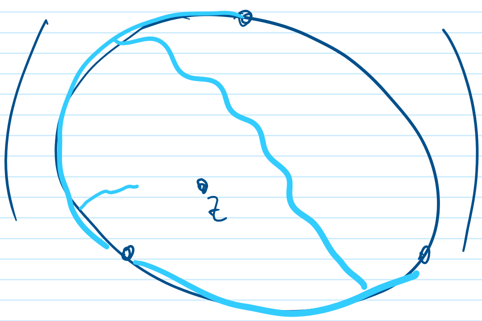
$\mathbb{P}\Big($  $\Big) = 1-\mathbb{P}\Big($  $\Big)$

$\mathbb{P}\Big($  $\Big) \geq \mathbb{P}\Big($  $\Big) \geq 1-c^{\#\text{ layers}}$



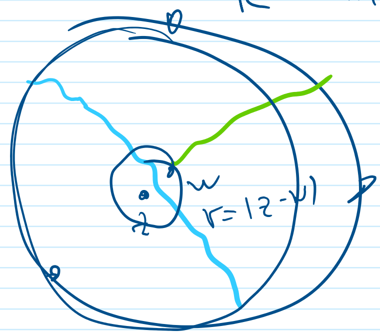
$$3^N \approx \frac{R}{r} \Rightarrow N \approx \ln R/r \qquad \geq 1 - C^{\ln R/r + \dots} \approx 1 - \left(\frac{R}{r}\right)^{-\alpha}$$

$$\mathbb{P}\Big(\ \ $$  $$\Big) \leq \left(\frac{R}{r}\right)^{-\alpha}$$

$$\mathbb{P}\Big(\ \ $$ 

$$\Big) = F(z)$$

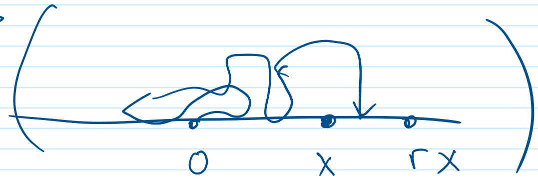
$R = \mathrm{dis}(z, \partial\Omega)$

$$F(z) - F(w) = $$ 

$$\leq \left(\frac{R}{r}\right)^{-\alpha}$$

$\Rightarrow$ F is Hölder with exp. α

<u>Convergence of one observable</u> $\Rightarrow$
<u>conv. to SLE</u>

$$\mathbb{P}\Big(\ \ $$ 

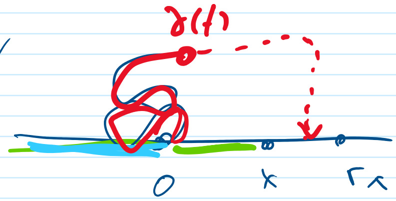
$$\Big) = \boxed{F(r)}$$ ← explicit function (in terms of ${}_2F_1$)

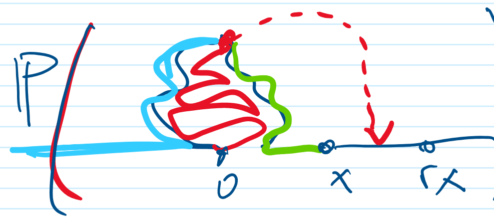
describe it in terms of LE

mesh = δ

describe it in terms of LE

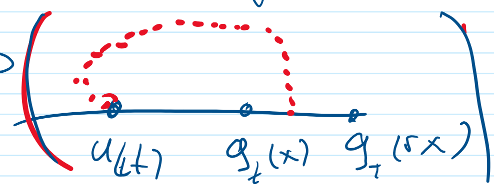
$\xrightarrow{g_{t,\delta}}$ $u_\delta(t)$ ← driver

$\mathbb{P}\Big($ 

$\Big| \; g_{t,\delta} \Big) \underset{\text{Domain Markov}}{=} \mathbb{P}\Big($ 

$\Big)$

$\downarrow \; \delta \to 0$

$$F\left(\frac{g_t(rx)-u(t)}{g_t(x)-u(t)}\right) = \mathbb{P}\Big($$ 

$\Big)$

Take $\mathbb{E}$ w.r.t law of g_t

$$F(r) = \mathbb{E}\, F\left(\frac{g_t(rx)-u(t)}{g_t(x)-u(t)}\right)$$

(expand g_t at ∞ $\quad g_t(x) = x + \frac{2t}{x} + \dots$

33 F at r)

$$F\left(\frac{g_t(rx)-u(t)}{g_t(x)-u(t)}\right) = F(r) + c_1\left(\frac{r-1}{r^2}\right)^{1/3}\frac{u(t)}{x} +$$

$$+ c_2\,\frac{(r+1)(r-1)^{1/3}}{r^{5/3}}\left(u(t)^2 - 6t\right)\frac{1}{x^2} + \dots$$

Take $\mathbb{E}$

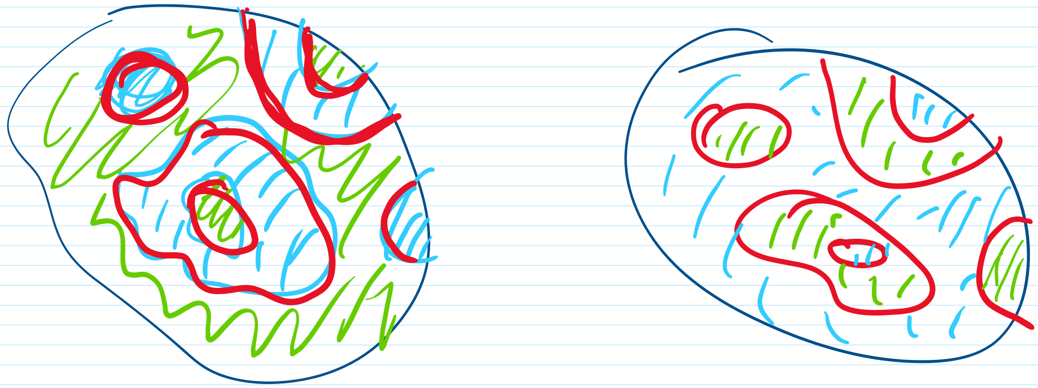
$$\mathbb{E}\,u(t) = 0\,, \qquad \mathbb{E}\left(u(t)^2 - 6t\right) = 0$$

D. Markov propert the same is true for

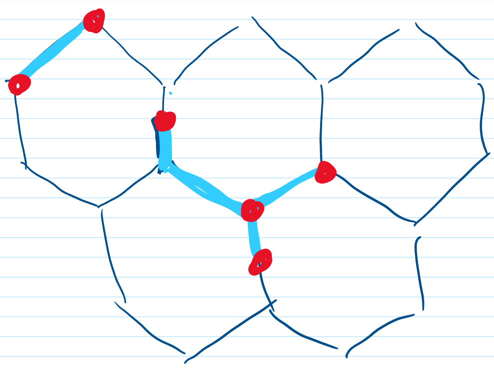
$$\mathbb{E}\left(u(t)-u(s)\right) = 0 \qquad \mathbb{E}\left[\left(u(t)-u(s)\right)^2 - 6(t-s)\right] = 0$$

$$\Rightarrow \boxed{u(t) = \sqrt{6}\, B_t}$$

$$\Rightarrow \boxed{u(t) = \sqrt{6}\, B_t}$$



Ω



Let E be a collection of half-edges then ∂E is the set of all vertices and mid-edges s.t. they touch odd number of half-edges from E

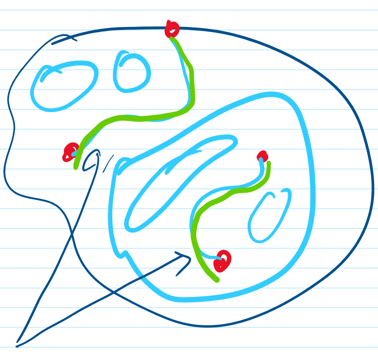
Let $\mathcal{U} = \{u_1, \dots, u_k\}$ of marked mid-edges

$$W_\Omega(u_1, \dots, u_k) = W_\Omega(\mathcal{U}) = W(\mathcal{U})$$

$$= \text{collection of all } \underbrace{E \text{ s.t. } \partial E = \mathcal{U}}_{\text{configuration with disorders at } u_i}$$

$W = \emptyset$ if k is odd

$IP(E)$ = Important part of E = union of all connected components of E containing at least one u_i

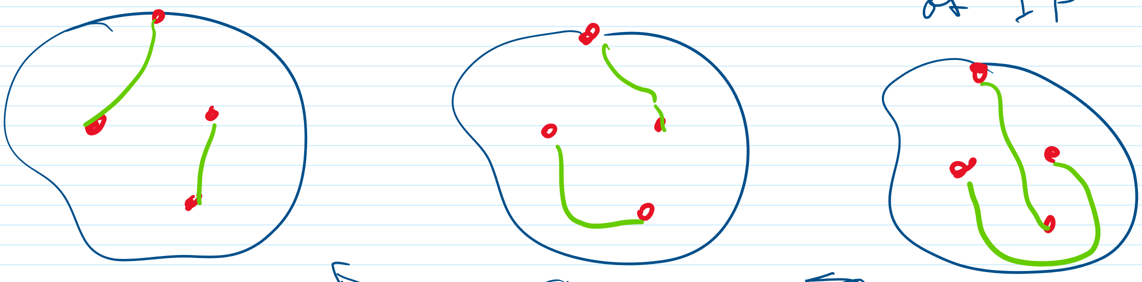


$IP(E)$

at least one u_i

$IP(E)$

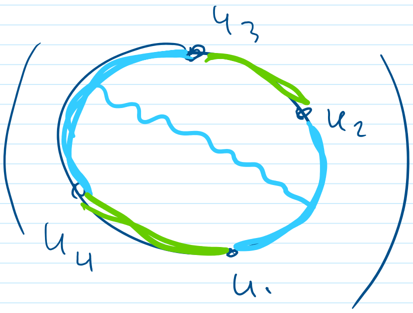
Connectivity pattern = homotopy class of IP

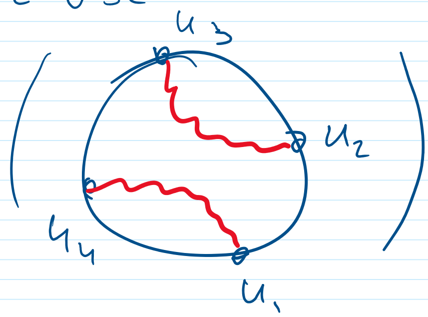


all possible connectivity patterns.

$\mathbb{P}^{loop}$ = uniform measure on $W_\Omega(u_1,\dots,u_k)$

Lemma $u_1,\dots,u_n \in \partial\Omega$

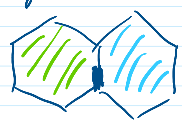
$\mathbb{P}^{perc}\Big($ 

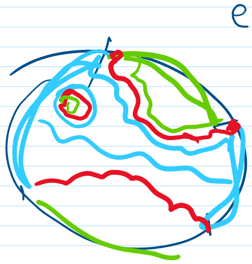
$\Big) = \mathbb{P}^{loop}\Big($ 

$\Big)$

Proof Given a percolation config

$\exists$ natural map to loop config.

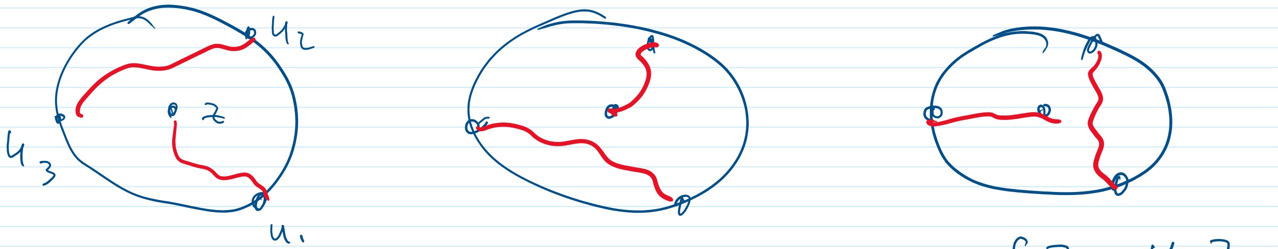
half-edge $e \in E$ if $e \notin \partial\Omega$ 



Claim This is a measure preserving config.

Proof this is a bijection $\blacksquare$

Discrete Holomorphicity



$[z \leftrightarrow u_1]$ $[z \leftrightarrow u_2]$ $[z \leftrightarrow u_3]$

Define $\tau = e^{2\pi i/3}$

Fix u_1, u_2, u_3, think that z is a variable

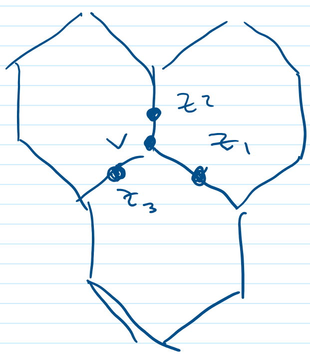
$F = F_{u_1,u_2,u_3}(z)$ function on mid-edges

$$F(z) = \mathbb{E}^{loop}\, H(E) = \sum_{k=1}^{3} \tau^k H_k(z)$$

$$H(E) = \sum \tau^k \mathbb{1}_{[z \leftrightarrow u_k]}$$

$$H_k(z) = \mathbb{P}^{loop}\, [z \leftrightarrow u_k]$$

Lemma



Then $\sum_{k=1}^{3} \tau^k F(z_k) = 0$

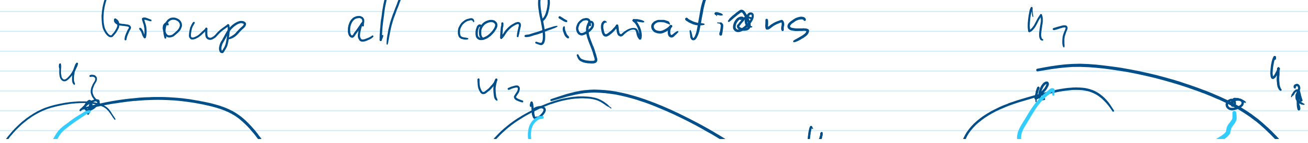
[This is a discrete version of Cauchy-Riemann equations

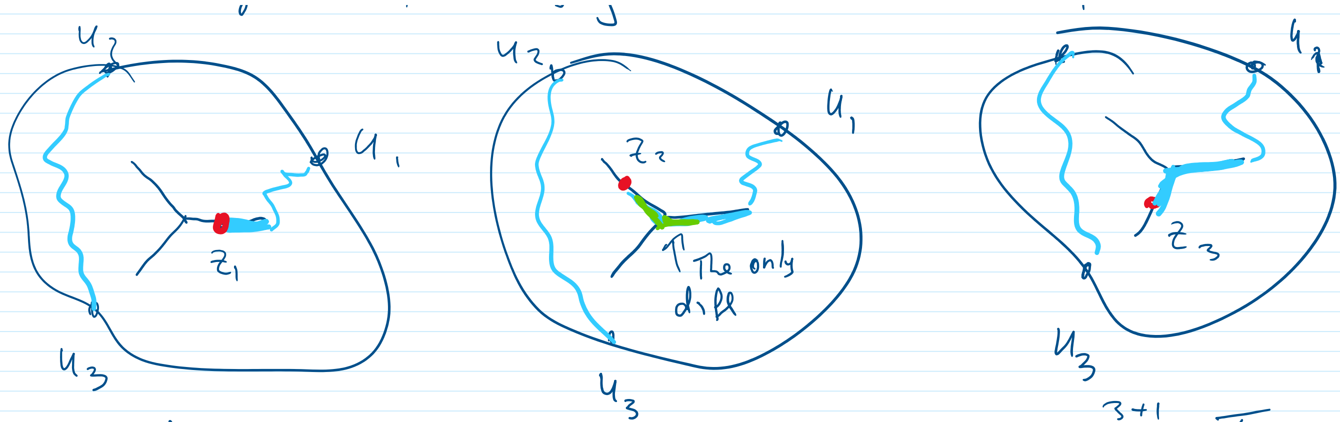
Proof $\sum \tau^k F(z_k) = \sum \tau^k \sum \tau^j H_j(z_k)$

$$= \sum_{k,j} \tau^{k+j}\, \mathbb{P}\,[z_k \leftrightarrow u_j] =$$

$$= c \sum \tau^{k+j} \#\text{ config s.t } z_k \leftrightarrow u_j$$

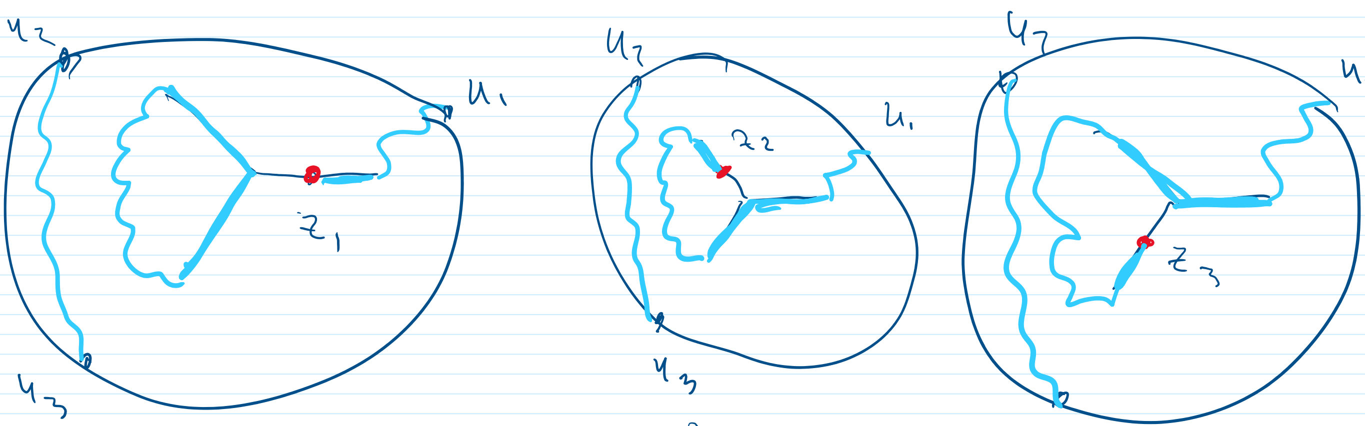
Group all configurations



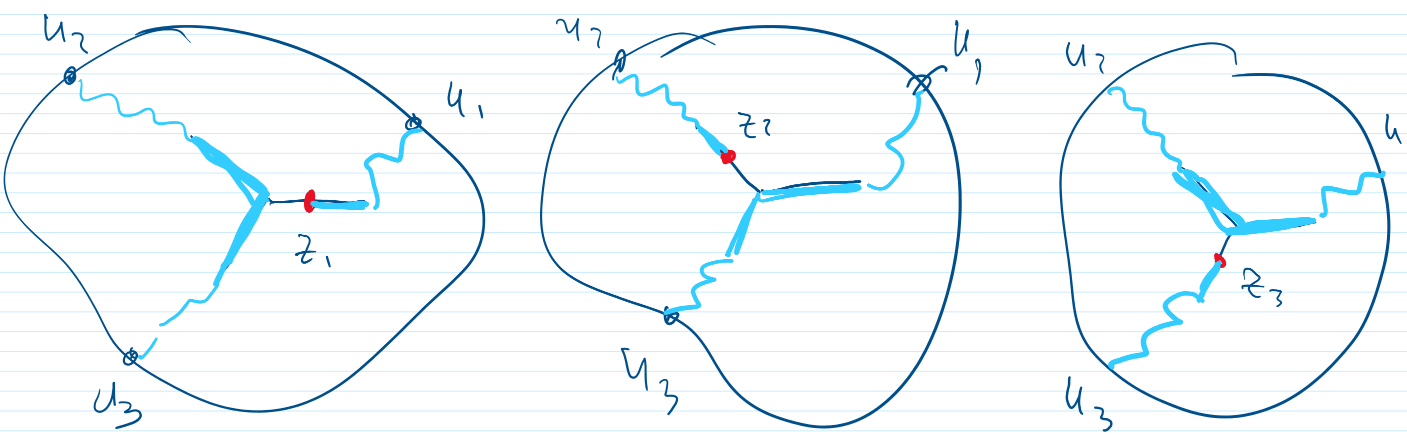


$\tau^{1+1} = \tau^2$ $\qquad$ $\tau^{2+1} = 1$ $\qquad$ $\tau^{3+1} = \tau$

Numbers of these config are the same



$\tau^{1+1} = \tau^2$ $\qquad$ $\tau^{2+1} = 1$ $\qquad$ u_3 $\quad$ $\tau^{3+1} = \tau$



$\tau^{1+1} = \tau^2$ $\qquad$ $\tau^{2+2} = \tau$ $\qquad$ $\tau^{3+3} = 1$

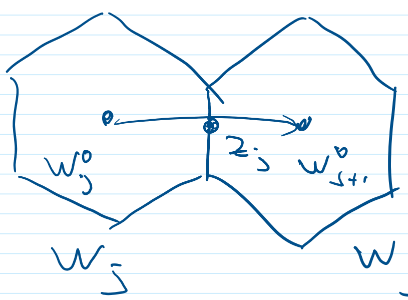
All config can be grouped into triplets and the weighted sum in each triplet $= 0$ $\blacksquare$

Discrete integration

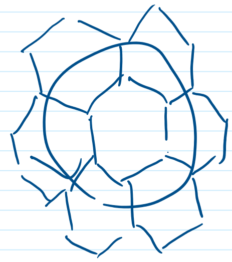
γ is a contour made of hexagons W_j

$W_0 = W_n \quad (W_{k \pm n} = W_k)$

W_j and W_{j+1} share an edge



$w_{j+1}^o - w_j^o$



Let F be a function defined on mid edges

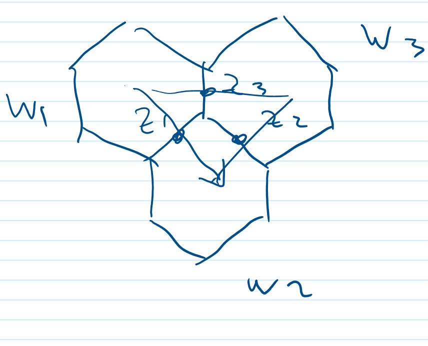
$$\int_\gamma^\# F(z)\, d^\# z = \sum_j F(z_j)(w_{j+1}^o - w_j^o)$$

<u>Th</u> Let F be a function s.t $\forall$ vertex v if z_1, z_2, z_3 are 3 mid-edges around v

$$\sum \tau^k F(z_k) = 0$$

<u>Then</u> $\forall$ contour γ $\quad \displaystyle\int_\gamma^\# F(z)\, d^\# z = 0$

<u>Proof</u> Consider an elementary γ

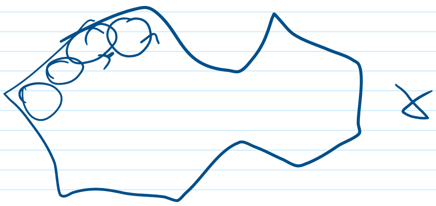


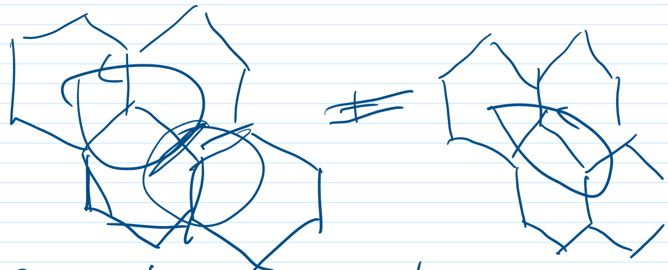
$$\int_\gamma^\# F\, d^\# z = F(z_1)(-\tau) + F(z_2)(-\tau^2) + F(z_3)(-1)$$

$$= -(\tau F(z_1) + \tau^2 F(z_2) + \tau^3 F(z_3))$$

$$= 0$$

C-R equation

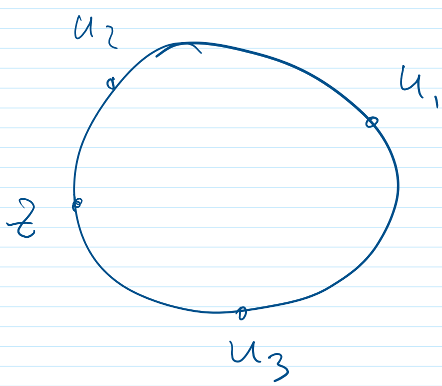




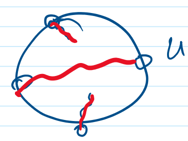
any $\gamma = \Sigma$ of elementary contours

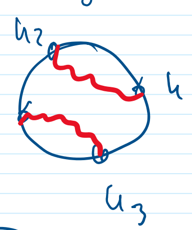
$$\Rightarrow \oint_{\gamma}^{\#} F \, d^{\#} z = 0 \quad \forall \gamma \qquad \blacksquare$$

$$F(z) = \sum \tau^k \, \mathbb{P}[z \leftrightarrow u_k]$$

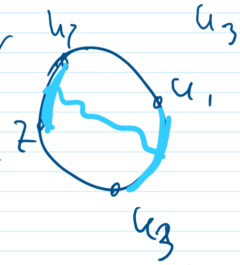


if $z \in (u_2, u_3)$ then

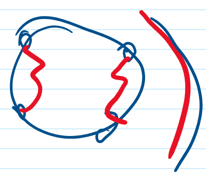
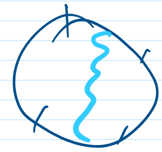
$H_1 = 0$ 

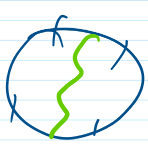
$\mathbb{P}[z \leftrightarrow u_3] = \mathbb{P}\Big($ 

$\Big)$

$= \mathbb{P}^{perc}\Big($ 

$\Big)$

$\mathbb{P}[z \leftrightarrow u_2] = \mathbb{P}^{loop}\Big($  $\Big) = \mathbb{P}^{perc}\Big($  $\Big) =$

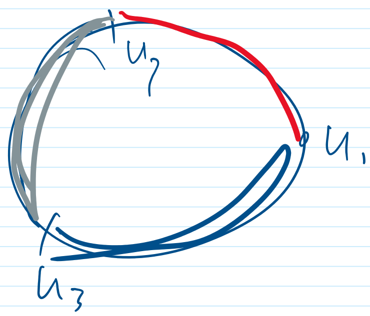
$= \mathbb{P}\Big($  $\Big) = 1 - \mathbb{P}\Big($  $\Big) \qquad \blacksquare$

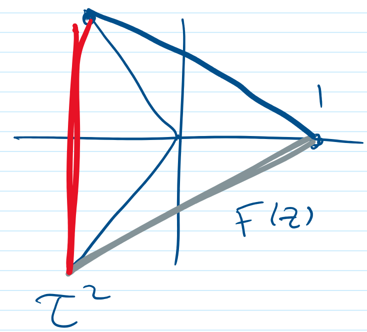
$z \in (u_2, u_3)$ then $F(z) = \boxed{\lambda \tau^2 + (1-\lambda)\tau^3}$

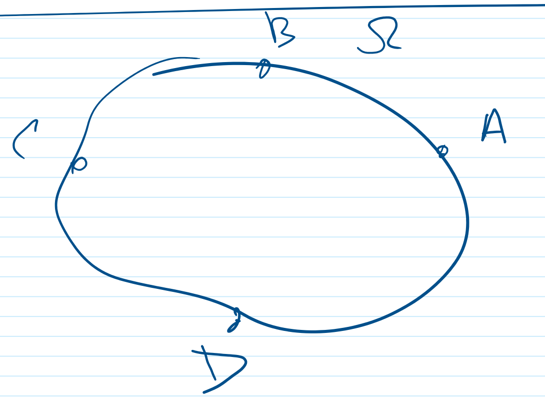
$z \in (u_2, u_3)$ then $F(z) = \lambda \tau + (1-\lambda)\tau$

$z \in (u_3, u_1)$ ——— " " " " "

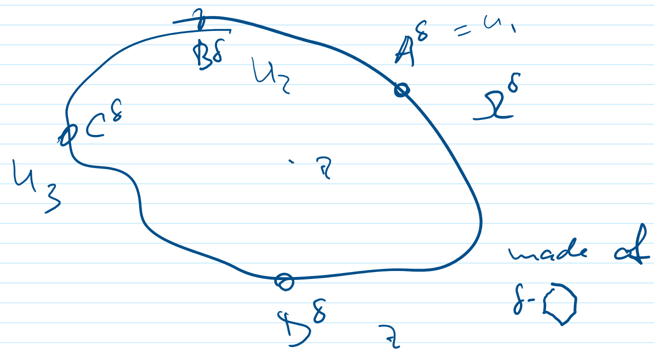
$z \in (u_1, u_2)$ ——— " " " τ







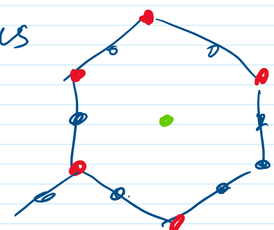
Take small δ, approximate by ⬡ domain with mesh δ



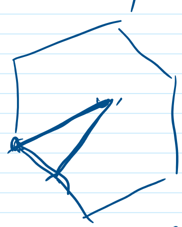
Define $F^{\delta}_{\Omega^\delta, A^\delta, B^\delta, D^\delta}(z)$

Extend to Ω^δ as a domain in $\mathbb{C}$

$f_\delta(\text{vertex})$ = average of values at neighbouring mid-edges



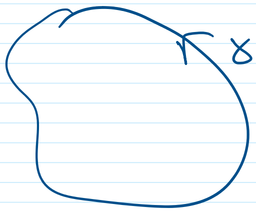
$f(\text{center of a face})$ = average of values of mid-edges

inside  it is an affine function

$\underline{\text{RSW}} \Rightarrow \{f_\delta\}_\delta$ is Hölder bounded.

RSW $\Rightarrow$ $\{f_\delta\}_\delta$ is Holder continuous

Arzela-Ascoli theorem $\Rightarrow$ if $\delta_n \to 0$ then $\exists\, \delta_{n_k} \to 0$ s.t. $f_{\delta_{n_k}} \longrightarrow f$ unif on compacts

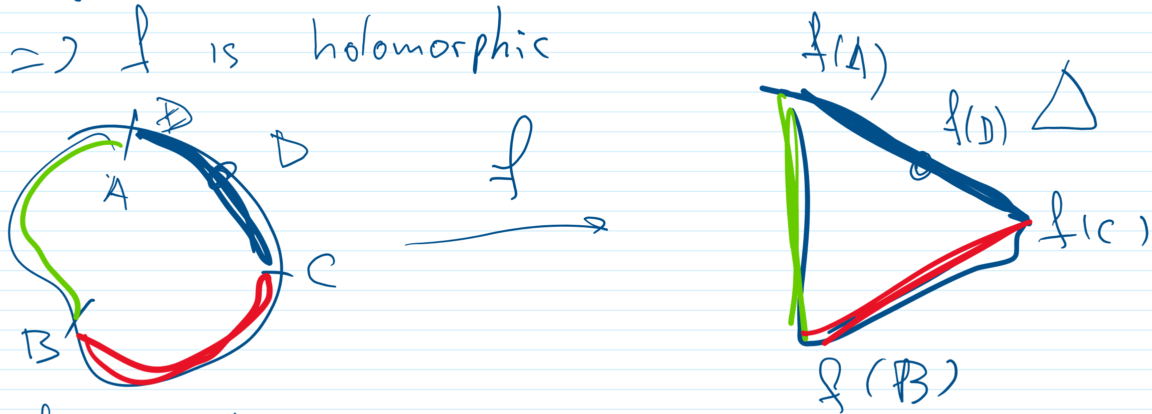


$$\int_\gamma f(z)dz = \int_{\gamma_\delta} f(z)dz + o(1) =$$

$$= \int^{\#}_{\gamma_\delta} \underbrace{F_\delta(z)\, d^{\#}z}_{=0} + o(1) \qquad \text{take } \lim \delta_{n_k} \to 0$$

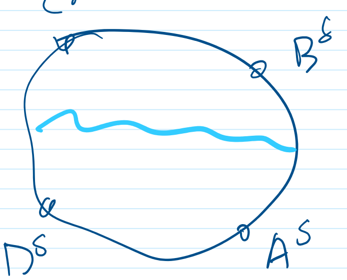
$\int_\gamma f(z)dz = 0 \quad \forall \gamma$. By Morera's theorem

$\Rightarrow$ f is holomorphic



f is hol, $\partial\Omega \to \partial\Delta$

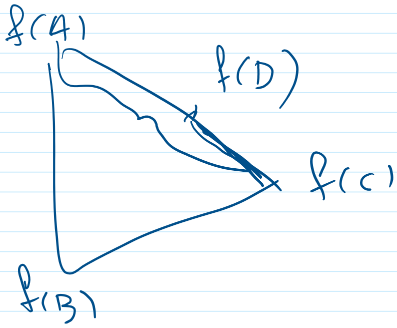
Argument principle implies that f is the conf map from (Ω, A, B, C) onto Δ with marked vertices

$\lim_{\delta_{n_k} \to 0} \mathbb{P}\Big($  $\Big) = \lim \dfrac{\tau^3 - F_\delta(D^\delta)}{\tau^3 - \tau^1}$

$$= \frac{f(C) - f(D)}{[illegible]}$$

$$= \frac{f(C) - f(D)}{f(C) - f(A)}$$

= Carleson - Cardy formula



All conv. subseq conv to the same limit

$\Rightarrow f_\delta \to f$ as $\delta \to 0$ $\blacksquare$