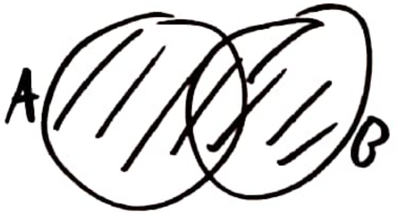
§1 The first moment method.

The union bound

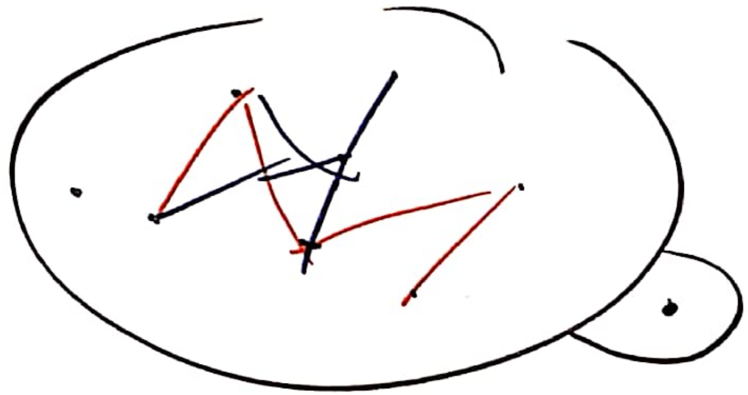
$$\mathbb{P}(A \cup B) \le \mathbb{P}(A) + \mathbb{P}(B)$$



If $A_1, \dots, A_n$ are events

$$\mathbb{P}\left(\bigcup A_i\right) \le \sum_{i=1}^{n} \mathbb{P}(A_i).$$

For $k, l \geq 1$ integers the Ramsey number $R(k,l) = \min\{n :$ every red/blue colouring of the edges of K_n contains a red K_k or a blue $K_l\}$

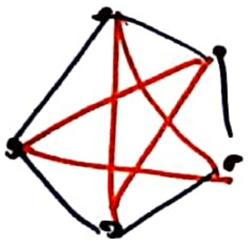


× × × × × × ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓

n $n+1$

$\times \; \times \; \times \; \cdots \times \; \checkmark \; \checkmark \; \checkmark$

n

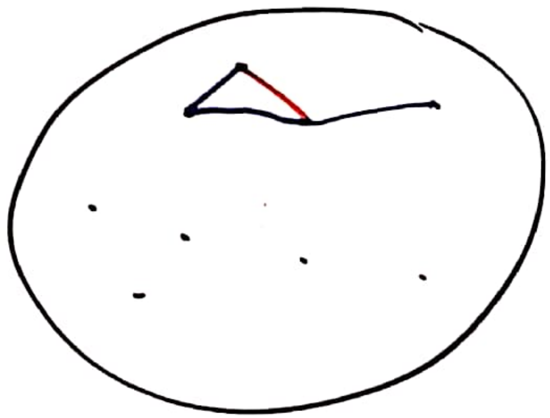


$R(3,3) > 5$

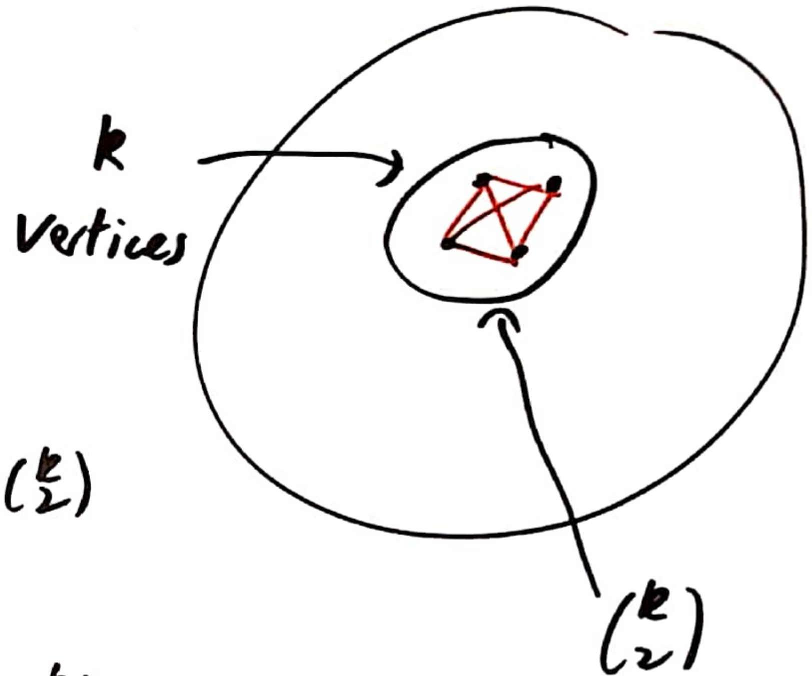
Theorem 1.1 (Erdős 1947).
Let $n, k \geq 1$ be integers s.t.
$\binom{n}{k} 2^{1-\binom{k}{2}} < 1$. Then $R(k,k) > n$.

__Pf__ Suffices to show K_n has a 'good' colouring, ie. a colouring with no monochromatic K_k.

Colour edges of K_n randomly each red with prob. $\frac{1}{2}$, blue otherwise with the colours of the edges independent.



Let A_i be the event that i^{th} copy of K_k is monochromatic, $i = 1, 2, \ldots \binom{n}{k}$



$$\mathbb{P}(A_i) = \left(\tfrac{1}{2}\right)^{\binom{k}{2}} + \left(\tfrac{1}{2}\right)^{\binom{k}{2}} = 2^{1-\binom{k}{2}}$$

$$\mathbb{P}\left(\bigcup A_i\right) \le \sum_{i=1}^{m} \mathbb{P}(A_i) = \binom{n}{k} 2^{1-\binom{k}{2}} < 1$$

ie. $\mathbb{P}$(random colouring is 'bad') < 1

ie. $\mathbb{P}$(random colouring is 'good') > 0

$\therefore \exists$ a good colouring $\square$

Theorem 1.1 If $n, k \geq 1$ are integers s.t.

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1 \quad \text{then} \quad R(k,k) > n.$$

Cor 1.2 For $k \geq 3$, $R(k,k) \geq 2^{k/2}$.

$k \to k+1$

Pf Let $n = \lfloor 2^{k/2} \rfloor$. Then

$$\binom{n}{k} 2^{1-\binom{k}{2}} \leq \frac{(2^{k/2})^k}{k!} 2^{1-\binom{k}{2}} = \frac{2^{k^2/2}}{k!} 2^{1-\frac{k^2}{2}+\frac{k}{2}} = \frac{2^{1+\frac{k}{2}}}{k!} < 1 \text{ for } k \geq 3$$

$\times \sqrt{2}$

$\times (k+1)$

decreases as $k \nearrow$ $(k \geq 2)$

$$\therefore R(k,k) > \lfloor 2^{k/2} \rfloor \qquad \square$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!} \leq \frac{n^k}{k!}$$

$$\binom{k}{2} = \frac{k(k-1)}{2} = \frac{k^2}{2} - \frac{k}{2}$$

§1 The first moment method

If X is a RV (random variable) its first moment is just its mean / expectation $\mathbb{E}[X] = \sum_x x\,\mathbb{P}(X=x)$. $= \mathbb{E}(X) = \mathbb{E}X$

Key property: $\mathbb{E}$ is LINEAR

If X and Y are RVs, $\mathbb{E}[X+Y] \overset{(*)}{=} \mathbb{E}[X] + \mathbb{E}[Y]$

If λ is a constant $\mathbb{E}[\lambda X] = \lambda\mathbb{E}[X]$.

(*) ALWAYS HOLDS - no independence or other condition.

If A is an event, its indicator function

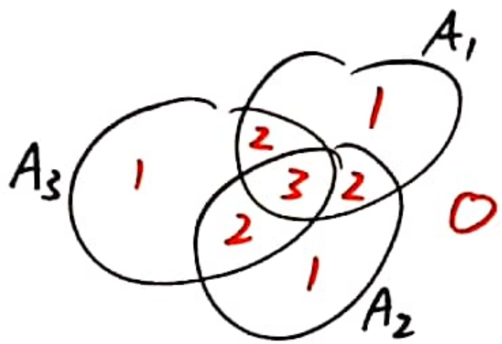
I_A or 1_A is the RV taking value 1

when A holds, 0 otherwise.

$$\mathbb{E}[I_A] = 1 \cdot \mathbb{P}(A) + 0 \cdot \mathbb{P}(A^c) = \mathbb{P}(A).$$

Let $A_1, \dots, A_n$ be events and let $I_1, \dots, I_n$ be their indicator functions. $\quad (I_j = I_{A_j})$

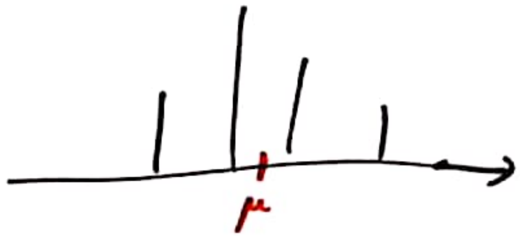
Let $X = \sum_{j=1}^{n} I_j$ counts how many of the A_j hold.



Then $\mathbb{E}[X] = \sum_{j=1}^{n} \mathbb{E}[I_j] = \sum_{j=1}^{n} \mathbb{P}(A_j).$

Observation: if X is a RV with $\mathbb{E}[X] = \mu$

then $\mathbb{P}(X \geq \mu) > 0$

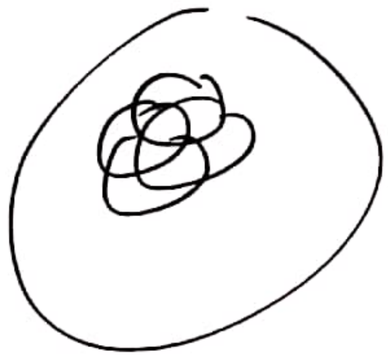


Similarly $\mathbb{P}(X \leq \mu) > 0$.

__Theorem 1.3__ Let $n, k \geq 1$ be integers.

Then $R(k,k) > n - \binom{n}{k} 2^{1 - \binom{k}{2}}$.

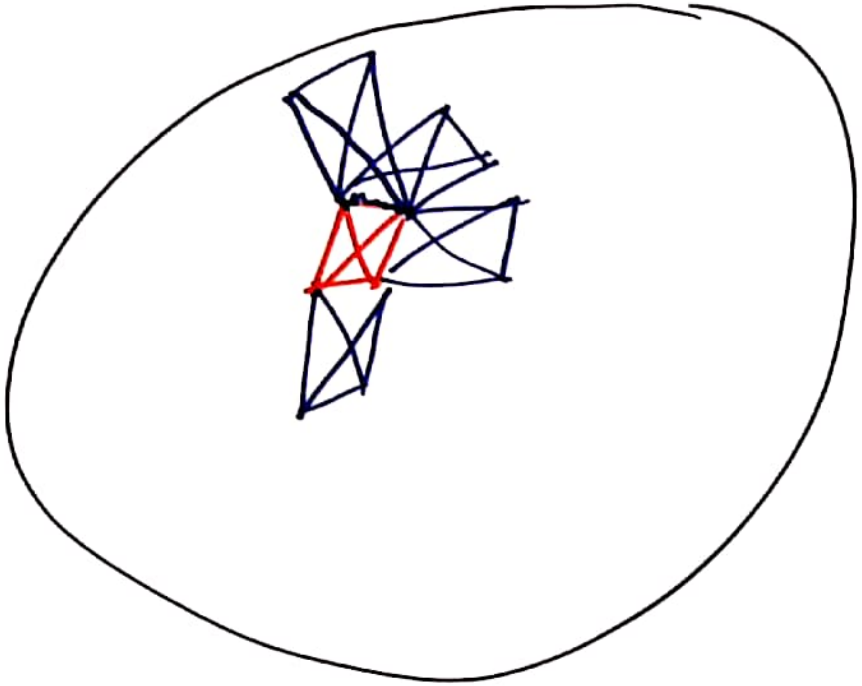
__Pf__ Colour edges of K_n red/blue as before.

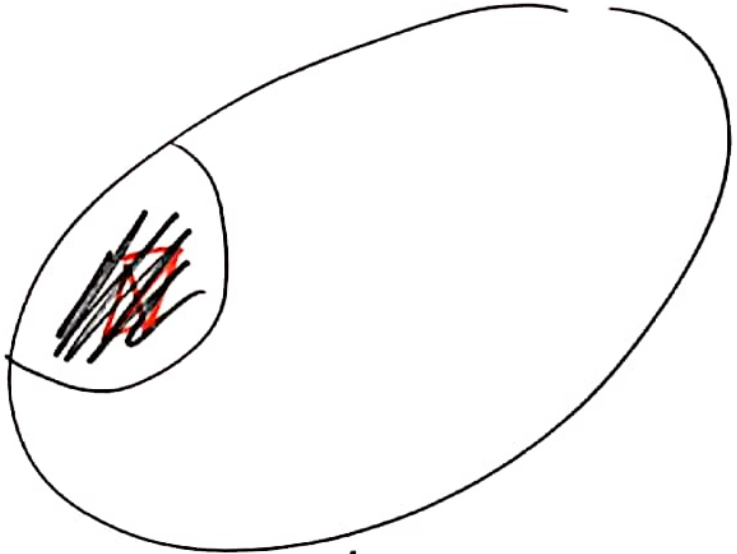


Let $X = \#$ monochromatic copies of K_k

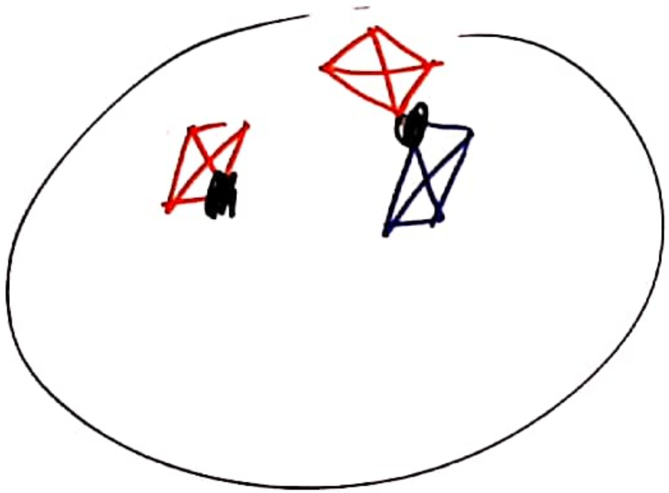
Define A_i as before

Then $\mathbb{E}[X] = \sum \mathbb{P}(A_i) = \binom{n}{k} 2^{1 - \binom{k}{2}} =: \mu$





K_{n-4}



By observation $\mathbb{P}(X \leq \mu) > 0$

i.e. a colouring with $t \leq \mu$ defects (mono. K_k)

Pick such a colouring. Pick a vertex of each defect to delete.

In total delete $\leq t \leq \mu$ vertices

Obtain a 'good' colouring of K_m where $m \geqslant n - \mu$

$\therefore \quad R(k,k) > m$

$\square$

Theorem 1.3 Let $k, n \geq 1$ be integers.

Then $R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$.

Cor 1.4 $R(k,k) \geq (1-o(1)) \frac{k}{e} 2^{k/2}$ as $k \to \infty$.

i.e. $\forall \varepsilon > 0 \; \exists k_0 \; \forall k \geq k_0 \quad R(k,k) \geq (1-\varepsilon)$

Pf Exercise. Take $n = \left\lfloor \frac{k}{e} 2^{k/2} \right\rfloor$ $\square$

Remarks on calculation

Considering stuctures of a cetain type.

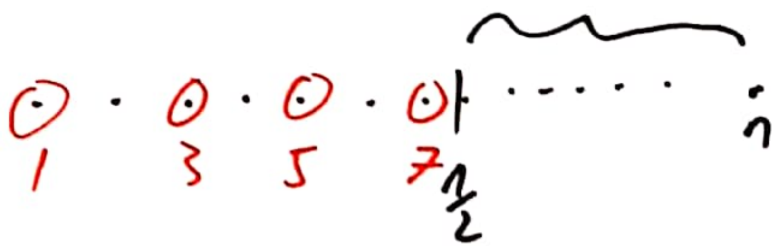
Aim: show $\exists$ a good one.

$$\left.\begin{array}{lll} \text{Total \# structures} & \overset{\geq}{=} & \text{some formula} \\ & & \vee \\ \text{\# bad structures} & \underset{\leq}{=} & \text{some other formula} \end{array}\right\} \Rightarrow \exists \text{ good structure.}$$

A set $S \subseteq \mathbb{R}$ or $\mathbb{Z}$ is sum-free if $\nexists\, a,b,c \in S$ s.t. $a+b=c$. eg. $\{1,2,6\}$ is NOT sum-free as $1+1=2$

$\{1, 3, 7, 9, 285\}$ is sum-free.

Consider $[n] = \{1, 2, \dots, n\}$. How big a sum-free set can we find in $[n]$?

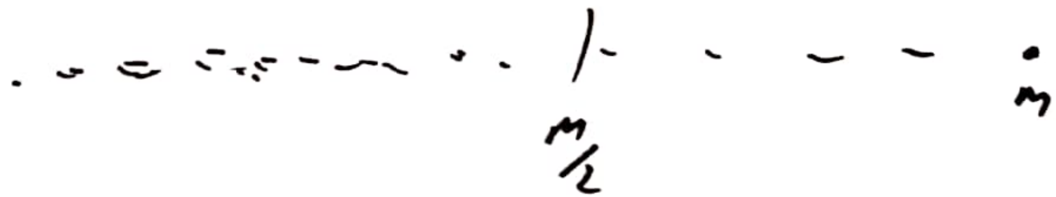


Around $n/2$ — odd numbers

— lage numbers

S a set of integers: can we find large sum-free $A \subseteq S$?

eg try $A = \{a \in S : a \text{ odd}\}$



<u>Theorem 1.5</u> (Erdős 1965). Let S be a set of $n \geq 1$ non-zero integers. There is some $A \subseteq S$ s.t. A is sum-free and $|A| > \frac{n}{3}$.

<u>Pf</u> Let $S = \{s_1, \dots, s_n\}$.

Trick: work mod p prime, p of the form $3k+2$

choose $p > 2 \max |s_i|$ so $s_1 \cdots s_n$ are distinct & $\neq 0$ mod p.

A set $A \subseteq \mathbb{Z}$ is sum-free mod p if

$$\nexists\ a, b, c \in A \text{ s.t. } a + b \equiv c \pmod{p}$$

$$\Uparrow$$

$$a + b = c$$

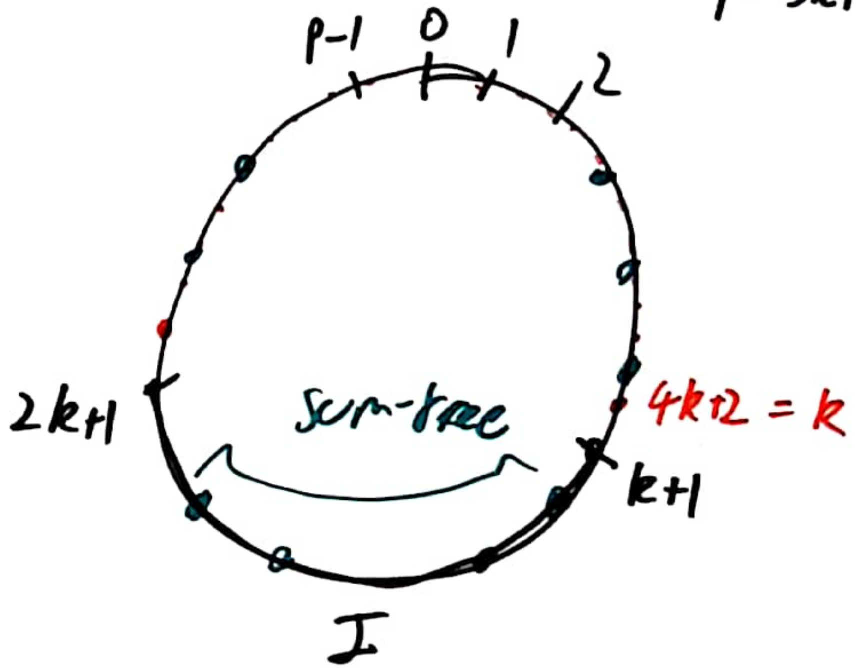
SUM-FREE mod p $\Rightarrow$ SUM-FREE

$p = 3k+2$

For $r = 1, 2, \cdots p-1$ the map

$a \mapsto ra \mod p$ is a bijection

from $\mathbb{Z}_p^* = \{1, 2, \cdots p-1\}$ to itself.



I is sum free mod p.

Let R be random with

$$\mathbb{P}(R = r) = \frac{1}{p-1} \qquad r = 1, 2, \cdots p-1$$

Let $X = \#\{i : R s_i \overset{\text{mod } p}{\in} I\}$

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{P}(R s_i \overset{\text{mod } p}{\in} I)$$

Fix i: since $x \mapsto x s_i$ is a bijection $\mathbb{Z}_p^*$ to itself

Rs_i takes all values in $\mathbb{Z}_p^*$ with equal probability.

$$\therefore \quad \mathbb{P}(Rs_i \bmod p \in I) = \frac{|I|}{p-1} = \frac{k+1}{3k+1} > \frac{1}{3}$$

$$\therefore \quad \mathbb{E}[X] = \sum \qquad = n \times \bigcirc > \frac{n}{3}$$

$\therefore$ $\exists$ value r s.t. when $R = r$, $X > \frac{n}{3}$.

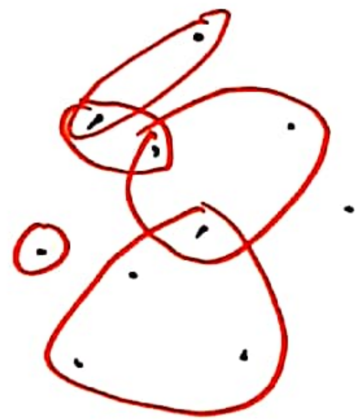
Let $A = \{ s_i : rs_i \bmod p \in I \}$. Then $|A| > \frac{n}{3}$

A is sum-free: if $s_i, s_j, s_k \in A$ with $s_i + s_j = s_k$

then $\underbrace{rs_i}_{\in I} + \underbrace{rs_j}_{\in I} \equiv \underbrace{rs_k}_{\in I} \mod (p)$

$\cancel{\#}$ I sum-free mod p $\square$

A hypergraph ~~G~~ H is an ordered pair (V, E) where V is a (finite) set (of vertices) and E, the set of hyperedges is a set of ~~2-element~~ subsets of V.



Note E is a set: eg. $\{1,2,3\}$ either present or not.

Often $V = [n]$

Write 125 for hyperedge $\{1, 2, 5\}$

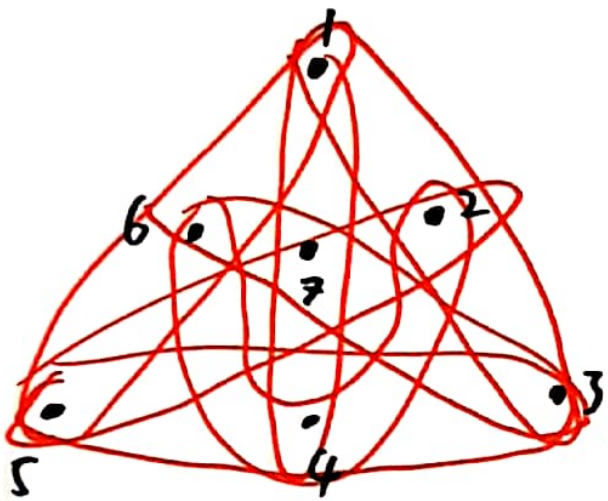
((could consider multi-hypergraphs)

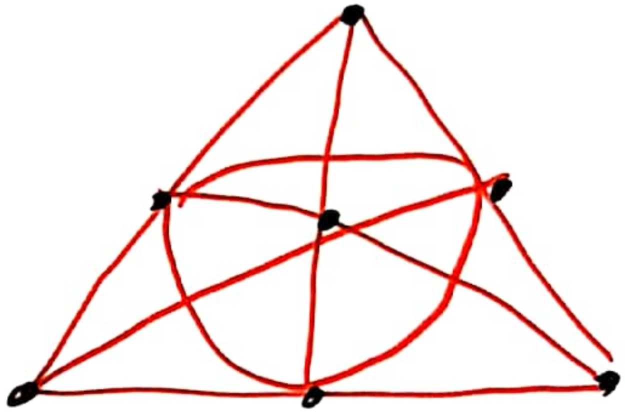
of

H is r-uniform if $|e| = r$ for all $e \in E(H)$

2-uniform hypergraph $\equiv$ graph.

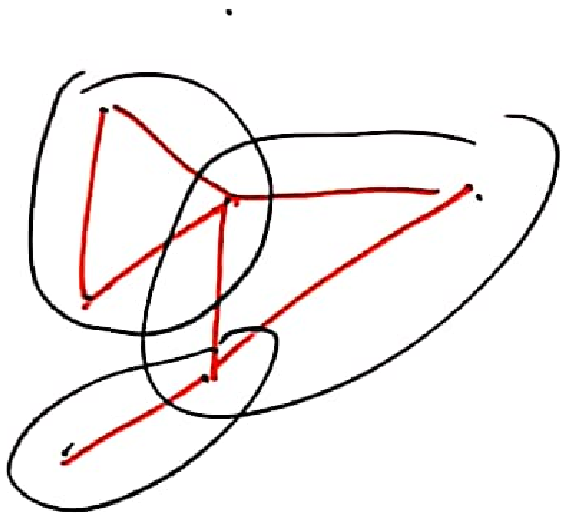
eg. 3-uniform H : Fano plane

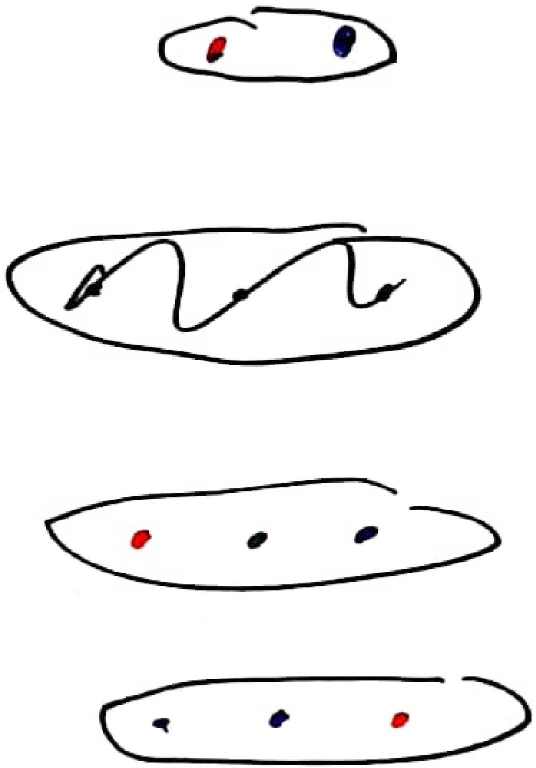




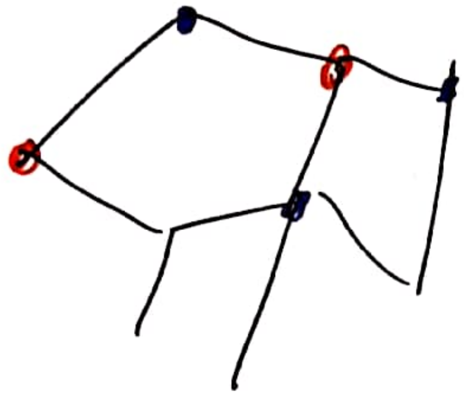
$V = [7]$ $E = \{123, 345, \ldots$

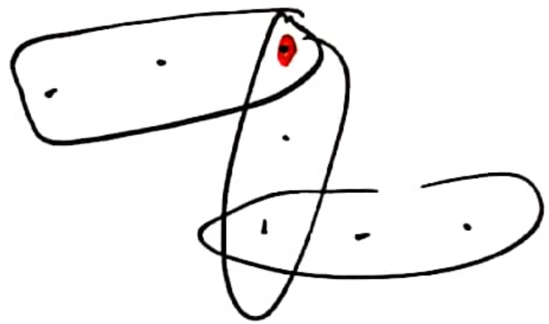
Colouring hypergraphs



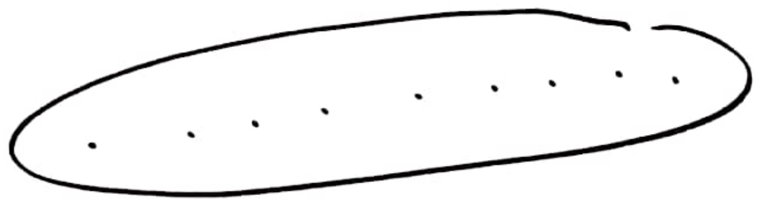


A proper k-coloring of a hypergraph H is a map $c: V(H) \to S$ where $|S| = k$ s.t. no edge e of H is monochromatic.

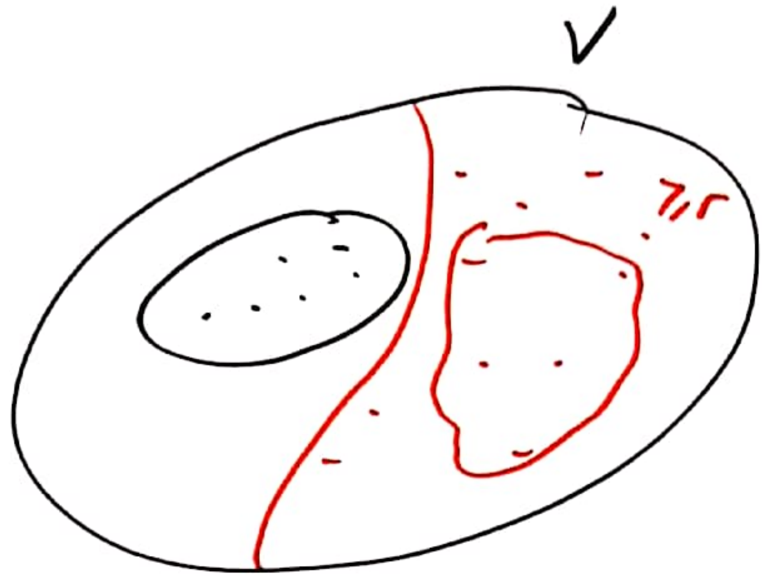




A hypergraph H is k-colourable if it has a proper k-colouring.



only forbid 2 combinations out of 2^r.



complete r-uniform hypergraph on V

E = all $\binom{n}{r}$ r-element subsets $n = |V|$

If $n = 2r-1$ not 2-colourable

$$e(H) = \binom{2r-1}{r} \approx 4^r$$

Let $m(r) = \min\{e(H) : H \text{ is } r\text{-uniform \& NOT 2-colourable}\}$.

$m(2) = 3$ $\quad$ $m(3) = 7$ (Fano plane)

Theorem 1.6 For $r \geq 2$, $m(r) \geq 2^{r-1}$.

Pf. Let H be r-uniform with $e(H) = m < 2^{r-1}$.

Suffices to show H is 2-colourable.

Colouring vertices of H randomly, each red with prob. $\frac{1}{2}$, blue otherwise, independently.

Let A_i be event that i^{th} edge is mono.

$\mathbb{P}(A_i) = 2/2^r$. Union bound $\mathbb{P}(\cup A_i) \leq \sum \mathbb{P}(A_i) = \frac{m}{2^{r-1}} < 1$

$\mathbb{P}(\text{colouring bad}) < 1$. $\therefore \exists$ good colouring. $\square$

Theorem 1.7 (Erdős 1964). For r large enough $m(r) \leq 3r^2 2^r$.

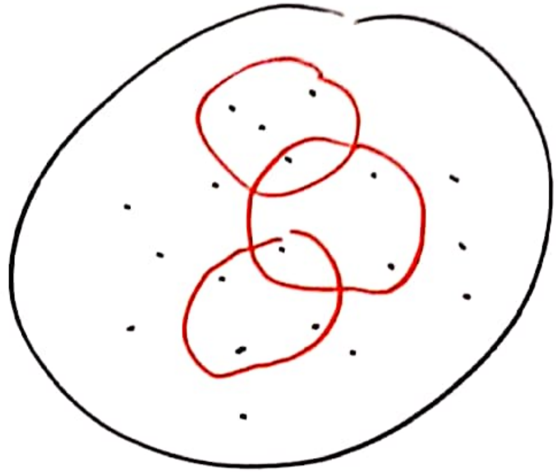
<u>Proof</u> Fix $r \geq 3$

Let $m = 3r^2 2^r$

Task: show $\exists$ r-uniform H with $e(H) \leq m$ s.t. H is not 2-colourable.

Idea: let H be random.

Fix a set of $n = r^2$ vertices



$|V| = n$

Pick $e_1, \dots, e_m$ independently and uniformly at random from all $\binom{n}{r}$ possible hyperedges.

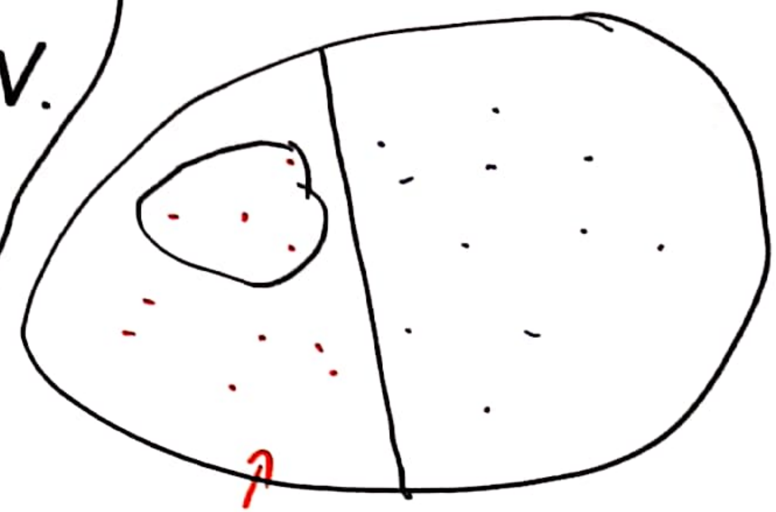
Let $H = (V, \{e_1, \dots, e_m\})$

$\{1,1,2\} = \{1,2\}$

N.B. duplicate edges only 'there once'

so $e(H) \le m$.

Fix a (not random) 2-coloring of V.



wlog $\geq \lceil \frac{n}{2} \rceil$

Let $p = p_c = \mathbb{P}(e_1 \text{ is } c\text{-mono})$

Then

$$p \geq \frac{\binom{\lceil n/2 \rceil}{r}}{\binom{n}{r}} \geq \frac{(\frac{n}{2})(\frac{n}{2}-1)\cdots(\frac{n}{2}-r+1)\cancel{r!}}{n\,(n-1)\cdots(n-r+1)\cancel{r!}}$$

ratios decrease

$\geq \binom{\lceil n/2 \rceil}{r}$ possible hyperedges are mono. w.r.t. c.

$$\geq \left(\frac{\frac{n}{2}-r}{n-r}\right)^r = \frac{1}{2^r}\underbrace{\left(1-\frac{r}{n-r}\right)^r}_{\approx 1}$$

$\approx \frac{1}{r}$

r^2

$\frac{x-1}{y-1}$ $\frac{x}{y}$

$$p \geq \frac{1}{2^r}\left(1-\frac{r}{r^2-r}\right)^r = \frac{1}{2^r}\left(1-\frac{1}{r-1}\right)^r \geq \frac{1}{3\cdot 2^r} =: p_0 \quad \text{if } r \geq r_0$$

assume from now on

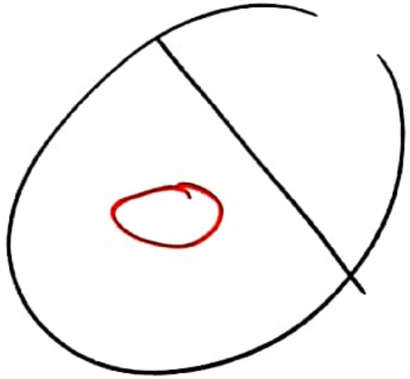
$$\mathbb{P}(e_1 \text{ is } c\text{-mono}) \geq p_0$$

$\mathbb{P}(c$ is a proper colouring of $H)$

$$= \mathbb{P}(\text{none of } e_1 \cdots e_n \text{ are } c\text{-mono})$$

$$= \prod_{i=1}^{n} \mathbb{P}(e_i \text{ is not } c\text{-mono})$$

$$\leq (1-p_0)^m.$$



Let $A_c = \{c \text{ is a proper colouring of } H\}$

$$\mathbb{P}(H \text{ is 2-colourable}) = \mathbb{P}(\cup A_c) \leq \sum \mathbb{P}(A_c)$$

$$\leq 2^n (1-p_0)^m$$

$1 - x \leq e^{-x}$

$$\leq 2^n e^{-p_0 m}$$

$$\leq 2^{r^2} e^{-\frac{m}{3 \cdot 2^r}} = 2^{r^2} e^{-r^2} < 1$$

H is always r-uniform & $e(H) \leq m$

Shown possible H is NOT 2-colourable.

$\square$

$A_1, \cdots, A_n$ events

IF disjoint $\mathbb{P}(\bigcup A_i) = \sum \mathbb{P}(A_i)$

IF indep. $\mathbb{P}(\bigcap A_i) = \prod \mathbb{P}(A_i)$ $\mathbb{P}(\bigcup A_i) = 1 - \prod_i (1 - \mathbb{P}(A_i))$

We <u>ALWAYS</u> have $\mathbb{P}(\bigcup A_i) \leq \sum \mathbb{P}(A_i)$

$$\{H \text{ is 2-colourable}\} = \bigcup_c \bigcap_i \{e_i \text{ is mono. w.r.t. } c\}^c$$