C8.5 Introduction to SLE

Sheet 3

Problem 1.

Properties of Bessel process. Let Z_t be a stochastic process such that

$$dZ_t = \frac{\delta - 1}{2Z_t} dt + dB_t,$$

where $\delta \in \mathbb{R}$ and B_t is the Brownian motion. This process is called the Bessel process of dimension δ. By Z_t^x we denote the Bessel process started at $x > 0$. This process is well defined up to the stopping time

$$T_x = \sup\{t > 0 | \inf_{s \in [0,t]} Z_s^x > 0\}.$$

(1) Let $\delta = n$ be an integer number. Show that the modulus of n-dimensional Brownian motion is equal to n-dimensional Bessel process.
(2) Let $W_t = \lambda Z^x_{t/\lambda^2}$. Show that W_t is a Bessel process.
(3) Show that

$$\mathbb{P}[\inf_{0 \le t \le T_x} Z_t^x > 0] = 1$$

if and only if $\delta > 2$. Show that in this case $Z_t^x \to \infty$ a.s. for every $x > 0$.

Problem 2.

Suppose that g_t is a chordal Loewner evolution with the driving function u_t and the corresponding hulls K_t

$$\partial_t g_t(z) = \frac{2}{g_t(z) + u_t}, \qquad g_0(z) = z.$$

Let $x_0 = u_0$ and let Φ be a map which is univalent in some $\mathbb{H}$-neighbourhood U of x_0 and $\Phi(z) = a_0 + a_1(z - x_0) + a_2(z - x_0)^2 + \dots$ with real coefficients a_n. *(Such maps are called locally real at x_0).* Let us assume that there is $t_0 > 0$ such that $K_t \subset U$ for all $0 \le t < t_0$.

Define $\tilde{K}_t = \Phi(K_t)$ and let $\tilde{g}_t : \mathbb{H} \setminus \tilde{K}_t \to \mathbb{H}$ be the corresponding conformal transformations. Define $\Phi_t = \tilde{g}_t \circ \Phi \circ g_t^{-1}$.

(1) Show that the maps $\tilde{g}_t$ satisfy the Loewner equation

$$\partial_t \tilde{g}_t = \frac{2\Phi_t'(u_t)^2}{\tilde{g}_t(z) - \tilde{u}_t}$$

where $\tilde{u}_t = \Phi_t(u_t)$.
(2) Show that

$$\dot{\Phi}_t(z) = 2\left(\Phi_t'(u_t)\frac{\Phi_t'(u_t)}{\Phi_t(z) - \Phi_t(u_t)} - \Phi_t'(z)\frac{1}{z - u_t}\right).$$

(3) Show that one can pass to the limit in the formula above and obtain

$$\dot{\Phi}_t(u_t) = \lim_{z \to u_t} \dot{\Phi}_t(z) = -3\Phi''(u_t).$$

(4) Show that

$$\dot{\Phi}_t'(z) = 2\left(-\frac{\Phi_t'(u_t)^2\Phi_t'(z)}{(\Phi_t(z) - \Phi_t(u_t))^2} + \frac{\Phi_t'(z)}{(z - u_t)^2} - \frac{\Phi_t''(z)}{z - u_t}\right).$$

(5) Show that

$$\dot{\Phi}_t'(u_t) = \lim_{z \to u_t} \dot{\Phi}_t'(z) = \frac{\Phi_t''(u_t)^2}{2\Phi_t'(u_t)} - \frac{4\Phi_t'''(u_t)}{3}.$$