This is a preliminary problem sheet, to get the ball rolling. There is a 'bonus' problem for MFoCS students overleaf. Hints/solutions will be put on the website near the start of week 2. Problem sheet 1 (based on the first two week's lectures) will be for the first class, in week 3 or 4.

Estimates and asymptotics, union bound and first-moment method

1. Prove the following inequalities:

   (a) $1 + x \leqslant e^x$ for all real x.

   (b) $(1 + a)^n \leqslant e^{an}$ for $a > -1$, $n \geqslant 0$.

   (c) $k! \geqslant k^k/e^k$ for $k \geqslant 1$.

   (d) $\left(\frac{n}{k}\right)^k \leqslant \binom{n}{k} \leqslant \frac{n^k}{k!} \leqslant \left(\frac{en}{k}\right)^k$ for $1 \leqslant k \leqslant n$.

2. For the following functions $f(n)$ and $g(n)$, decide whether $f = o(g)$ or $g = o(f)$ or $f = \Theta(g)$ as $n \to \infty$:

   (a) $f(n) = \binom{n}{k}$, $g(n) = n^k$, first for k fixed and then for the case where $k = k(n) \to \infty$ as $n \to \infty$;

   (b) $f(n) = (\log n)^{1000}$, $g(n) = n^{1/1000}$;

3. In lectures we saw that the kth diagonal Ramsey number satisfies

$$R(k, k) > n - \binom{n}{k} 2^{1-\binom{k}{2}},$$

   for each integer n. By considering $n = \lfloor e^{-1} k 2^{k/2} \rfloor$, deduce that

$$R(k, k) \geqslant \big(1 - o(1)\big) e^{-1}\, k 2^{k/2}.$$

4. Show that if $n, k, \ell \geqslant 1$ are integers and $0 < p < 1$, then

$$R(k, \ell) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{\ell} (1 - p)^{\binom{\ell}{2}}.$$

5. Let H be an r-uniform hypergraph with fewer than $\frac{3^{r-1}}{2^r}$ edges. Prove that the vertices of H can be coloured using three colours in such a way that in each edge, all three colours are represented.

6. Let F be a collection of binary strings ("codewords") of finite length, where the ith codeword has length c_i. Suppose that no member of F is a prefix of another member (so you can decode any string made up by concatenating codewords as you go along, without looking ahead). Show that $\sum_i 2^{-c_i} \leqslant 1$ (the *Kraft inequality* for prefix-free codes).

Bonus question (for MFoCS students, optional for others):

A (finite, or infinite and convergent) sum $S = \sum_{i \geqslant 0} a_i$ is said to *satisfy the alternating inequalities* if the partial sum $\sum_{i=0}^{t} a_i$ is at least S for all even t and at most S for all odd t; that is, the partial sums alternately over- and under-estimate the final result.

7. Let $I_1, \ldots, I_n$ be the indicator functions of n events $E_1, \ldots, E_n$. For $0 \leqslant r \leqslant n$ let $S_r = \sum_{A \subseteq [n],\, |A|=r} \prod_{i \in A} I_i$, where $[n] = \{1, 2, \ldots, n\}$. Show that

$$\prod_{i=1}^{n} (1 - I_i) = \sum_{r=0}^{n} (-1)^r S_r, \tag{0.1}$$

and that the sum satisfies the alternating inequalities. [Both sides are random; the statement is that the relevant inequalities *always* hold. You may want to consider different cases according to how many of the events E_i hold.] Deduce that

$$\mathbb{P}(\text{no } E_i \text{ holds}) = \sum_{r=0}^{n} (-1)^r \sum_{A \subseteq [n],\, |A|=r} \mathbb{P}\left(\cap_{i \in A} E_i\right), \tag{0.2}$$

and that the sum satisfies the alternating inequalities. [This is a form of the inclusion–exclusion formula.]