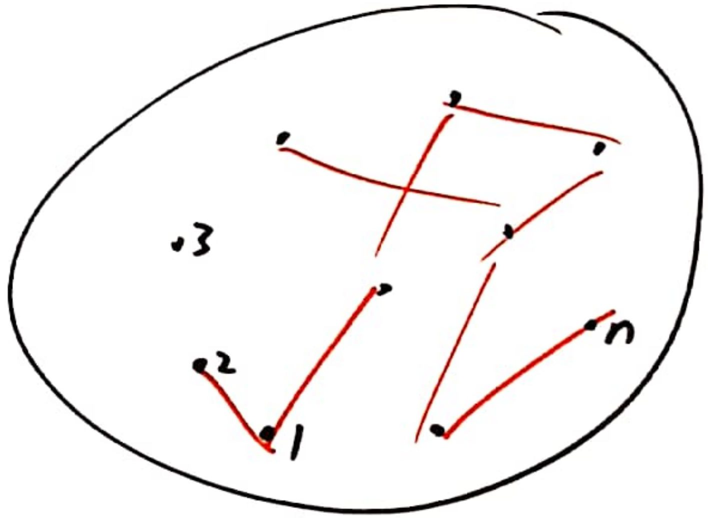
**Definition** For $n \geq 1$ an integer and $0 \leq p \leq 1$, the random graph $G(n,p)$ is the random graph on $[n] = \{1, 2, \dots, n\}$ in which each of the $\binom{n}{2}$ possible edges is present indep. with prob. $p$.




$|V| = n$

$V = [n]$

Or $\Omega = \{ \text{all } 2^{\binom{n}{2}} \text{ graphs on } [n] \}$

For $H \in \Omega$

$$\mathbb{P}(H) = \mathbb{P}(G(n,p) = H) \overset{\circledast}{=} p^{e(H)} (1-p)^{\binom{n}{2} - e(H)}$$

$\mathcal{G}(n,p)$ = this prob. space

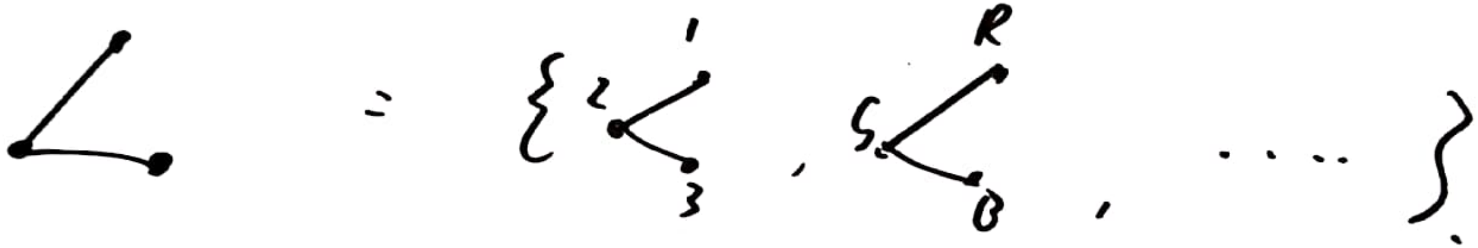
$G \in \mathcal{G}(n,p)$ or $G = G(n,p)$ or $G \sim G(n,p)$

all mean $G$ is a random graph s.t. $\circledast$ holds.

If $p = \frac{1}{2}$ all graphs on $[n]$ equally likely.

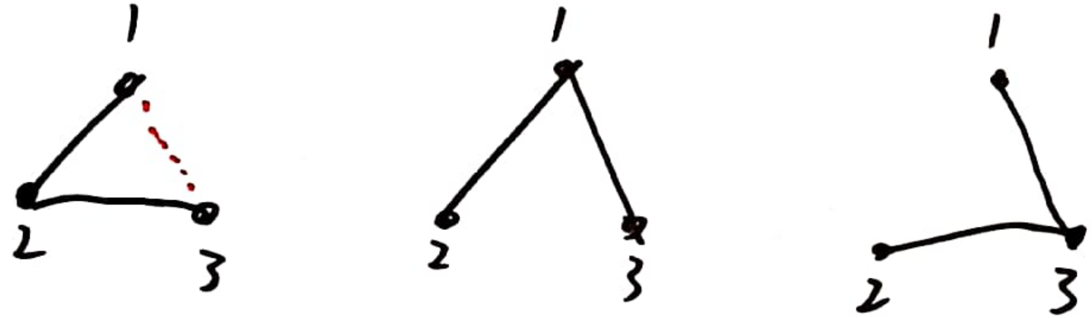
"random labelled graph" $G=(V,E)$


"unlabelled graph" - IM class

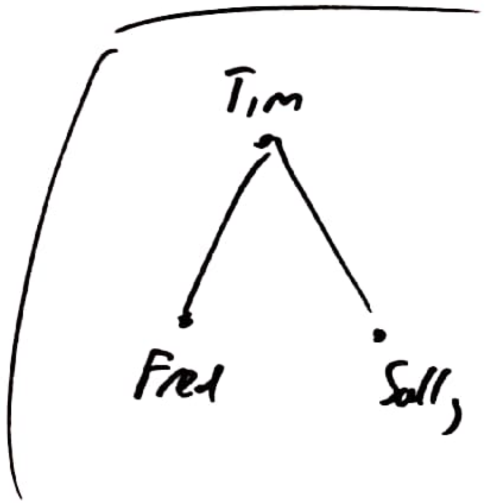



$\mathbb{P}\Big(G(3,p) \cong$ 

$\Big) = 3p^2(1-p).$



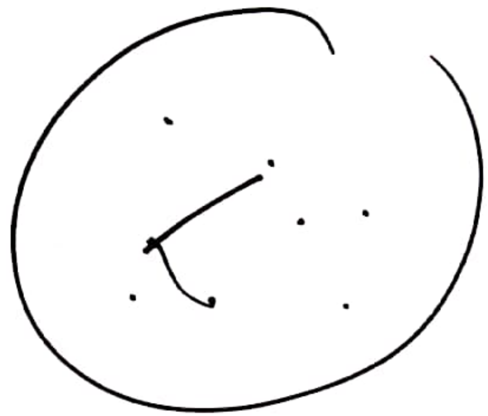

$p^2(1-p)$



Model $G(n,p)$ often called the binomial random graph model

$e(G(n,p)) \sim Bi(N,p)$ $\quad N = \binom{n}{2}$

Sometimes called Erdős-Rényi model



Expected # of cycles of a given length in $S(n,p)$.

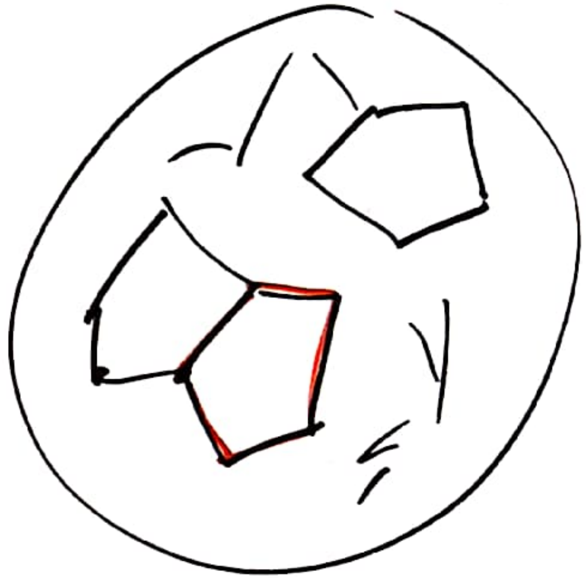
Fix $r \geq 3$, let $X_r$ = # $r$-cycles in $S(n,p)$

$X_r$ counts how many of the events $A_i$ hold where

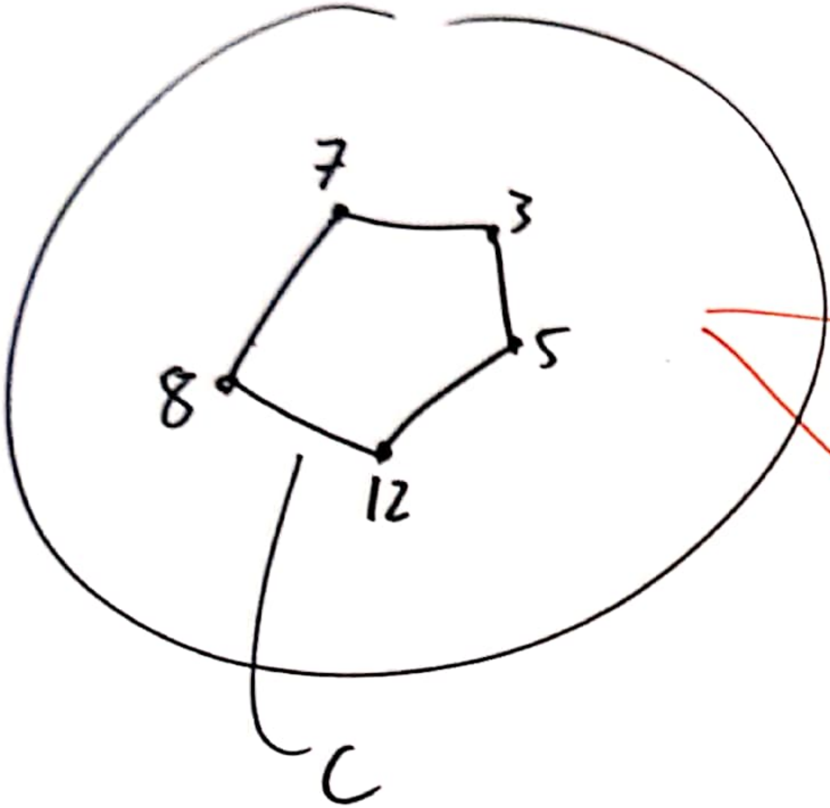
$A_i = \{ i^{th}$ possible cycle is present $\}$.

cycle in $K_n$ of length $r$

$$\mathbb{E}[X_r] = \sum_i \underbrace{\mathbb{P}(A_i)}_{p^r}$$



$V = [n] = V(K_n)$




Can describe all by listing vertices

eg $(7, 3, 5, 12, 8)$

distinct

$(12, 5, 3, 7, 8)$

2r lists

There are $n(n-1)(n-2)\cdots(n-r+1)$ lists of $r$ distinct vxs.

$\therefore$ there are $\dfrac{n(n-1)\cdots(n-r+1)}{2r}$ r-cycles in $K_n$

$\mathbb{E}[X_r] = \dfrac{n(n-1)\cdots(n-r+1)}{2r} p^r.$

**Lemma 1.8 (Markov's Inequality).** Let $X$ be a RV taking non-negative values, and $t > 0$ a constant. Then $\mathbb{P}(X \geq t) \leq \dfrac{\mathbb{E}[X]}{t}$.

<u>Proof</u> $X \geq t I_{\{X \geq t\}}$ always holds.

$\therefore \mathbb{E}[X] \geq \mathbb{E}[\qquad] = t \mathbb{P}(X \geq t)$ $\square$

Notation / definitions

$\chi(G)$ = chromatic number

= min {k : G has a proper vertex colouring using k colours}

$\alpha(G)$ = independence number

= size of largest of independent set


$g(G)$ = girth of G. = length of shortest cycle

or $\infty$ if no cycle


$$\forall G: \chi(G) \geq \frac{|G|}{\alpha(G)}$$

$g(G) \geqslant \ell \iff$ no cycles in $G$ of length $3, 4, \ldots, \ell-1$.

Theorem 1.9 (Erdős 1959).

For any $k$ and $l$ there exists $G$ with $\chi(G) \geq k$ and $g(G) \geq l$.

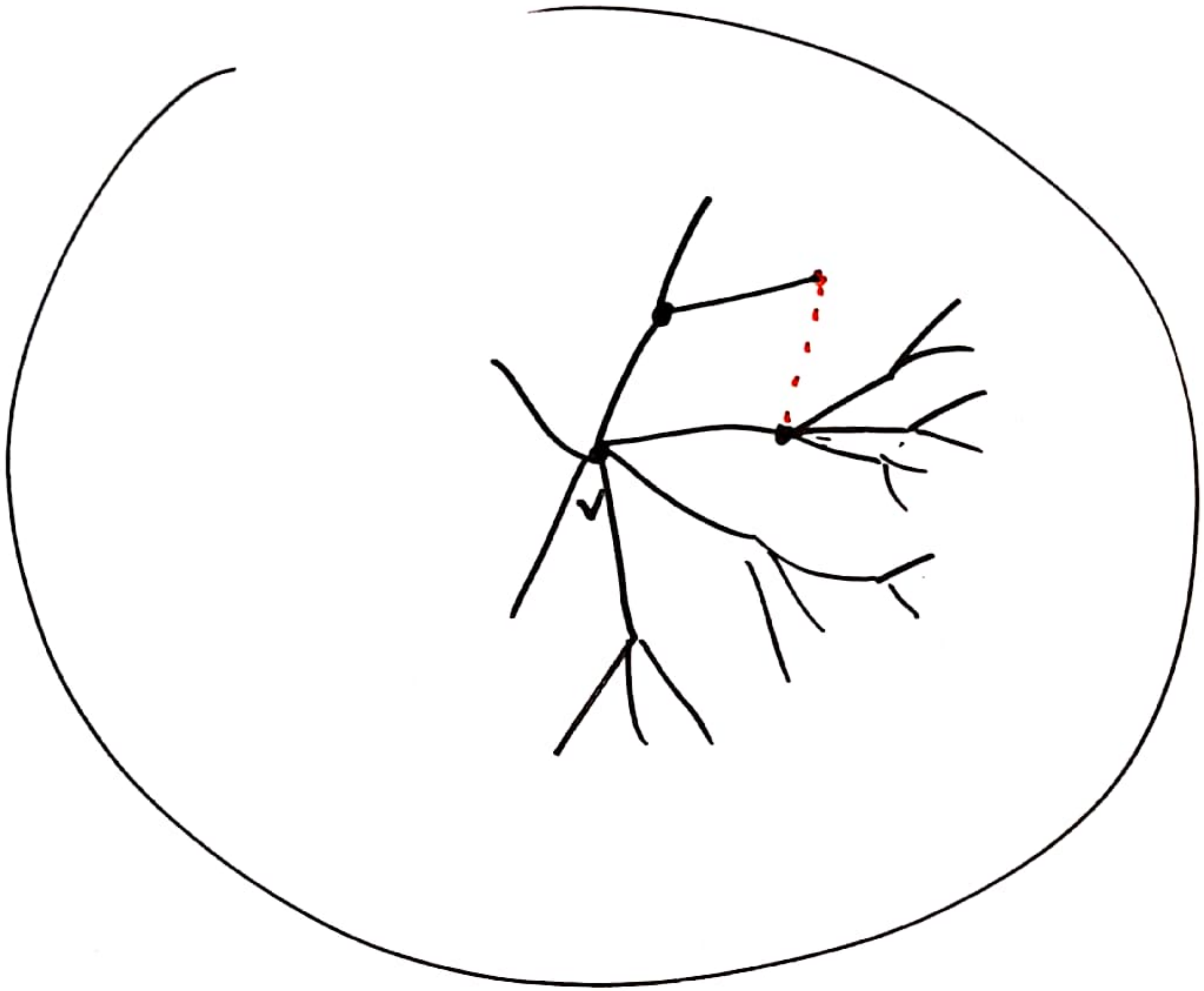
---

$K_{10}$

$g(G) \geq 20$

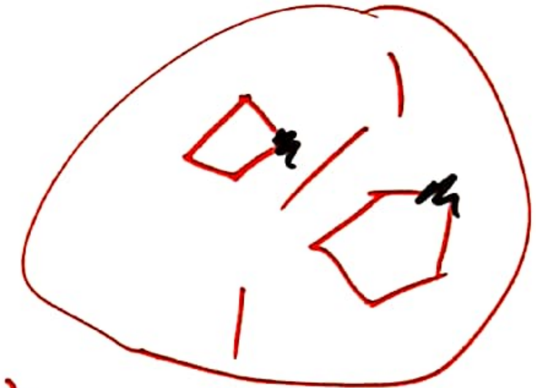
No 3-cycle, no $\triangle$

$g(G) \geq 20$



## Theorem 1.9 (Erdős 1959)

For any $k, l$ $\exists\, G$ s.t. $\chi(G) \geq k$, $g(G) \geq l$.

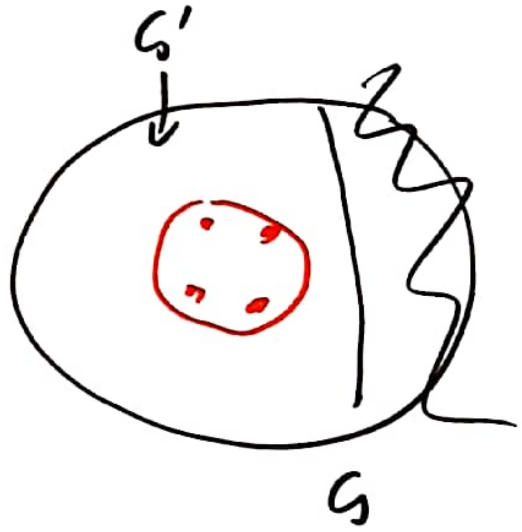


**Idea** Consider $G(n,p)$

Show separately

(i) very likely $G(n,p)$ has ~~no~~ FEW ($< \frac{n}{2}$) short cycles (need $p \ll \frac{1}{n}$)

(ii) very likely $G(n,p)$ has ~~large $\chi$~~. small $\alpha$




$\alpha(G') \leq \alpha(G)$

<u>Theorem 1.9</u> (Erdős 1959) $\forall k, l \; \exists G : \chi(G) \geq k, \; g(G) \geq l.$

<u>Pf</u> Fix $k, l \geq 3$.

Let $p = p(n) = n^{\frac{1}{l}-1}$

$(np)^{l-1} \ll n$

$np = n^{\frac{1}{l}}$

$p = n^{\frac{1}{l}-1}$

(i) Let $X$ = # short cycles in $G(n,p)$

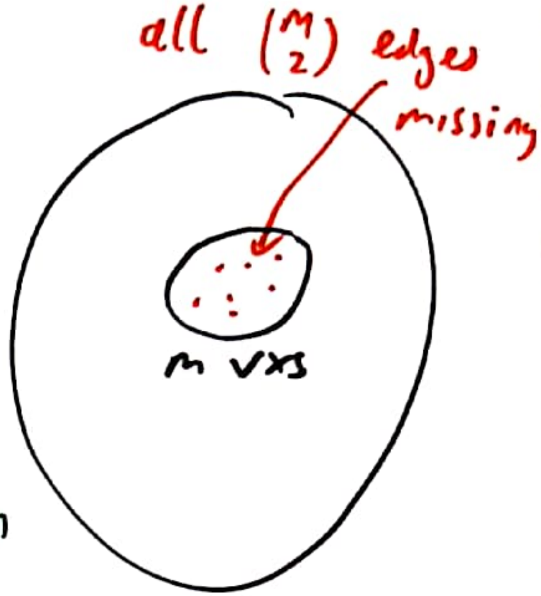
Then $X = \sum_{r=3}^{l-1} X_r$ , $X_r$ = # $r$-cycles.

so $$\mathbb{E}[X] = \sum_{r=3}^{l-1} \mathbb{E}[X_r] = \sum_{r=3}^{l-1} \frac{n(n-1)\cdots(n-r+1)}{2r} p^r$$

$$\leq \sum_{r=3}^{l-1} n^r p^r = \sum_{r=3}^{l-1} n^{r/l} = O(n^{\frac{l-1}{l}}) = o(n)$$

$\therefore$ By Markov $\mathbb{P}(X \geq n/2) \leq \frac{\mathbb{E}(X)}{n/2} \to 0$

(ii) Let $m = m(n) = \lfloor n^{1-\frac{1}{2c}} \rfloor$




Consider $Y$ = # indep. sets in $G(n,p)$ of size exactly $m$

$$\mathbb{E}[Y] = \binom{n}{m}(1-p)^{\binom{m}{2}}$$

$$\leq \left(\frac{en}{m}\right)^m e^{-p\binom{m}{2}} = \Big(\overbrace{\frac{en}{m}\underbrace{e^{-\frac{p(m-1)}{2}}}_{n^{-10}}}^{<1}\Big)^m \qquad \frac{p(m-1)}{2} > 10\log n$$

$$1-x \leq e^{-x}$$

$$\binom{m}{2} = \frac{m(m-1)}{2}$$

Now $\frac{p(m-1)}{2} \sim \frac{pm}{2} \sim \frac{n^{\frac{1}{c}-1}n^{1-\frac{1}{2c}}}{2}$

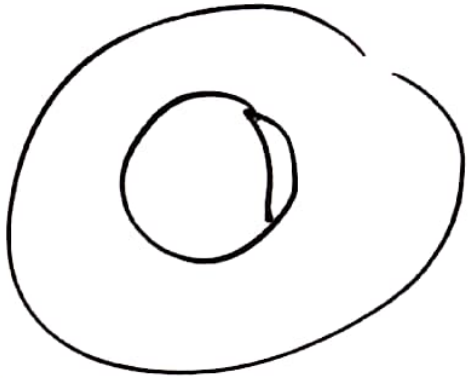
$\sim \frac{n^{\frac{1}{2c}}}{2} > 20 \log n$ if $n$ large.

$$pm = \frac{n^{\frac{1}{c}-1}n^{1-\frac{1}{2c}}}{n^{\varepsilon}} > 10\log n$$

If $n$ large enough $\frac{p(m-1)}{2} > 10\log n$ $\quad\therefore\quad \mathbb{E}[Y] \leq \left(\frac{en}{m}n^{-10}\right)^m \to 0$

∴ By Markov $\mathbb{P}(Y \geq 1) \leq \frac{\mathbb{E}[Y]}{1} \to 0$

$$\mathbb{P}(\alpha(G) \geq m) \to 0$$



By union bound

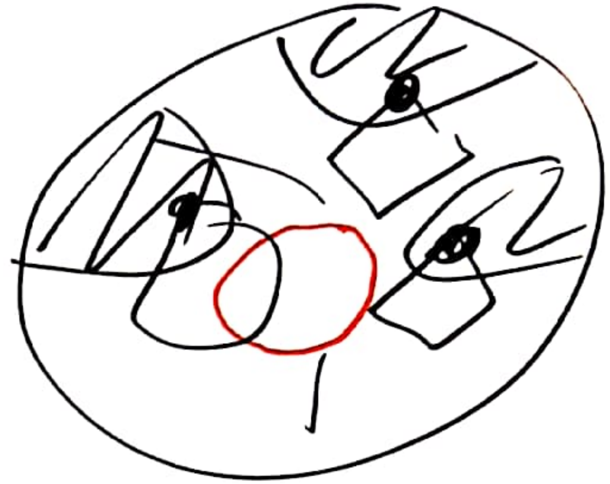
$$\mathbb{P}\left(X \geq \frac{n}{2} \text{ or } \alpha(G(n,p)) \geq m\right) \to 0$$

$$\mathbb{P}\left(X < \frac{n}{2} \text{ AND } \alpha(G(n,p)) < m\right) \to 1.$$

If $n$ large enough

$$\exists \text{ graph } G \text{ s.t. } \begin{cases} G \text{ contains } < \frac{n}{2} \text{ short cycles} \\ \alpha(G) < m. \end{cases}$$

Form $G^*$ by deleting a vertex in each short cycle of $G$



Then $g(G^*) \geqslant l$

$|G^*| > n/2$ (deleted $< \frac{n}{2}$ vxs)

$\alpha(G^*) \leq \alpha(G) < m$

$$\chi(G^*) \geqslant \frac{|G^*|}{\alpha(G^*)} \geqslant \frac{n}{2m} \geqslant \frac{n}{2n^{1-\frac{1}{2l}}} = \frac{n^{\frac{1}{2l}}}{2} > k$$

if $n$ large enough.

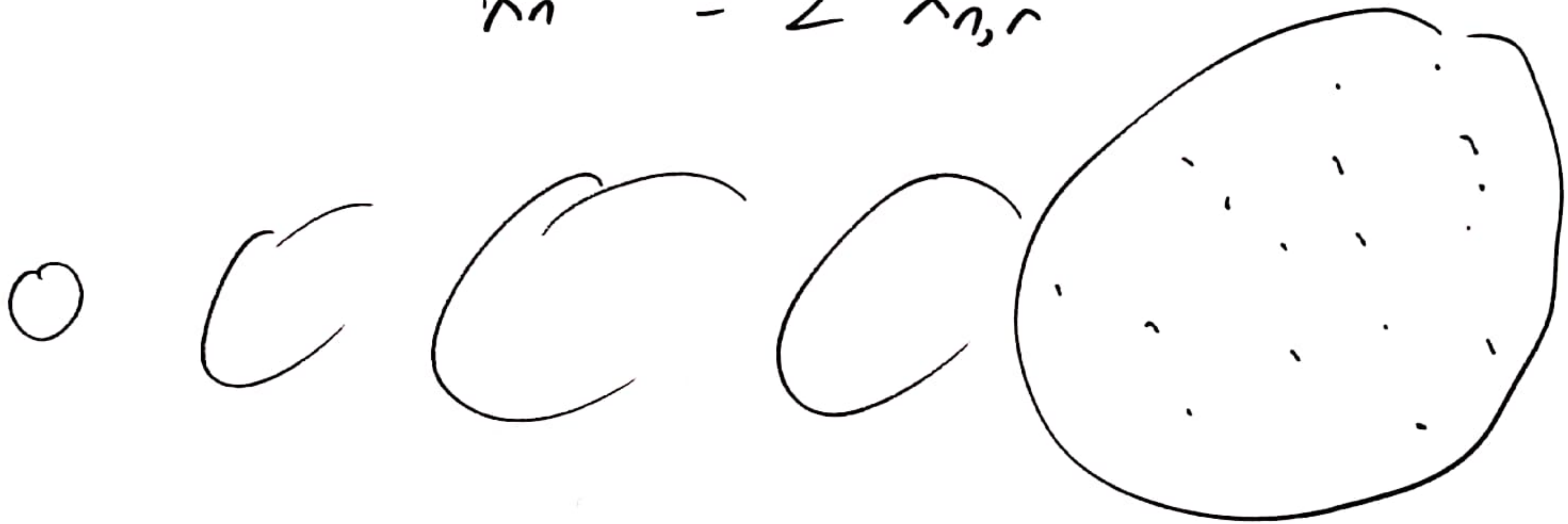
$\square$

$\mathbb{E}[Y] \to 0$ as $n \to \infty$

Had $p = p(n)$ $G_n = G(n, p(n))$

$Y_n = \# \cdots\cdots$ in $G_n$

$X_n = \sum X_{n,r}$



## §2 The second moment method.

**Defn** A counting RV is a RV taking non-negative integer values.

Suppose $(X_n)$ is a sequence of counting RVs.

Would like conditions s.t. can deduce

(i) • $\mathbb{P}(X_n = 0) \to 1$

(ii) • $\mathbb{P}(X_n > 0) \to 1$.

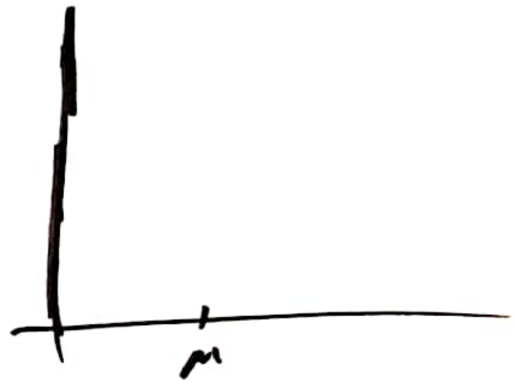
For (i): if $\mathbb{E}(X_n) \to 0$, by Markov

$$\mathbb{P}(X_n > 0) = \mathbb{P}(X_n \geq 1) \leq \frac{\mathbb{E}[X_n]}{1} \to 0.$$

For (ii) might hope $\mathbb{E}[X_n] \to \infty$ is enough.

e.g. $X_n = \begin{cases} n^2 & \text{with prob. } \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$

$\mathbb{E}[X_n] \to 0$ but $\mathbb{P}(X_n > 0) = \frac{1}{n} \to 0$.



**Defn** The variance $Var[X]$ of a RV $X$ is

$$Var[X] = \mathbb{E}[(X-\mathbb{E}X)^2] = \mathbb{E}[X^2] - (\mathbb{E}X)^2$$

**Lemma 2.1** Chebyshev's Ineq.

Let $X$ be a RV, $t>0$. Then $\mathbb{P}(|X-\mathbb{E}X| \geq t) \leq \frac{Var[X]}{t^2}$

**Proof** Apply Markov to $Y = (X-\mathbb{E}X)^2$ $\square$

**Cor 2.2** Let $(X_n)$ be a sequence of RVs with $\mu_n = \mathbb{E}[X_n] > 0$.

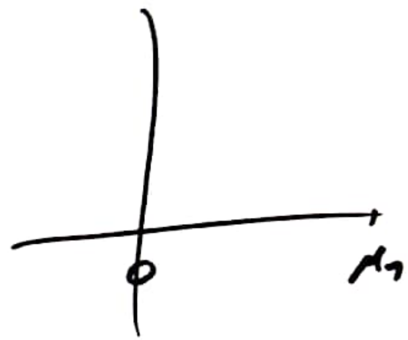
If $Var[X_n] = o(\mu_n^2)$ then $\mathbb{P}(X_n = 0) \to 0$

$\mathbb{P}(X_n > 0) \to 1$.

**Pf**

$$\mathbb{P}(X_n = 0) \le \mathbb{P}(|X_n - \mu_n| \ge \varepsilon \mu_n)$$

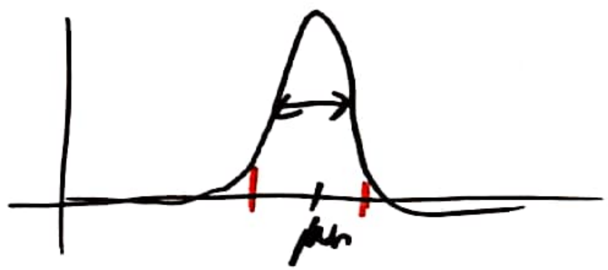
$$\le \frac{Var(X_n)}{\varepsilon^2 \mu_n^2} \to 0.$$




$\square$

Same conditions

$$\mathbb{P}((1-\varepsilon)\mu_n \le X_n \le (1+\varepsilon)\mu_n) \to 1.$$



$$\mathrm{Var}\, X = \mathbb{E}(X^2) - (\mathbb{E}X)^2$$

$$\mathrm{Var}[X_n] = o(\mu_n^2) \iff \underbrace{\mathbb{E}(X_n^2)}_{\text{second moment.}} \sim \mu_n^2$$

Suppose $X = \#\, A_i$ that hold

i.e. $X = \sum_{i=1}^{k} I_i$ where $I_i = I_{A_i}$

$$\mathbb{E}[X] = \sum \mathbb{P}(A_i)$$

$$\mathbb{E}[X^2] = \mathbb{E}\Big[\sum_i I_i \sum_j I_j\Big] = \mathbb{E}\Big[\sum_i \sum_j I_i I_j\Big]$$

$$= \sum_i \sum_j \underbrace{\mathbb{E}[I_i I_j]}_{1 \times 1} = \sum_{i=1}^{k} \sum_{j=1}^{k} \mathbb{P}(A_i \cap A_j)$$

$i=1 \quad j=1$

$i=1 \quad j=2$

$i=2 \quad j=1$

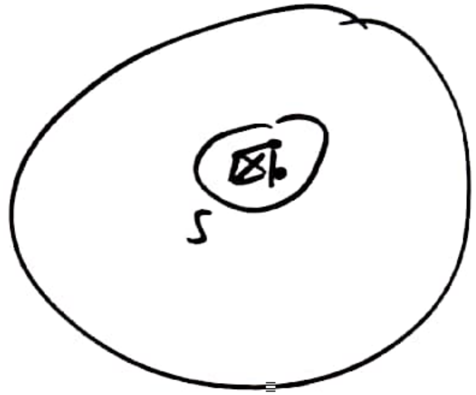
# $K_4$s in $G(n,p)$

**Theorem 2.3** Let $p = p(n)$ be a function of $n$.

(i) If $n^{2/3}p \to 0$, then $\mathbb{P}(G(n,p)$ contains a $K_4) \to 0$

(ii) If $n^{2/3}p \to \infty$, then $\mathbb{P}($ " $) \to 1$.

**Proof** Let $X$ $(= X_n)$ be the RV counting # of copies of $K_4$ in $G(n,p)$. For each $S \subseteq [n]$ with $|S| = 4$ let $A_S$ be the event that $S$ induces a $K_4$ in $G$.




$$\mathbb{E}[X] = \sum_S \mathbb{P}(A_S) = \binom{n}{4} p^6 = \frac{n(n-1)(n-2)(n-3)}{4!} p^6 \sim \frac{n^4 p^6}{4!}$$

$\mu \, (= \mu_n) := \mathbb{E}(X) \sim \dfrac{n^4 p^6}{4!}$ , $\mu = \Theta(n^4 p^6)$

$n^4 p^6 = (n^{2/3} p)^6$ $\left[ n^{2/3} p \to 0 \Leftrightarrow \dfrac{p}{n^{-2/3}} \to 0 \Leftrightarrow p = o(n^{-2/3}) \right]$

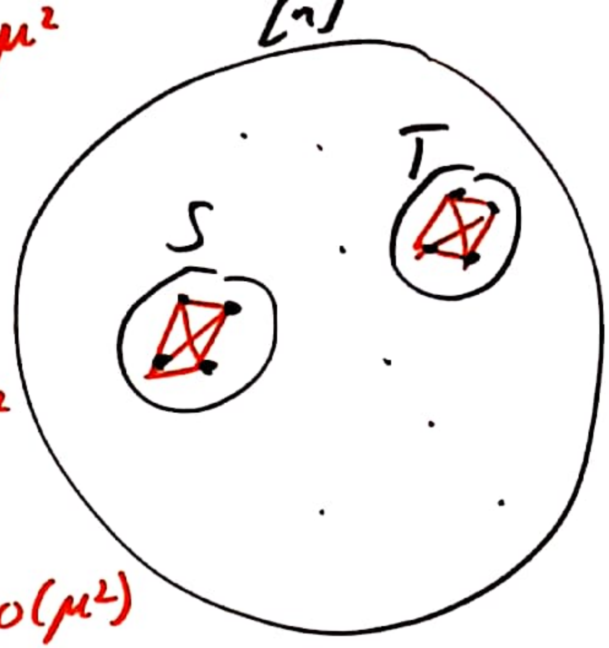
(i) by assumption $n^{2/3} p \to 0 \quad \therefore \quad \mu \to 0$

$\therefore \; \mathbb{P}(X > 0) = \mathbb{P}(X \geq 1) \to 0$ ✓

(ii) $n^{2/3} p \to \infty \quad \therefore \quad \boxed{\mu \to \infty}$

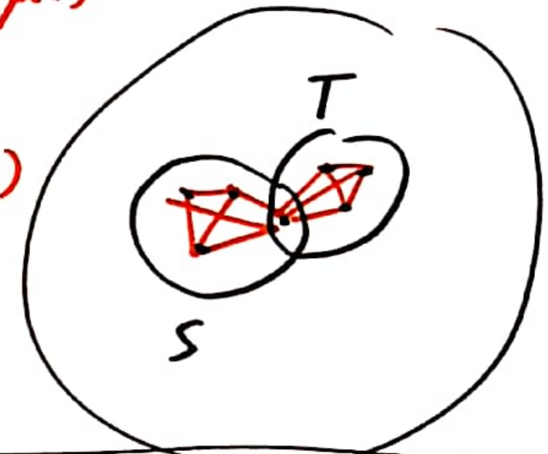
Aim show $Var(X) = o(\mu^2)$ or, equiv $\boxed{\mathbb{E}[X^2] \sim \mu^2}$

$$\mathbb{E}[X^2] = \sum_S \sum_T \mathbb{P}(A_S \cap A_T) \qquad \text{AIM } \sum\sum \sim \mu^2$$




| $\lvert S \cap T \rvert$ | contribution to $\sum\sum$ |
|---|---|
| 0 | $\binom{n}{4}\binom{n-4}{4} p^{12} \sim \frac{n^4}{4!} \cdot \frac{n^4}{4!} p^{12} \sim \mu^2$ |
| 1 | $\binom{n}{4} 4 \binom{n-4}{3} p^{12} = \Theta(n^7 p^{12}) = o(\mu^2)$ |
| 2 | $\binom{n}{4}\binom{4}{2}\binom{n-4}{2} p^{11} = \Theta(n^6 p^{11}) = o(\mu^2)$ |
| 3 | $\binom{n}{4}\binom{4}{3}\binom{n-4}{1} p^{9} = \Theta(n^5 p^9) = o(\mu^2)$ |
| 4 | $\binom{n}{4} p^6 = \mu = \mu = o(\mu^2)$ |

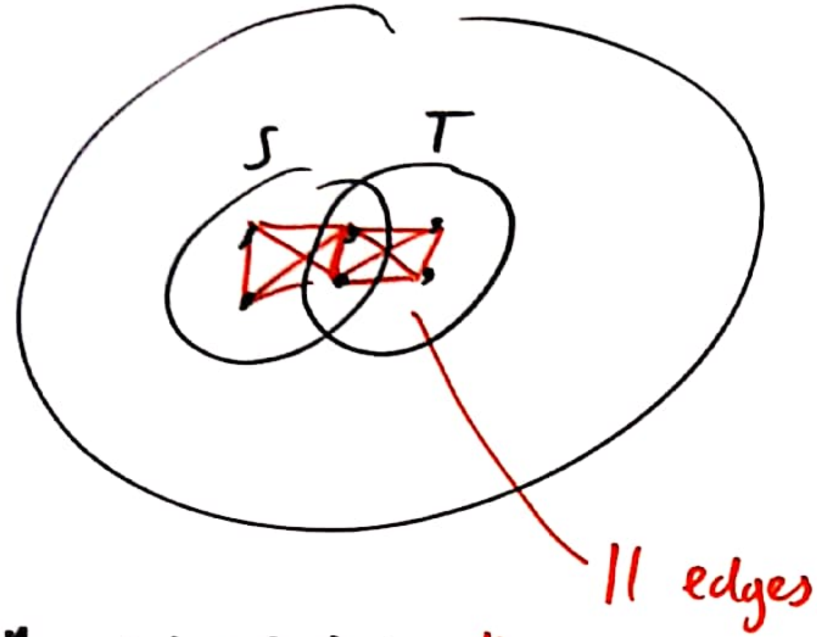
$o(\mu^2)$?!



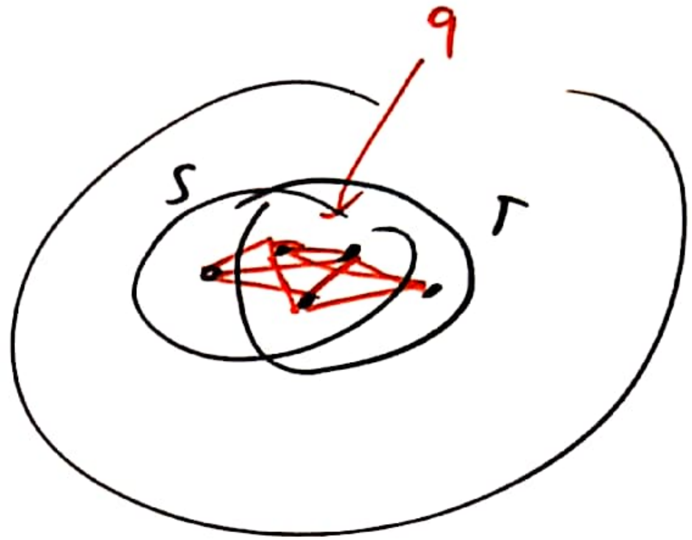

Recall $\mu = \Theta(n^4 p^6) \to \infty$

$\therefore \mathbb{E}[X^2] \sim \mu^2, \quad \mathrm{Var}(X) = o(\mu^2)$

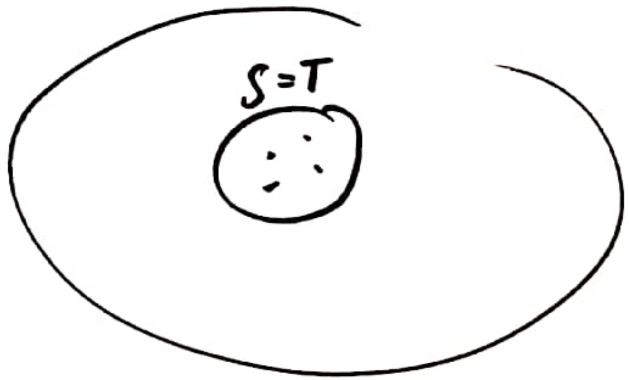
$\therefore$ by Cor 2.2 $\mathbb{P}(X > 0) \to 1$. $\square$




$$\binom{n}{4}\binom{4}{2}\binom{n-4}{2}p^{11}$$




$$\binom{n}{4}\binom{4}{3}\binom{n-4}{1}p^{9}$$




$$\binom{n}{4}p^{6}=\mu$$

$$A_S \cap A_S = A_S$$

$$\frac{n^6 p^{11}}{n^8 p^{12}} = \frac{1}{n^2 p} = \frac{1}{\underbrace{n^{4/3}}_{\to\infty}\underbrace{(n^{2/3}p)}_{\to\infty}} \to 0$$

$$\frac{n^5 p^9}{n^8 p^{12}} = \frac{1}{n^3 p^3} = \frac{1}{\underbrace{n}_{\to\infty}\ \underbrace{(n^{2/3}p)^3}_{\to\infty}} \to 0$$

Let $\mathcal{P}$ be a property of graphs (eg is connected).

A function $p^*(n)$ is a <u>threshold ~~function~~</u> for $\mathcal{P}$ in the model $G(n,p)$ if

- $\dfrac{p(n)}{p^*(n)} \to 0 \Rightarrow \mathbb{P}(G(n,p(n)) \text{ has } \mathcal{P}) \to 0$

& - $\dfrac{p(n)}{p^*(n)} \to \infty \Rightarrow \mathbb{P}(G(n,p(n)) \text{ has } \mathcal{P}) \to 1$.

Theorem 2·3 says $p^*(n) = n^{-2/3}$ is a threshold for 'contains a $K_4$'.

Not unique eg $10n^{-2/3}$ also works

$$q(n) = \mathbb{P}(G(n, p(n)) \text{ has } P) \to 0$$

$$\to 1$$

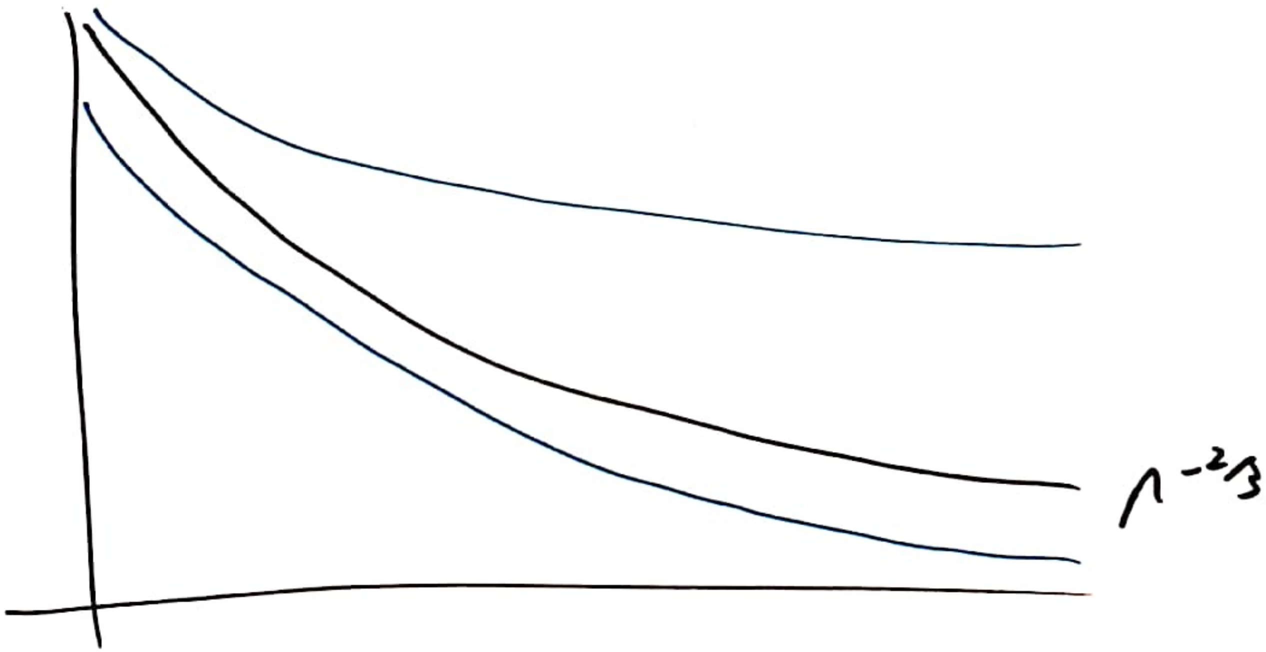
$$\to \tfrac{1}{2}$$

$$\to 0.372$$

For 'sensible $p(n)$' get $\to 0$ or $\to 1$

unless $\Theta(n^{-2/3})$

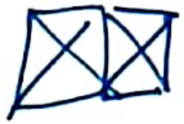
$$n^{-\alpha}(\log n)^{\beta}(\log\log n)^{\gamma}\, e^{\delta(\sqrt{\log n})} \quad \cdots\cdots$$




$$\frac{p}{n^{-2/3}} \to 0 \qquad \frac{p}{n^{-2/3}} \to \infty$$

"Monotone properties"

$X = \# A_i$ hold, $\quad X = \sum_{i=1}^{k} I_i \qquad I_i = I_{A_i}$



$$\mathrm{Var}[X] = \mathbb{E}(X^2) - (\mathbb{E}X)^2$$

$$= \sum_i \sum_j \left( \mathbb{P}(A_i \cap A_j) - \mathbb{P}(A_i)\mathbb{P}(A_j) \right)$$

Write $i \sim j$ if $i \neq j$ and $A_i$ and $A_j$ are dependent

$$= \sum_i \left( \mathbb{P}(A_i) - \mathbb{P}(A_i)^2 \right) + \sum_i \sum_{j \sim i} \left( \mathbb{P}(A_i \cap A_j) - \mathbb{P}(A_i)\mathbb{P}(A_j) \right) \qquad \begin{matrix} i = j \\ i \sim j \end{matrix}$$

$$\leq \underbrace{\sum_i \mathbb{P}(A_i)}_{\mathbb{E}[X]} + \underbrace{\sum_i \sum_{j \sim i} \mathbb{P}(A_i \cap A_j)}_{\Delta}$$

$i \neq j$, $i \not\sim j$ $\quad A_i, A_j$ independent

**Cor 2.4** If $\mu := \mathbb{E}[X] \to \infty$ and $\Delta = \sum_i \sum_{j \sim i} \mathbb{P}(A_i \cap A_j) = o(\mu^2)$

then $\mathbb{P}(X > 0) \to 1$.

**Proof** $Var[X] \leq \mu + \Delta$

$$\therefore \frac{Var(X)}{\mu^2} \leq \frac{1}{\mu} + \frac{\Delta}{\mu^2} \to 0$$

$\mathbb{P}(X = 0) \to 0$.