C8.5 Introduction to SLE

Sheet 4

Let g_t be the Loewner Evolution given by

$$\partial_t g_t(z) = \frac{a}{g_t(z) - B_t}, \quad g_0(z) = z, \ z \in \mathbb{H}$$

where B_t is a standard Brownian motion and $a = 2/\kappa$. Define $f_t(z) = g_t^{-1}(z)$ and $\hat{f}_t(z) = g_t^{-1}(z + B_t)$. We can consider the *backward* SLE

$$\partial_t h_t(z) = -\frac{a}{h_t(z) - B_t}, \quad h_0(z) = z, \ z \in \mathbb{H}.$$

Define $Z_t = X_t + iY_t = h_t(z) - B_t$.

(1) How g_t is related to the usual SLE(κ)? Show that for fixed time $t \geq 0$, the function $z \mapsto \hat{f}_t'(z)$ and the function $z \mapsto h_t'(z)$ have the same law (but it is not true that the joint law of $(\hat{f}_t'(z))_{t\geq 0}$ and the joint law of $(h_t'(z))_{t\geq 0}$ would be the same!).

(2) Make a time change $\tau(t)$ and introduce

$$\tilde{Z}_t = \tilde{X}_t + i\tilde{Y}_t = Z_{\tau(t)}$$

such that

$$d\tilde{Y}_t = \tilde{Y}_t dt, \qquad d(\tilde{X}_t) = -a\tilde{X}_t dt + |\tilde{Z}_t| d\tilde{B}_t$$

where $\tilde{B}$ is a standard Brownian motion.

(3) Let

$$\tilde{K}_t = \frac{\tilde{X}_t}{\tilde{Y}_t}, \qquad \tilde{L}_t = \frac{|\tilde{Z}_t|}{\tilde{Y}_t}$$

and define $\tilde{J}_t$in such a way that $\tilde{K}_t = \sinh \tilde{J}_t$ and $\tilde{L}_t = \cosh \tilde{J}_t$.

Show that

$$d\tilde{J}_t = -(1/2 + 2a) \tanh \tilde{J}_t dt + d\tilde{B}_t.$$

(4) Let $\tilde{h}_t = h_{\tau(t)}$. Show that

$$\partial_t \log |\tilde{h}_t'(z)| = a(2(\tanh \tilde{J}_t)^2 - 1).$$

(5) Let $\tilde{\theta}_t = \arg \tilde{Z}_t$. Show that

$$\tilde{M}_t = |\tilde{h}_t'(z)|^p \tilde{Y}_t^{p-r/a} (\sin \tilde{\theta}_t)^{-2r}$$

is a martingale if $r^2 - (1 + 2a)r + ap = 0$ and deduce that

$$\mathbb{E}[|\tilde{h}_t'(z)|^p (\sin \tilde{\theta}_t)^{-2r}] = \left(\frac{Y_0}{|Z_0|}\right)^{-2r} \exp(-at(p - r/a)).$$

Assuming that $p, r \geq 0$ and $\lambda > 0$ show that

$$\mathbb{P}[|\tilde{h}_t'(z)| \geq \lambda] \leq \lambda^{-p} \left(\frac{Y_0}{|Z_0|}\right)^{-2r} \exp(-at(p - r/a)).$$

For the rest of the question you may assume that the estimate above implies an estimate for the derivative h_t' in the original time. Namely, for every $r \in [0, 1+2a]$, there is $c = c(\kappa, r) \in (0, \infty)$ such that for all $t \in [0, 1]$, $x \in \mathbb{R}$, $y \in (0, 1]$ and $\lambda \in [e^6, 1/y]$

$$\mathbb{P}[|h_t'(x + iy)| \geq \lambda] \leq c\lambda^{-p} \left(\frac{y}{|x + iy|}\right)^{-2r} \delta(y, \lambda),$$

where $p = p(r) = ((1+2a)r - r^2)/a \geq 0$ and

$$\delta(y,\lambda) = \begin{cases} \lambda^{-p+r/a}, & p - r/a > 0, \\ 1 - \log(\lambda y), & p - r/a = 0, \\ y^{p-r/a}, & p - r/a < 0. \end{cases}$$

(6) We set

$$r = r_0 = \frac{1+4a}{4} = \frac{1}{4} + \frac{2}{\kappa}.$$

This value maximizes $2p - r/a$. We have

$$2p(r_0) - \frac{r_0}{a} = \kappa r_0 \left(\left(\frac{1}{2} + \frac{4}{\kappa} \right) - r_0 \right) = \kappa r_0^2 \geq 2.$$

Note that $\kappa r_0^2 = 2$ if and only if $\kappa = 8$. From now on, we assume that $r = r_0$ and $p = p(r_0)$. We also assume that $\kappa \neq 8$.

Show that there is $\epsilon > 0$ such that if $0 < \alpha < 1 - 2/(2p(r_0) - r_0/a)$ is small enough then

$$\mathbb{P}[|h_t'(i2^{-n})| \geq 2^{n(1-\alpha)}] \leq c2^{n(2+\epsilon)}.$$

(7) Show that for α small enough there is a random $C > 0$ which is finite a.s. such that

$$|h_t'(i2^{-n})| \leq C2^{n(1-\alpha)},$$

for all $n \in \mathbb{N}$ and $t = k2^{-2n}$, $k = 0, 1, \ldots, 2^{2n}$.