1. Consider $G(n,p)$ with $p = \frac{\log n + c}{n}$ for a constant $c \in \mathbb{R}$, and let Y_r be the number of components with r vertices.

   (i) Show that $\mathbb{E}Y_2 \to 0$ as $n \to \infty$.

   (ii) Using Cayley's formula, show that
   $$\mathbb{E}Y_r \leqslant \binom{n}{r} r^{r-2} p^{r-1} (1-p)^{r(n-r)}.$$
   Hence show that
   $$\mathbb{E} \sum_{r=3}^{\lfloor n/2 \rfloor} Y_r \to 0 \text{ as } n \to \infty.$$
   [Try on your own first; there is a hint below.]

   (iii) Using (i) and (ii), show that as $n \to \infty$,
   $$\mathbb{P}(\text{all non-isolated vertices are in a single component}) \to 1$$
   as $n \to \infty$, for any fixed c.

   (iv) Assuming (or, if keen, proving) that the bound above remains valid if c is allowed to vary with n, tending to $+\infty$ or $-\infty$ sufficiently slowly, deduce that $p(n) = \frac{\log n}{n}$ is a threshold function for $G(n,p)$ to be connected.

   [Hint for part (ii): using the standard bounds $\binom{n}{r} \leqslant (en/r)^r$ and $1-p \leqslant e^{-p}$, show that the sum of the expectations is at most $\sum_{r \geqslant 3} r^{-2} p^{-1} X^r$ for some expression X. Using $n - r \geqslant n/2$ and $3\log n/4 \leqslant np \leqslant 2\log n$ (for n large), show that $X = o(n^{-1/3})$. Use $r^{-2}p^{-1} \leqslant p^{-1} \leqslant n$ to complete the calculation. For part (iv): recall the result of Sheet 1 Q6.]

2. Give an example of three events A, B_1 and B_2 such that $\mathbb{P}(A \mid B_i) > \mathbb{P}(A)$ for $i = 1, 2$ but $\mathbb{P}(A \mid B_1 \cup B_2) < \mathbb{P}(A)$.

3. Let $\mathcal{U}_1, \mathcal{U}_2, \ldots$ be up-sets (increasing events) in $\mathcal{P}(X)$, let $\mathcal{D}$ be a down-set, and let $\mathbb{P}_p$ denote the probability measure associated to selecting each element of X independently with probability p.

   (i) Show that $\mathbb{P}_p(\mathcal{U}_1 \cap \cdots \cap \mathcal{U}_k) \geqslant \prod_{i=1}^k \mathbb{P}_p(\mathcal{U}_i)$.

   (ii) Show that $\mathbb{P}_p(\mathcal{U}_1 \mid \mathcal{D}) \leqslant \mathbb{P}_p(\mathcal{U}_1)$.

   (iii) Show that if $\mathcal{U}_1$ and $\mathcal{U}_2$ are independent, then $\mathbb{P}_p(\mathcal{U}_1 \mid \mathcal{U}_2 \cap \mathcal{D}) \leqslant \mathbb{P}_p(\mathcal{U}_1 \mid \mathcal{U}_2)$.

   (iv) Does the last inequality always hold, i.e., without assuming independence?

4. Let $\mathcal{U} \subset \mathcal{P}(X)$ be an up-set. Show that the function $f(p)$ defined by $f(p) = \mathbb{P}_p(\mathcal{U})$ is increasing on $[0, 1]$. Is it continuous? Differentiable?

5. When does equality hold in Harris's Lemma? [Hint: you may wish to consider more than one way of splitting $\mathcal{P}(X)$ into $(n-1)$-dimensional sub-cubes.]

6. Let $0 < p < 1$ be fixed, and let X denote the number of triangles in $G(n,p)$. Show that there is a constant $c = c(p) > 0$ such that $\mathbb{P}(X = 0) \leqslant e^{-cn^2}$ for all $n \geqslant 3$. Is $\mathbb{P}(X = 0)$ at least $e^{-c'n^2}$ for some constant c'?

   Let $G = G(n,p)$, where $0 < p < 1$ is constant. Call a subset V of $V(G)$ *triangle-free* if the subgraph of G induced by V contains no triangles. Show that there is a constant A such that whp (i.e., with probability tending to 1) G contains no triangle-free subset with more than $A \log n$ vertices.

   Do you expect $G(n, 1/2)$ to have a triangle-free subset of size at least $B \log n$ for some constant B?

7. Let H be a fixed strictly balanced graph, let $0 < \alpha < v(H)/e(H)$ be fixed, let $p = p(n) = n^{-\alpha}$, and let X denote the number of copies of H in $G(n,p)$. Use Janson's inequality to show that there is some constant $\beta > 0$ such that $\mathbb{P}(X = 0) \leqslant \exp(-n^{\beta})$ for n sufficiently large.

   [Hint: evaluate $\mu = \mathbb{E}[X]$ and Δ approximately. Have we considered Δ before in a different context?]

8. *Buffon's needle.* A floor is made up of parallel strips of wood, each of width 1. A needle of length $l < 1$ is dropped onto the floor. What is the probability $p(l)$ that it crosses the boundary between two strips? (If the picture is not clear, Google for "Buffon's needle"!). The answer leads to an experimental method for estimating π.

   The traditional solution is to average over the position of the end of the needle and over the angle at which the needle lies, using a double integral. This is reasonably straightforward, but an alternative approach to the calculation is to use the principle of linearity of expectation (which underlies everything we did with the first moment method).

   Instead of $p(l)$, consider $e(l)$, the *expected number* of times that the needle crosses a line.

   - Show that if $l < 1$ then $p(l) = e(l)$.
   - Show that $e(l) = cl$ for some constant c (for all positive l).

   If we replace a straight needle by some other shape with the same length, the probability of crossing a line will change; however, note that the expectation of the number of line-crossings doesn't! – so to calculate $e(l)$ we may consider "needles" which are not straight lines.

   - Find a "needle" for which the expectation is very easy to calculate. Hence find c.

Bonus question (compulsory for MFoCS students, optional for others):

9. Fix $k \geqslant 1$. Show that if $p(n)$ is chosen so that the expected number of vertices of $G(n,p)$ with degree strictly less than k tends to a constant c, then

$$\mathbb{P}(\delta(G(n,p)) \geqslant k) \to e^{-c},$$

where $\delta(G)$ denotes the minimum degree of a graph G.

Deduce that if $p = \frac{\log n + c}{n}$ where c is constant, then

$$\mathbb{P}(G(n,p) \text{ is connected}) \to e^{-e^{-c}}.$$

10. Let $\mathcal{U} \subset \mathcal{P}(X)$ be an up-set. The *influence* $I_p(i,\mathcal{U})$ of an element $i \in X$ on $\mathcal{U}$ is the probability that exactly one of $X_p \setminus \{i\}$ and $X_p \cup \{i\}$ is in $\mathcal{U}$. Defining $f(p)$ as in Question 4, show that $\frac{\mathrm{d}}{\mathrm{d}p} f(p) = \sum_{i \in X} I_p(i,\mathcal{U})$.

If you find an error please check the website, and if it has not already been corrected, e-mail riordan@maths.ox.ac.uk